Stochastic Differential Equations and Diffusion Processes

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1 Notion of solutions to stochastic differential equations

The goal of this section is to give a notion of solutions to stochastic differential equations (SDE). We will discuss here different types of solutions to SDE and consider the uniqueness.

1.1 A basic notion of stochastic analysis

Let B_t, $t \geq 0$, be a **Brownian motion** in $\mathbb{R}^n$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, B satisfies the following properties

1. $B_0 = 0$;
2. $B_t - B_s \sim N(0, I_m(t-s))$, where I_m denotes the identity $m \times m$-matrix;
3. B has independent increments, i.e. for every $t_1 < \dots < t_n$ the random variables B_{t_1}, $B_{t_2} - B_{t_1}$,…,$B_{t_n} - B_{t_{n-1}}$ are independent;
4. B is a continuous process in $\mathbb{R}^n$.

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. We will call the process B an $(\mathcal{F}_t)$-**Brownian motion** if additionally B is $(\mathcal{F}_t)$-adapted and $B_t - B_s$ is independent of $\mathcal{F}_s$ for every $0 \leq s < t$. Considering a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, we always understand that it is an $(\mathcal{F}_t)$-Brownian motion.

Exercise 1.1. Let $(\mathcal{F}_t^B)_{t\geq 0}$ be the natural filtration generated by a Brownian motion B, that is,

$$\mathcal{F}_t^B := \sigma(B_s,\ s \leq t), \quad t \geq 0.$$

Show that B is an $(\mathcal{F}_t^B)$-Brownian motion.

We define the complete extension of a filtration $(\mathcal{F}_t)_{t\geq 0}$ as follows

$$\bar{\mathcal{F}}_t = \sigma(\mathcal{F}_t, \mathcal{N}), \quad t \geq 0,$$

where $\mathcal{N} = \left\{ A \subset \Omega :\ \exists \tilde{A} \in \mathcal{F}_\infty \text{ such that } \mathbb{P}\left\{\tilde{A}\right\} = 0 \text{ and } A \subseteq \tilde{A} \right\}$ and $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$. Once can easily seen that $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and it is the smallest complete extension of $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1.2. Let B be an $(\mathcal{F}_t)$-Brownian motion. Show that B is an $(\bar{\mathcal{F}}_t)$-Brownian motion.

We now discuss the integration with respect to an $(\mathcal{F}_t)$-Brownian motion.

Definition 1.1. A process φ_t, $t \geq 0$, is called $(\mathcal{F}_t)$**-progressive**[1] if, for every $t \geq 0$, the restriction of the map $(s, \omega) \mapsto \varphi_s(\omega)$ to $[0,t] \times \Omega$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

Exercise 1.3. Every $(\mathcal{F}_t)$-adapted càdlàg process with values in a metric space is $(\mathcal{F}_t)$-progressive.

If φ_t, $t \geq 0$, is an $(\mathcal{F}_t)$-progressive process satisfying

$$\mathbb{E} \int_0^t \varphi_s^2 ds < \infty, \quad t \geq 0,$$

Then one can define a stochastic integral

$$I_t := \int_0^t \varphi_s dB_s, \quad t \geq 0,$$

with respect a Brownian motion B in $\mathbb{R}$, which is a continuous square integrable martingale with quadratic variation

$$\langle I \rangle_t = \int_0^t \varphi_s^2 ds, \quad t \geq 0.$$

Moreover, the class of integrands can be extended to $(\mathcal{F}_t)$-progressive processes satisfying

$$\mathbb{P} \left\{ \int_0^t \varphi_s^2 ds < \infty, \quad t \geq 0 \right\} = 1.$$

In this case the stochastic integral I_t, $t \geq 0$, is a continuous local martingale with the same quadratic variation. For the definition of stochastic integral and its properties see e.g. [Bov18, Section 3]. We recall here only Itô's formula.

Consider a stochastic process X_t, $t \geq 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ of the form

$$X_t = x_0 + V_t + M_t,$$

where $V = (V^l)_{l \in [m]}$ is a continuous $(\mathcal{F}_t)$-adapted process whose coordinate processes have bounded variation, the coordinate processes of $M = (M^l)_{l \in [m]}$ are continuous local $(\mathcal{F}_t)$-martingales, and $V_0 = M_0 = 0$. Let $f : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable in the first and twice continuously differentiable in the second argument. Then **Itô's formula** holds:

$$f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial}{\partial s} f(s, X_s) ds + \sum_{l=1}^m \int_0^t \frac{\partial}{\partial x_l} f(s, X_s) dV_s^l + \sum_{l=1}^m \int_0^t \frac{\partial}{\partial x_l} f(s, X_s) dM_s^l + \frac{1}{2} \sum_{k,l=1}^m \int_0^t \frac{\partial^2}{\partial x_l \partial x_k} f(s, X_s) d\langle M^l, M^k \rangle_s, \quad t \geq 0.$$

[1]The notion of a progressive process is stronger than that of an adapted process. The important property of an $(\mathcal{F}_t)$-progressive process is that the τ-stopped is $\mathcal{F}_\tau$-measurable (for more details see Section 2.7 [Bov18])

1.2 Definition of solutions to SDE

We will consider the following stochastic differential equation in $\mathbb{R}^n$

$$\begin{aligned} dX_t^k &= b_k(t, X_t)dt + \sum_{l=1}^{m} \sigma_{k,l}(t, X_t)dB_t^l, \\ X_0 &= x_0, \quad k = 1, \ldots, n, \end{aligned} \tag{1}$$

where b_k, $\sigma_{k,l}$, $k \in [n]$, $l \in [m]$, are Borel measurable function from $[0, \infty) \times \mathbb{R}^n$ to $\mathbb{R}$, $B = (B^l)_{l \in [m]}$ is an m-dimensional Brownian motion, and $X_t = (X_t^k)_{k \in [n]}$. Shortly, we will write equation (1) in the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0.$$

Definition 1.2. A **weak solution** to SDE (1) is a pair (X, B) of adapted processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that

(a) B is an m-dimensional $(\mathcal{F}_t)$-Brownian motion;

(b) for any $t \geq 0$,
$$\int_0^t \left(\sum_{k=1}^{n} |b_k(s, X_s)| + \sum_{k=1}^{n} \sum_{l=1}^{m} \sigma_{k,l}^2(s, X_s) \right) ds < \infty \quad \mathbb{P}\text{-a.s.}$$

(c) for any $t \geq 0$ and $k \in [n]$
$$X_t^k = x_0^k + \int_0^t b_k(s, X_s)ds + \sum_{l=1}^{m} \int_0^t \sigma_{k,l}(s, X_s)dB_s^l \quad \mathbb{P}\text{-a.s.}$$

Remark 1.1. A weak solution to SDE (1) is the date $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, X, B)$. We will say that a **weak solution to equation** (1) **esixts** if there exist a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a weak solution (X, B) defined on this space.

Definition 1.3. A weak solution (X, B) to SDE (1) is called a **strong solution** if X is adapted to the complete extension $(\bar{\mathcal{F}}_t^B)$ of the filtration generated by the Brownian motion B.

Since for every $t \geq 0$ the random vector X_t is measurable with respect to the σ-algebra $(\bar{\mathcal{F}}_t^B)$, one can see that there exists a Borel measurable map

$$\Phi_t : \mathcal{C}([0, t], \mathbb{R}^m) \to \mathbb{R}^n$$

such that $X_t = \Phi_t \left((B_s)_{s \in [0,t]} \right)$ $\mathbb{P}$-a.s. This implies that there exists a Borel measurable map

$$\Psi : \mathcal{C}([0, \infty), \mathbb{R}^m) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$$

such that $X = \Psi(\mathcal{B})$ $\mathbb{P}$-a.s. This observation leads to the following statement.

Proposition 1.1. *Let (X, B) be a strong solution to SDE (1). Then*

(i) there exists a measurable map

$$\Psi : \mathcal{C}([0, \infty), \mathbb{R}^m) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$$

such that the process $\Psi(B)$ is $(\bar{\mathcal{F}}_t^B)$-adapted and $X = \Psi(B)$ $\mathbb{P}$-a.s.

(ii) If $\tilde{B}$ is an m-dimensional $(\tilde{\mathcal{F}}_t)$-Brownian motion on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ and $\tilde{X} = \Psi(\tilde{B})$, then $(\tilde{X}, \tilde{B})$ is a strong solution to (1).

Proof. The detailed proof of the proposition can be found e.g. in [Che00, Teorem 2.1]. □

Now we will formulate different types of uniqueness of solutions to (1).

Definition 1.4. There is **uniqueness in law** for (1) if for any solutions (X, B) and $(\tilde{X}, \tilde{B})$ (that may be defined on different filtered probability spaces), one has $\mathrm{Law}\, X = \mathrm{Law}\, \tilde{X}$.

Definition 1.5. We say that there is **pathwise uniqueness** for (1) if for any solutions (X, B) and $(\tilde{X}, B)$ (that are defined on the same filtered probability space) one has $\mathbb{P}\left\{X_t = \tilde{X}_t, \quad t \geq 0\right\} = 1$.

1.3 Some examples

In this section we will consider some important examples of stochastic differential equations.

Example 1.1 (No solutions)**.** There exists no weak solution to the equation

$$dX_t = -\frac{1}{2X_t}\mathbb{I}_{\{X_t\neq 0\}}dt + dB_t, \quad X_0 = 0. \tag{2}$$

Indeed, suppose that there exists a weak solution (X, B) to SDE (2). Then

$$X_t = -\int_0^t \frac{1}{2X_s}\mathbb{I}_{\{X_s\neq 0\}}ds + B_t, \quad t \geq 0.$$

By Itô's formula,

$$\begin{aligned} X_t^2 &= -\int_0^t 2X_s\frac{1}{2X_s}\mathbb{I}_{\{X_s\neq 0\}}ds + \int_0^t 2X_s dB_s + t \\ &= \int_0^t \mathbb{I}_{\{X_s=0\}}ds + \int_0^t 2X_s dB_s, \quad t \geq 0. \end{aligned}$$

Since the process X is a continuous semimartingale with the quadratic variation $\langle X\rangle_t = t$, the occupation times formula (we will discuss this formula later in the topic devoted to the local time) implies that

$$\int_0^t \mathbb{I}_{\{X_s=0\}}ds = 0, \quad t \geq 0.$$

This implies that X_t^2 is a positive local martingale which started at 0. This implies that $X_t^2 = 0$, $t \geq 0$, a.s. But then (X, B) is not a solution to (2).

Exercise 1.4. Let X_t, $t \geq 0$, be a positive local martingale. Define the hitting time

$$\tau := \inf\{t \geq 0 : \ X_t = 0\}.$$

Show that

$$\mathbb{P}\{X_t = 0, \quad t \geq \tau\} = 1.$$

Example 1.2 (No strong solution; Tanaka)**.** For the SDE

$$dX_t = \operatorname{sgn} X_t dB_t, \quad X_0 = 0, \tag{3}$$

where

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0, \end{cases}$$

we have no strong solution and no pathwise uniqueness.

Weak existence. Let W_t, $t \geq 0$, be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. We set

$$X_t = W_t, \quad B_t = \int_0^t \operatorname{sgn} W_s dW_s, \quad t \geq 0$$

and take $\mathcal{F}_t = \mathcal{F}_t^W$. It is trivial that (X, B) is a solution to (3) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Uniqueness in law. If (X, B) is a solution to (3) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, then X is a continuous $(\mathcal{F}_t)$-local martingale with $\langle X \rangle_t = t$. The Levy characterization theorem implies that X is a Brownian motion. This implies the uniqueness in law.

No strong existence. If (X, B) is a solution to (3), then

$$B_t = \int_0^t \operatorname{sgn} X_s dX_s, \quad t \geq 0.$$

Studying the local time, we will show that this equality implies that $\mathcal{F}_t^B = \mathcal{F}_t^{|X|}$. Hence, there exists no strong solution.

No pathwise uniqueness. If (X, B) is a solution to (3), then $(-X, B)$ is a solution to (3). Indeed,

$$-X_t = -\int_0^t \operatorname{sgn} X_s dB_s = \int_0^t \operatorname{sgn}(-X_s) dB_s + 2\int_0^t \mathbb{I}_{\{X_s = 0\}} dB_s,$$

where we have used the equality $-\operatorname{sgn} x = \operatorname{sgn}(-x) + 2\mathbb{I}_{\{x=0\}}$. Let us show that the square integrable martingale $M_t := \int_0^t \mathbb{I}_{\{X_s=0\}} dB_s$, $t \geq 0$, equals 0. Remark that its quadratic variation vanishes

$$\langle M \rangle_t = \int_0^t \mathbb{I}_{\{X_s=0\}} ds = 0, \quad t \geq 0,$$

because X is a Brownian motion. Thus, $\mathbb{E}\, M_t^2 = \mathbb{E}\, \langle M \rangle_t = 0$, that implies that $(-X, B)$ is a solution to (3). Therefore, there is no pathwise uniqueness.

Exercise 1.5. Show that the equation

$$dX_t = \mathbb{I}_{\{X_t \neq 0\}} dB_t, \quad X_0 = 0,$$

has a strong solution and a weak solution (which is not a strong solution). Show that the uniqueness in law and the pathwise uniqueness fail.

Other interesting examples can be found in [CE05, Section 1.3].

2 First existence results of solutions to SDEs

2.1 Lipschitz continuous coefficients

In this section, we will discuss the existence (and also uniqueness) of solutions to equation (1) whose conditions have a "good" regularity. For simplicity, we will consider the equation on the time interval $[0, T]$ for a fixed $T > 0$.

For a vector $a = (a_k)_{k\in[n]}$ and a matrix $A = (A_{k,l})_{k\in[n],l\in[m]}$ we introduce the norms

$$\|a\| := \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|A\| := \left(\sum_{k=1}^{n}\sum_{l=1}^{m} A_{k,l}\right)^{\frac{1}{2}}.$$

Theorem 2.1. *Let functions $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be Lipschitz continuous functions which have at most a linear growth, that is, there exists $C > 0$ such that*

$$\|b(t, x) - b(t, y)\|^2 + \|\sigma(t, x) - \sigma(t, y)\|^2 \le C\|x - y\|^2, \quad t \in [0, T], \quad x, y \in \mathbb{R}^n,$$

and

$$\|b(t, x)\|^2 + \|\sigma(t, x)\|^2 \le C(1 + \|x\|^2), \quad t \in [0, T], \quad x \in \mathbb{R}^n.$$

Then for every m-dimensional Brownian motion B_t, $t \ge 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a continuous $(\bar{\mathcal{F}}_t^B)$-adapted process with

$$\sup_{t\in[0,T]} \mathbb{E}\,\|X_t\|^2 < \infty,$$

such that (X, B) is a strong solution to SDE (1):

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0.$$

Moreover, the solution is pathwise unique.

Idea of proof. The proof of the statement follows from a kind of the Picard iteration argument for ordinary differential equations. Considering the sequence of continuous $(\bar{\mathcal{F}}_t^B)$-adapted processes

$$X_t^{(n+1)} = x_0 + \int_0^t b(s, X_s^{(n)})ds + \int_0^t \sigma(s, X_s^{(n)})dB_s, \quad t \in [0, T], \quad n \ge 0$$
$$X_t^{(0)} = x_0, \quad t \in [0, T],$$

one can check the inequality

$$\mathbb{E}\left[\sup_{s\in[0,T]} \|X_s^{(n+1)} - X_s^{(n)}\|^2\right] \le \tilde{C}\frac{D^n T^n}{n!}, \quad n \ge 1,$$

for some positive constants $\tilde{C}$ and D, by Gronwall's lemma and the assumption of the theorem. This implies that the sequence $\{X^{(n)},\ n \ge 1\}$ converges in $\mathcal{C}([0, T], \mathbb{R}^n)$ to a (continuous) $(\bar{\mathcal{F}}_t^B)$-adapted process X which solves SDE (1). In order to prove the pathwise uniqueness one needs to use Gronwall's lemma again.

For the detailed proof in the one dimensional case ($n = m = 1$) see e.g. [Tra20, Theorem 4.3]. The proof of the theorem in more general case can be found e.g. in [KS91, Theorem 5.2.9], [IW89, p.178-182] or [EK86, Theorem 5.3.11]. □

2.2 Weak solutions via Girsanov's theorem

Theorem 2.1 implies that SDE (1) admits a (strong) solution if its coefficients are Lipschitz continuous with at most a linear growth. From Section 1.3 we know that this is not true in a general case. We will consider here a class of equations which admits a weak solutions even the drift b is only bounded and measurable.

Proposition 2.1. *Consider the stochastic differential equation*

$$\begin{aligned} dX_t &= b(t, X_t)dt + dB_t, \quad 0 \leq t \leq T, \\ X_0 &= x_0, \end{aligned} \tag{4}$$

where $T > 0$ is fixed, B is an n-dimensional Brownian motion, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Borel-measurable bounded function and $x_0 \in \mathbb{R}^n$. Then there exists a weak solution to equation (4).

We will prove the proposition using Girsanov's transform. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $B_t = (B_t^k)_{k\in[n]}$, $t \geq 0$, be an n-dimensional $(\mathcal{F}_t)$-Brownian motion on that space. Given an $(\mathcal{F}_t)$-progressive process $Y = (Y_t^k)_{k\in[n]}$, $t \geq 0$, satisfying for every $k \in [n]$

$$\mathbb{P}\left\{\int_0^t \left(Y_s^k\right)^2 ds < \infty, \quad t \geq 0\right\} = 1,$$

we define the continuous process

$$Z_t = \exp\left\{\sum_{k=1}^n \int_0^t Y_s^k dB_s^k - \frac{1}{2}\int_0^t \|Y_s\|^2 ds\right\}, \quad t \geq 0. \tag{5}$$

Setting $N_t = \sum_{k=1}^n \int_0^t Y_s^k dB_s^k$, we can define the process Z_t, $t \geq 0$, as follows

$$Z_t = \exp\left\{N_t - \frac{1}{2}\langle N\rangle_t\right\}, \quad t \geq 0.$$

Exercise 2.1. Applying Itô's formula, show that

$$Z_t = 1 + \sum_{k=1}^n \int_0^t Z_s Y_s^k dB_s^k, \quad t \geq 0.$$

According to Exercise 2.1, the process Z is a positive local $(\mathcal{F}_t)$-martingale. If $\mathbb{E}\, Z_t < \infty$, $t \geq 0$, i.e. it is a martingale, then $\mathbb{E}\, Z_t = 1$ for every $t \geq 0$. In this case, we can define a new probability measure on $(\Omega, \mathcal{F}_T)$ by

$$\tilde{\mathbb{P}}_T(A) := \mathbb{E}\, \mathbb{I}_A Z_T, \quad A \in \mathcal{F}_T.$$

Theorem 2.2 (Girsanov)**.** *Let the process Z_t, $t \geq 0$, defined by (5), be a martingale, i.e. $\mathbb{E}\, Z_t = 1$, $t \geq 0$. Then the process*

$$\tilde{B}_t = B_t - \int_0^t Y_s ds, \quad t \in [0, T],$$

is an n-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \tilde{\mathbb{P}}_T)$.

Proof. For the proof of the theorem in the one-dimensional case see e.g. [Tra20, Theorem 6.2]. The proof in the general case can be found e.g. in [KS91, Theorem 3.5.1]. ☐

Remark 2.1. The fact that the process Z_t, $t \geq 0$, defined by (5), is a martingale follows from Novikov's condition

$$\mathbb{E}\, e^{\frac{1}{2}\langle N\rangle_t} < \infty, \quad t \geq 0.$$

See e.g. [KS91, Proposition 3.5.12].

Proof of Proposition 2.1. Let X_t, $t \geq 0$, be an n-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. By Remark 2.1, the process

$$Z_t = \exp\left\{ \sum_{k=1}^n \int_0^t b_k(s, X_s) dX_s^k - \frac{1}{2}\int_0^t \|b(s, X_s)\|^2 \right\}, \quad t \geq 0,$$

is a continuous martingale due to the fact that the quadratic variation of the martingale $N_t = \sum_{k=1}^n \int_0^t b_k(s, X_s) dX_s^k$, $t \geq 0$, equals

$$\langle N\rangle_t = \sum_{k=1}^n \int_0^t b_k^2(s, X_s) ds = \int_0^t \|b(s, X_s)\|^2 ds,$$

and is bounded in ω for every $t \geq 0$. Therefore, we can define a new probability measure on $(\Omega, \mathcal{F}_T)$ as follows

$$\tilde{\mathbb{P}}_T(A) = \mathbb{E}\, \mathbb{I}_A Z_T, \quad A \in \mathcal{F}_T.$$

Using Girsanov's Theorem 2.2, we obtain that the process

$$B_t := X_t - x_0 - \int_0^t b(s, X_s) ds, \quad t \in [0, T],$$

is an $(\mathcal{F}_t)$-Brownian motion. Thus, (X, B) is a weak solution to SDE (4) on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \tilde{\mathbb{P}})$, since

$$X_t = x_0 + \int_0^t b(s, X_s) ds + B_t, \quad t \in [0, T].$$

□

Remark 2.2. In Proposition 2.1, the assumption on the boundedness of b is stronger than necessary but it simplify the proof. This assumption can be relaxed to

$$\|b(t, x)\| \leq C(1 + \|x\|), \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

for some constant $C > 0$. For more details see [KS91, Proposition 5.3.6].

3 Martingale problem

3.1 Connection with well-possedness of SDEs

We know that any continuous local martingale M_t, $t \geq 0$, with quadratic variation $\langle M\rangle_t = t$, $t \geq 0$, is a Brownian motion. This is Levy's characterisation of Brownian motion. It turns out that weak solutions to SDEs can be describe in a similar fashion via so called a *martingale problem*.

In this section, we will consider SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0, \tag{6}$$

in general form (1), where $b : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be Borel measurable functions, and B_t, $t \geq 0$, be an m-dimensional Brownian motion. We define $a = \sigma\sigma^T$, that is,

$$a_{k,j} = \sum_{l=1}^{m} \sigma_{k,l}\sigma_{j,l}, \quad k, j \in [n].$$

For every $t \geq 0$, we also introduce the second-order differential operator

$$\mathcal{A}_t f(x) := \frac{1}{2}\sum_{k,j=1}^{n} a_{k,j}(t, x)\frac{\partial^2 f(x)}{\partial x_k \partial x_j} + \sum_{k=1}^{n} b_k(t, x)\frac{\partial f(x)}{\partial x_k}, \quad f \in \mathcal{C}^2(\mathbb{R}^n). \tag{7}$$

Here $\mathcal{C}^2(\mathbb{R}^n)$ denotes the space of all twice continuously differentiable functions on $\mathbb{R}^n$.

The applications of Itô's formula straightforward implies the following result.

Proposition 3.1. *Let (X, B) be a weak solution to equation (6) (defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})$). Then for every $f \in \mathcal{C}^2(\mathbb{R}^n)$, the process*

$$\begin{aligned} M_t^f :&= f(X_t) - f(X_0) - \int_0^t \left(\frac{1}{2}\sum_{k,j=1}^{n} a_{k,j}(s, X_s)\frac{\partial^2 f(X_s)}{\partial x_k \partial x_j} + \sum_{k=1}^{n} b_k(s, X_s)\frac{\partial f(X_s)}{\partial x_k}\right) ds \\ &= f(X_t) - f(X_0) - \int_0^t \mathcal{A}_s f(X_s)ds, \quad t \geq 0, \end{aligned}$$

is a continuous local $(\mathcal{F}_t^X)$-martingale with $M_0^f = 0$. Moreover, if f has a compact support and the coefficients b and σ are bounded, then M_t^f, $t \geq 0$, is a square integrable martingale.

Exercise 3.1. Let the coefficient of SDE (6) are locally bounded. Show that M_f is a martingale for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$.[2]

Exercise 3.2. Let X_t, $t \geq 0$, be a continuous process on $\mathbb{R}$ and $X_0 = 0$. Assume that for every $f \in \mathcal{C}^2(\mathbb{R})$

$$M_t^f := f(X_t) - f(0) - \int_0^t \frac{1}{2} f''(X_s)ds, \quad t \geq 0,$$

is a continuous local martingale. Show that X_t, $t \geq 0$, is a Brownian motion.

Exercise 3.3. Let (X, B) be a weak solution to SDE (6).

(i) Show that for every $f \in \mathcal{C}^{1,2}([0, \infty) \times \mathbb{R}^n)$ the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \left(\frac{\partial f(s, X_s)}{\partial s} + \mathcal{A}_s f(s, X_s)\right) ds, \quad t \geq 0,$$

is a continuous local $(\mathcal{F}_t^X)$-martingale.

[2] the space $\mathcal{C}_0^2(\mathbb{R}^n)$ denotes the space of functions from $\mathcal{C}^2(\mathbb{R}^n)$ with compact support

(ii) Assume that $n = m$, a is the identity matrix and $b = 0$, that is, $X_t = x_0 + B_t$, $t \geq 0$, is an n-dimensional Brownian motion started at x_0. Show that for every bounded $\varphi \in \mathcal{C}(\mathbb{R}^n)$,

$$\mathbb{E}\,\varphi(X_t) = u(t, x_0)$$

where u is the solution to the heat equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x), \quad t > 0, \quad x \in \mathbb{R}^n,$$
$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n.$$

Hint: Consider the function $f(t,x) = u(T-t,x)$, $t \in [0,T]$, $x \in \mathbb{R}^n$, and apply (i)

Definition 3.1. (i) Let a and b be as above. A continuous $\mathbb{R}^n$-valued stochastic process X_t, $t \geq 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a solution to the **martingale problem** associated with the family of second-order differential operators $\mathcal{A}_t$, $t \geq 0$, if for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}_s f(X_s) ds, \quad t \geq 0, \tag{8}$$

is a continuous $(\mathcal{F}_t^X)$-martingale.

(ii) We say that the solution to the martingale problem associated with $\mathcal{A}_t$, $t \geq 0$, is **unique** if for any two solutions X and $\tilde{X}$ (possibly defined on different probability spaces) and started from the same point x_0 the laws of X and $\tilde{X}$ coincide.

The fact that X solves the martingale problem associated with $\mathcal{A}_t$, $t \geq 0$, is a statement about its finite-dimensional distributions. This is explained in the following remark.

Remark 3.1. A continuous process X solves the martingale problem if and only if for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$, partition $0 \leq t_1 < \dots < t_N < t_{N+1}$, and bounded functions $h_1, \dots, h_N \in \mathcal{C}(\mathbb{R}^n)$ one has

$$\mathbb{E}\left[\left(f(X_{t_{N+1}}) - f(X_{t_N}) - \int_{t_N}^{t_{N+1}} \mathcal{A}_s f(X_s) ds\right) \prod_{i=1}^N h_i(X_{t_i})\right] = 0. \tag{9}$$

Therefore, we can transfer the formulation of the martingale problem to the following *canonical form*. We consider the probability space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = \left(\mathcal{C}([0,\infty), \mathbb{R}^n), \mathcal{B}\left(\mathcal{C}([0,\infty), \mathbb{R}^n)\right), \mathbb{P}^X\right),$$

where $\mathbb{P}^X = \operatorname{Law} X$ denotes the law of X. Define the canonical process $\tilde{X}_t(\omega) = \omega_t$, $t \geq 0$. Then $\tilde{X}$ has the same law as X and, consequently, $\tilde{X}$ solves the same martingale problem. In particular, we can identify a solution X to the martingale problem associated with $\mathcal{A}_t$, $t \geq 0$, with the probability measure $\mathbb{P}^X$ on the measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$. Thus, we will also say that a probability measure is a solution to the martingale problem associated with $\mathcal{A}_t$, $t \geq 0$.

Exercise 3.4. Show that a process X_t, $t \geq 0$, is a solution to the martingale problem associated with $\mathcal{A}_t$, $t \geq 0$, if and only if it satisfies property (9) in Remark 3.1.

Our further goal is the proof of the inverse statement to Proposition 3.1. We will before show that a continuous local martingales can be represented as a stochastic integral with respect to a Brownian motion. Let M_t, $t \geq 0$, be a continuous local martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t V_s^2 ds, \quad t \geq 0,$$

and $M_0 = 0$, where V_t, $t \geq 0$, is a progressive process. If a.s. $V_t^2 > 0$, $t \geq 0$, then we can define the following continuous local martingale

$$B_t = \int_0^t \frac{1}{V_s} dM_s, \quad t \geq 0.$$

Since its quadratic variation equals

$$\langle B\rangle_t = \int_0^t \frac{1}{V_s^2} d\langle M\rangle_s = \int_0^t \frac{V_s^2}{V_s^2} ds = t, \quad t \geq 0,$$

the process B_t, $t \geq 0$, is an one-dimensional Brownian motion, by Levy's characterization. Therefore, we can represent the continuous local martingale M as a stochastic integral with respect to the Brownian motion B:

$$M_t = \int_0^t V_s dB_s, \quad t \geq 0.$$

A similar representation is true if V_t equals zero for some t with positive probability. In this case, we need to take a Brownian motion $\tilde{B}_t$, $t \geq 0$, independent of M and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the process M is defined, and set

$$B_t = \int_0^t \frac{\mathbb{I}_{\{V_s^2 > 0\}}}{V_s} dM_t + \int_0^t \mathbb{I}_{\{V_s^2 = 0\}} d\tilde{B}_s, \quad t \geq 0. \tag{10}$$

A similar computation shows that B_t, $t \geq 0$, defined by (10), is a Brownian motion and M has the same representation as before. The problem could be that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is too small and there exists no a Brownian motion $\tilde{B}$ independent of M. Consequently, we need to extend the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 3.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $\hat{W}_t$, $t \geq 0$, be an m-dimensional Brownian motion on another filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, \hat{\mathbb{P}})$. The space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ defined by

$$\tilde{\Omega} := \Omega \times \hat{\Omega}, \quad \tilde{\mathcal{F}} := \mathcal{F} \otimes \hat{\mathcal{F}}, \quad \tilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \hat{\mathcal{F}}_t, \quad \tilde{\mathbb{P}} := \mathbb{P} \otimes \hat{\mathbb{P}}$$

is called an **m-extension of** $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We define $\tilde{W}_t(\omega, \hat{\omega}) := W_t(\hat{\omega})$, $(\omega, \hat{\omega}) \in \tilde{\Omega}$, $t \geq 0$, that is an $(\tilde{\mathcal{F}}_t)$-Brownian motion on the m-extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$. Moreover, any $(\mathcal{F}_t)$-adapted (or $(\mathcal{F}_t)$-progressive) process X_t, $t \geq 0$, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, can be extended to the $(\tilde{\mathcal{F}}_t)$-adapted (respectively, $(\tilde{\mathcal{F}}_t)$-progressive) process

$$\tilde{X}_t(\omega, \hat{\omega}) = X_t(\omega), \quad t \geq 0,$$

defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$.

Theorem 3.1. *Let $M = (M_t^k)_{k\in[n]}$, $t \geq 0$, be a continuous local $(\mathcal{F}_t)$-martingale defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $M_0 = 0$ and*

$$\langle M^i, M^j\rangle_t = \sum_{l=1}^m \int_0^t V_s^{i,l} V_s^{j,l} ds, \quad t \geq 0,$$

for $(\mathcal{F}_t)$-progressive processes $V_t^{k,l}$, $t \geq 0$, $k \in [n]$, $l \in [m]$. Then on any m-extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ there exists an m-dimensional $(\tilde{\mathcal{F}}_t)$-Brownian motion $B_t = (B_t^k)_{k\in[n]}$, $t \geq 0$, such that

$$M_t^k = \sum_{l=1}^m \int_0^t V_s^{k,l} dB_s^k, \quad t \geq 0.$$

Proof. The idea of the proof is similar to the proof above for one-dimensional case. For the proof see e.g. [Sch18, Theorem 3.14], [KS91, Theorem 3.4.2] or [IW89, Theorem 7.1']. □

Theorem 3.2. *Let X_t, $t \geq 0$, be a solution to the martingale problem associated with the family of second-order differential operators $\mathcal{A}_t$, $t \geq 0$, given by (7), that is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be a complete filtration generated by X. Then on any m-extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ there exists an m-dimensional Brownian motion B such that (X, B) is a weak solution to (6) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$.*

Proof. We fix $k \in [n]$ and choose for each $N \in \mathbb{N}$ a function $f_N^k \in \mathcal{C}_0^2(\mathbb{R}^n)$ with $f_N^k(x) = x_k$, $\|x\| \leq N$. We define a sequence τ_N, $N \geq 1$, of $(\mathcal{F}_t)$-stopping times[3] which increases to ∞ a.s. and such that $M_{t\wedge\tau_N}^{f_N^k}$, $t \geq 0$, is a martingale and $\|X_t\| \leq N$, $t \leq \tau_N$, for each $N \in \mathbb{N}$, where $M^{f_N^k}$ is defined by (8). Then

$$M_{t\wedge\tau_N}^{f_N^k} = X_{t\wedge\tau_N}^k - X_0^k - \int_0^{t\wedge\tau_N} b_k(s, X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale for each $N \in \mathbb{N}$. Consequently, the process

$$M_t^k := X_t^k - X_0^k - \int_0^t b_k(s, X_s) ds, \quad t \geq 0, \tag{11}$$

is a continuous local $(\mathcal{F}_t)$-martingale.

We next fix $i, j \in [n]$, and choose for each $N \in \mathbb{N}$ a function $f_N^{i,j} \in \mathcal{C}_0^2(\mathbb{R}^n)$ with $f_N^{i,j}(x) = x_i x_j$, $\|x\| \leq N$. Similarly, we can show that

$$M_t^{i,j} := X_t^i X_t^j - X_0^i X_0^j - \int_0^t \left(X_s^i b_j(s, X_s) + X_s^j b_i(s, X_s) + a_{i,j}(s, X_s) \right) ds, \quad t \geq 0,$$

[3]such a sequence can be defined e.g. as

$$\tau_N := \left(\inf\{t \geq 0 : \ \|X_t\| \geq N\} \wedge \gamma_N^N \right) \vee \tau_{N-1},$$

and $\tau_0 = 1$, where γ_l^N, $l \geq 1$, is a localization sequence of $(\mathcal{F})$-stopping times for $M^{f_N^k}$

is a continuous local martingale. Applying Itô's formula to $X_t^i X_t^j$ and using (11), we obtain

$$\begin{aligned} M_t^{i,j} &= \int_0^t X_s^i dX_s^j + \int_0^t X_s^j dX_s^i + \langle M^i, M^j \rangle_t \\ &- \int_0^t \left(X_s^i b_j(s, X_s) + X_s^j b_i(s, X_s) + a_{i,j}(s, X_s) \right) ds \\ &= \int_0^t X_s^i dM_s^j + \int_0^t X_s^j dM_s^i + \langle M^i, M^j \rangle_t - \int_0^t a_{i,j}(s, X_s) ds, \quad t \geq 0. \end{aligned}$$

Therefore, the process $\langle M^i, M^j \rangle_t - \int_0^t a_{i,j}(s, X_s) ds$ is a continuous local martingale started at 0. Since it also has a bounded variation a.s., we can conclude that

$$\langle M^i, M^j \rangle_t = \int_0^t a_{i,j}(s, X_s) ds, \quad t \geq 0, \tag{12}$$

(see Exercise 3.5 below).

We have proved that M^k, $k \in [n]$, are continuous local $(\mathcal{F}_t)$-martingales with joint quadratic variation defined by (12). By Theorem 3.1, for any m-extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ there exists an m-dimensional Brownian motion B_t, $t \geq 0$, such that

$$M_t^k = X_t^k - X_0^k - \int_0^t b_k(s, X_s) ds = \sum_{l=1}^m \int_0^t \sigma_{k,l}(s, X_s) dB_s^l, \quad t \geq 0.$$

This completes the proof of the theorem. □

Exercise 3.5. Let X_t, $t \geq 0$, be a continuous local martingale started at 0 whose variation is finite a.s. Show that $X_t = 0$, $t \geq 0$, a.s.

Corollary 3.1. *Let the coefficients b, σ of SDE (6) are locally bounded. Then the existence and uniqueness of a solution to the martingale problem associated with the second-order differential operators $\mathcal{A}_t$, $t \geq 0$, defined by (7), is equivalent to the existence and uniqueness of a weak solution to SDE (6).*

Proof. The corollary directly follows from Proposition 3.1, Exercise 3.1 and Theorem 3.2. □

3.2 Existence of solutions

In this section we will use the martingale problem approach in order to show the weak existence of solutions to a stochastic differential equation. We first discuss some results about the weak convergence and the tightness of probability measures. Let E be a metric space and $\mathcal{P}(E)$ denotes the space of probability measures on the metric space E. We also denote by $\mathcal{C}_b(E)$ the set of continuous bounded functions from E to $\mathbb{R}$.

Definition 3.3. (i) A sequence $\{\mu_k,\ k \geq 1\} \subset \mathcal{P}(E)$ is said to **converge weakly** to $\mu \in \mathcal{P}(E)$ if

$$\lim_{k \to \infty} \int_E f d\mu_k = \int_E f d\mu, \quad f \in \mathcal{C}_b(E).$$

(ii) A sequence $\{\xi_k,\ k \geq 1\}$ of random variables in E is said to **converge in distribution** to the random variable ξ in E if the distributions $\mathbb{P}^{\xi_k}$ of ξ_k, $k \geq 1$, converge weakly to the distribution $\mathbb{P}^{\xi}$ of ξ, or equivalently, if

$$\lim_{k\to\infty} \mathbb{E}\, f(\xi_k) = \mathbb{E}\, f(\xi), \quad f \in \mathcal{C}_b(E).$$

We will denote weak convergence by $\mu_k \Rightarrow \mu$ and convergence in distribution by $\xi_k \Rightarrow \xi$.

We remark that the weak convergence in a complete separable metric space is metrizable.

The proof of the following theorem can be found in [EK86, Theorem 3.3.1].

Theorem 3.3. *Let E be an arbitrary metric space, and let $\{\mu_k,\ k \geq 1\} \subset \mathcal{P}(E)$ and $\mu \in \mathcal{P}(E)$. The following conditions are equivalent:*

(a) $\mu_k \Rightarrow \mu$;

(b) $\lim_{k\to\infty} \int_E f d\mu_k = \int_E f d\mu$ *for all uniformly continuous* $f \in \mathcal{C}_b(E)$;

(c) $\overline{\lim}_{k\to\infty} \mu_k(F) \leq \mu(F)$ *for all closed sets* $F \subset E$;

(d) $\underline{\lim}_{k\to\infty} \mu_k(G) \geq \mu(G)$ *for all open sets* $G \subset E$;

(e) $\lim_{k\to\infty} \mu_k(A) = \mu(A)$ *for all μ-continuity*[4] *sets* $A \subset E$.

We will need the following result.

Proposition 3.2. *Let E and S be metric spaces, a sequence $\{\mu_k,\ k \geq 1\} \subset \mathcal{P}(E)$ converge weakly to $\mu \in \mathcal{P}(E)$ and $f, f_1, f_2, \dots : E \to S$ be Borel measurable mappings. Assume that there exists a Borel measurable set $C \subseteq E$ such that $\mu(C) = 1$ and $f_k(x_k) \to f(x)$ whenever $x_k \to x \in C$. Then the sequence $\{\mu_k \circ f_k^{-1},\ k \geq 1\} \subset \mathcal{P}(S)$ converges weakly to $\mu \circ f^{-1} \in \mathcal{P}(S)$.*[5]

Proof. In order to prove the statement we will use Theorem 3.3. Fix an open set $G \subset S$ and let $x \in f^{-1}(G) \cap C$ in case the set is non-empty. Then there exists an open neighborhood U of x and some $m \in \mathbb{N}$ such that $f_l(U) := \{f_l(x) :\ x \in U\} \subset G$ for all $l \geq m$. Indeed, let it be false. Then for every $k \in \mathbb{N}$ there exists a point x_k from the ball of radius $\frac{1}{k}$ with center at x and a number $l_k > l_{k-1}$ such that $f_{l_k}(x_k) \notin G$. Therefore, $x_k \to x$ but $f_{l_k}(x_k) \not\to f(x) \in G$. This contradicts of the assumption of the proposition. Hence, we can conclude that the open neighborhood U of x is a subset of $\bigcap_{l=m}^{\infty} f_l^{-1}(G)$. Consequently,

$$f^{-1}(G) \cap C \subset \bigcup_{m=1}^{\infty} \left(\bigcap_{l=m}^{\infty} f_l^{-1}(G) \right)^{\circ},$$

where A° denotes the interior of the set A.

Using Theorem 3.3, we get

$$\left(\mu \circ f^{-1}\right)(G) = \mu\left(f^{-1}(G)\right) \leq \mu\left(\bigcup_{m=1}^{\infty} \left(\bigcap_{l=m}^{\infty} f_l^{-1}(G) \right)^{\circ} \right) = \sup_{m\in\mathbb{N}} \mu\left(\left(\bigcap_{l=m}^{\infty} f_l^{-1}(G) \right)^{\circ} \right)$$

$$[\text{Th. } 3.3] \leq \sup_{m\in\mathbb{N}} \varliminf_{k\to\infty} \mu_k\left(\bigcap_{l=m}^{\infty} f_l^{-1}(G) \right) \leq \varliminf_{k\to\infty} \mu_k\left(f_k^{-1}(G)\right) = \varliminf_{k\to\infty} \left(\mu_k \circ f_k^{-1}\right)(G).$$

This completes the proof of the proposition. □

[4] $A \subset E$ is a μ-continuity set if $\mu(\partial A) = 0$

[5] Recall that $\mu \circ f^{-1}(A) = \mu\left(f^{-1}(A)\right)$, $A \in \mathcal{B}(S)$

We know that the relative compactness place an important role for convergence. In particular, a relative compact sequence as always an convergence subsequence. In the space of probability measures, the relative compactness is equivalent to tightness, according to the Prohorov theorem below.

Definition 3.4. (i) A family of probability measures $\mathcal{M} \subset \mathcal{P}(E)$ is **tight** if for each $\varepsilon > 0$ there exists a compact set $K \subset E$ such that

$$\mu(K) \geq 1 - \varepsilon, \quad \mu \in \mathcal{M}.$$

(ii) A family of random elements $\{\xi_i,\ i \in I\}$ in E is **tight** if the family if their distributions $\{\mathbb{P}^{\xi_i},\ i \in I\}$ is tight.

Exercise 3.6. Let $\{\xi_k,\ k \geq 1\}$ be a family of random variables in $\mathbb{R}^d$ such that $\sup_{k \in \mathbb{N}} \mathbb{E}\,\|\xi_k\| < \infty$. Show that this family is tight.

Theorem 3.4 (Prohorov)**.** *Let E be a complete and separable metric space and $\mathcal{M} \subset \mathcal{P}(E)$. The family $\mathcal{M}$ is tight if and only if $\mathcal{M}$ is relatively compact, that is, any sequence $\{\mu_k,\ k \geq 1\} \subseteq \mathcal{M}$ contains a weakly convergent subsequence.*

Proof. For the proof of the theorem see e.g. [Kal02, Theorem 16.3] or [EK86, Theorem 3.2.2]. □

We will further discuss a tightness on the metric space $\mathcal{C}([0,\infty),\mathbb{R}^d)$ of continuous function from $[0,\infty)$ to $\mathbb{R}^n$ equipped with the metric

$$d(f,g) := \sum_{l=1}^{\infty} \frac{1}{2^l} \left(\max_{t \in [0,l]} |f(t) - g(t)| \wedge 1 \right).$$

Exercise 3.7. (i) Show that the metric space $\mathcal{C}([0,\infty),\mathbb{R}^d)$ is complete and separable.

(ii) Show that a sequence $\{f_k,\ k \geq 1\}$ converges to f in $\mathcal{C}([0,\infty),\mathbb{R}^d)$ if and only if for every $l \geq 1$ the restrictions $\pi_k(f_k)$ of f_k to $[0,l]$, $k \geq 1$, converges to $\pi_l(f)$ in $\mathcal{C}([0,l],\mathbb{R}^d)$.

The following statement gives the sufficient conditions of tightness in the space $\mathcal{C}([0,\infty),\mathbb{R}^d)$.

Proposition 3.3. *Let $X^1, X^2, \ldots$ be continuous processes taking values in $\mathbb{R}^d$. Assume that the family $\{X_0^k,\ k \geq 1\}$ is tight in $\mathbb{R}^d$ and for every $T > 0$*

$$\mathbb{E}\,\|X_t^k - X_s^k\|^\alpha \leq C_T |t-s|^{1+\beta}, \quad s,t \in [0,T], \quad k \in \mathbb{N}.$$

for some positive constants α, β, C_T depending on T. Then $\{X^k,\ k \geq 1\}$ is tight in $\mathcal{C}([0,\infty),\mathbb{R}^d)$.

The proof of the statement in more general settings can be found in [Kal02, Corollary 16.9]. See also [KS91, Problem 4.11].

We will use Proposition 3.3 and Corollary 3.1 in order to prove the existence of solution to stochastic differential equation (6).

Theorem 3.5. *Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be bounded and continuous functions and $x_0 \in \mathbb{R}^n$. Then the martingale problem associated with $\mathcal{A}$, has a solution started at x_0, where*

$$\mathcal{A}f(x) := \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial f(x)}{\partial x_i}, \quad f \in \mathcal{C}^2(\mathbb{R}^n),$$

and $a = \sigma\sigma^T$.

Proof. The idea of proof of the proposition is approximate the coefficients of the equation by coefficients which are Lipschitz continuous and use the existence result form Section 2.1.

We choose functions $b_i^{(k)} : \mathbb{R}^n \to \mathbb{R}$ and $\sigma_{i,j}^{(k)} : \mathbb{R}^n \to \mathbb{R}$, $k \geq 1$, such that they are Lipschitz continuous and uniformly converge to b_i and $\sigma_{i,j}$ on every rectange $[-K, K]^n$ for every $i \in [n]$ and $j \in [m]$ (see also Exercise 3.8). Since b_i and $\sigma_{i,j}$ are bounded, we may assume that $b_i^{(k)}$ and $\sigma_{i,j}^{(k)}$, $k \geq 1$, are bounded as well, that is, there exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |b_i^{(k)}(x)| \leq C, \quad k \geq 1, \quad i \in [n],$$

and

$$\sup_{x \in \mathbb{R}^n} |\sigma_{i,j}^{(k)}(x)| \leq C, \quad k \geq 1, \quad i \in [n], \quad j \in [m].$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space and $B_t = (B_t^j)_{j \in [m]}$ be an m-dimensional Brownian motion defined on it. By Theorem 2.1, the equation

$$dX_t^{(k)} = b^{(k)}(X_t^{(k)})dt + \sigma^{(k)}(X_t^{(k)})dB_t, \quad X_0 = x_0,$$

has a unique (strong) solution, where $b^{(k)} = (b_i^{(k)})_{i \in [n]}$ and $\sigma^{(k)} = (\sigma_{i,j}^{(k)})_{i \in [n], j \in [m]}$. We will next show that the family $\{X^{(k)},\ k \geq 1\}$ is tight in $\mathcal{C}([0, \infty), \mathbb{R}^n)$. Using the Burkholder-Davis-Gundy inequality[6] [KS91, Theorem 3.3.28], we can estimate for $0 \leq s < t \leq T$ and $p \geq 1$

$$\begin{aligned}
\mathbb{E}\left(\|X_t^{(k)} - X_s^{(k)}\|^{2p}\right) &= \mathbb{E}\left(\sum_{i=1}^n |X_t^{i,(k)} - X_s^{i,(k)}|^2\right)^p \\
&\leq n^{p-1} \sum_{i=1}^n \mathbb{E}\left(\left|\int_s^t b_i^{(k)}(X_r^{(k)})dr + \sum_{j=1}^m \int_s^t \sigma_{i,j}^{(k)}(X_r^{(k)})dB_r^j\right|^{2p}\right) \\
\text{[Hölder in.]} &\leq n^{p-1} \sum_{i=1}^n (m+1)^{2p-1} \left(\mathbb{E}\left(\left|\int_s^t b_i^{(k)}(X_r^{(k)})dr\right|^{2p}\right) + \sum_{j=1}^m \mathbb{E}\left(\left|\int_s^t \sigma_{i,j}^{(k)}(X_r^{(k)})dB_r^j\right|^{2p}\right)\right) \\
\text{[B-D-G in.]} &\leq n^{p-1}(m+1)^{2p-1} \sum_{i=1}^n \left(C^{2p}(t-s)^{2p} + \sum_{j=1}^m C_p \mathbb{E}\left(\left[\int_s^t \left(\sigma_{i,j}^{(k)}(X_r^{(k)})\right)^2 dr\right]^p\right)\right) \\
&\leq n^{p-1}(m+1)^{2p-1} \sum_{i=1}^n \left(C^{2p}(t-s)^{2p} + \sum_{j=1}^m C_p C^{2p}(t-s)^p\right).
\end{aligned}$$

Thus, choosing $p > 2$, Proposition 3.3 implies that $\{X^{(k)},\ k \geq 1\}$ is tight in $\mathcal{C}([0, \infty), \mathbb{R}^n)$. By Prohorov's Theorem 3.4 there exists a distribution P on $\mathcal{C}([0, \infty), \mathbb{R}^n)$ and a subsequence $N \subseteq \mathbb{N}$ such that $\mathbb{P}^{X^{(k)}} \Rightarrow P$ along N. Without loss of generality we assume that the sequences $b^{(k)}$, $k \geq 1$, and $\sigma^{(k)}$, $k \geq 1$, are such that $\mathbb{P}^{X^{(k)}} \Rightarrow P$. Let X be the canonical process on $(\mathcal{C}([0, \infty), \mathbb{R}^n), \mathcal{B}(\mathcal{C}([0, \infty), \mathbb{R}^n)), P)$

[6] For every $p > 0$ there exist positive constants C_p and c_p (depending only on p) such that for every continuous local martingale M_t, $t \geq 0$, one has

$$c_p \mathbb{E}\left(\langle M \rangle_\tau^p\right) \leq \mathbb{E} \max_{t \in [0, \tau]} |M_t|^{2p} \leq C_p \mathbb{E}\left(\langle M \rangle_\tau^p\right)$$

for every stopping time τ

as in Remark 3.1. Our goal is to show that X_t, $t \geq 0$, solves the martingale problem associated with $\mathcal{A}$. Then Corollary 3.1 will imply that X is a weak solution to SDE (6).

Using Proposition 3.1 and Exercise 3.1, we can conclude that $X_t^{(k)}$, $t \geq 0$, is a solution to the martingale problem associated with

$$\mathcal{A}^{(k)} f(x) := \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}^{(k)}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^{(k)}(x) \frac{\partial f(x)}{\partial x_i}, \quad f \in \mathcal{C}^2(\mathbb{R}^n),$$

that is, for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ the process

$$M_t^{f,(k)} := f(X_t^{(k)}) - f(x_0) - \int_0^t \mathcal{A}^{(k)}(X_s^{(k)}) ds, \quad t \geq 0,$$

is a continuous $(\mathcal{F}_t^{X^{(k)}})$-martingale, where $a^{(k)} = \sigma^{(k)} \left(\sigma^{(k)}\right)^T$. Therefore, for every $0 \leq s \leq t$ and counded continuous function $\Phi : \mathcal{C}([0,s], \mathbb{R}^n) \to \mathbb{R}$ one has

$$\mathbb{E}\left(\left(f(X_t^{(k)}) - f(X_s^{(k)}) - \int_s^t \mathcal{A}^{(k)}(X_r^{(k)}) dr \right) \Phi\left(X_r^{(k)},\ r \in [0,s]\right) \right) = 0.$$

Passing to the limit as $k \to \infty$, and using the boundedess of f, convergence $\mathbb{P}^{X^{(k)}} \Rightarrow P$, $\mathcal{A}^{(k)} f(x) \to \mathcal{A} f(x)$ for every x and Proposition 3.2, we get

$$\mathbb{E}\left(\left(f(X_t) - f(X_s) - \int_s^t \mathcal{A}(X_r) dr \right) \Phi\left(X_r,\ r \in [0,s]\right) \right) = 0.$$

This impliest that X is a solution to the martingale problem associated with $\mathcal{A}$. This completes the proof of the theorem. □

Corollary 3.2. *Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be bounded and continuous functions and $x_0 \in \mathbb{R}^n$. Then the equation*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0 \tag{13}$$

has a weak solution.

Proof. The statement directly follows from Theorem 3.5 and Corollary 3.1. □

Exercise 3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and $\varphi \in \mathcal{C}_0^1(\mathbb{R})$ such that $\operatorname{supp} \varphi \in [-1,1]$, $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi(x)dx = 1$. Show that for every $\varepsilon > 0$ the function

$$f_\varepsilon(x) := \int_{\mathbb{R}} \varphi(x-y) f(y) dy, \quad x \in \mathbb{R},$$

is bounded and Lipschitz continuous and $f_\varepsilon \to f$ uniformly on every interval $[-K, K]$ as $\varepsilon \to 0$, where $\varphi_\varepsilon(x) := \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right)$.

3.3 Uniqueness in law

The goal of this section is to study the uniqueness in law of a solution to SDE (6) that is equivalent to the uniqueness of the martingale solution according to Corollary 3.1.

We recall that the martingale problem associated with the family of second-order differential operators

$$\mathcal{A}f(x) := \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad f \in \mathcal{C}^2(\mathbb{R}^n). \tag{14}$$

admits a unique solution if for any two solutions X and Y started from the same point x_0 the laws of X and Y in $\mathcal{C}([0,\infty),\mathbb{R}^n)$ coincide.

We first prove the following theorem.

Theorem 3.6. *Assume that $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally bounded measurable functions and the Kolmogorov forward PDE*

$$\begin{aligned} \frac{\partial \psi(t,x)}{\partial t} &= \mathcal{A}\psi(t,x), \quad x \in \mathbb{R}^n, \quad t > 0, \\ \psi(0,x) &= \varphi(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{15}$$

has a solution $\psi \in \mathcal{C}^{1,2}([0,\infty)\times\mathbb{R}^n)$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ which is bounded on any set $[0,T]\times\mathbb{R}^n$. Then the martingale problem associated with $\mathcal{A}$ has a unique solution. In particular, there is uniqueness in law for equation (13).

Proof. Let $\psi \in \mathcal{C}^{1,2}([0,\infty)\times\mathbb{R}^n)$ be a solution to PDE (15) which is bounded on every set $[0,T]\times\mathbb{R}^n$, and let X_t, $t \geq 0$, be a solution to the martingale problem associated with $\mathcal{A}$ and started at x_0. We fix $t > 0$ and remark that the chain rule yields

$$\frac{\partial \psi(t-s,X_s)}{\partial s} + \mathcal{A}\psi(t-s,X_s) = 0, \quad s \in [0,t),$$

for every $t > 0$. Therefore, the process

$$\begin{aligned} M_r^t &= \psi(t-r,X_r) - \psi(t,X_0) - \int_0^r \left(\frac{\partial \psi(t-s,X_s)}{\partial s} + \mathcal{A}\psi(t-s,X_s)\right) ds \\ &= \psi(t-r,X_r) - \psi(t,x_0), \quad r \in [0,t], \end{aligned}$$

is a martingale. Since the expectation of a martingale is constant, for every $r \in [0,t]$,

$$\mathbb{E}\,(M_t^t) = \mathbb{E}\,M_0^t = \mathbb{E}\,(\psi(t,X_0) - \psi(t,x_0)) = 0.$$

Hence,

$$\mathbb{E}\,\psi(t-t,X_t) = \mathbb{E}\,\varphi(X_t) = \psi(t,x_0). \tag{16}$$

We obtained that the value of the expectation of $\varphi(X_t)$ do not depends the choise of a solution to the martingale problem, i.e. if Y_t, $t \geq 0$, is another solution to the same martingale problem started from x_0, then

$$\mathbb{E}\,\varphi(Y_t) = \psi(t,Y_0) = \psi(t,x_0).$$

Consequently, $\mathbb{E}\,\varphi(X_t) = \mathbb{E}\,\varphi(Y_t)$ for every $t \geq 0$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. This implies that processes X and Y have the same one-dimensional distributions.

We will now show that the two-dimensional distributions of X and Y coincide. Using the fact that M_r^t, $r \in [0,t)$, is a martingale, we compute

$$\begin{aligned}\mathbb{E}\left(\varphi(X_t)|\mathcal{F}_r\right) &= \mathbb{E}\left(\psi(t-t,X_t) - \psi(t-0,x_0)|\mathcal{F}_r\right) + \mathbb{E}\left(\psi(t,x_0)|\mathcal{F}_r\right) = \mathbb{E}\left(M_t^t|\mathcal{F}_r\right) + \psi(t,x_0)\\ &= \mathbb{E}\left(M_r^t|\mathcal{F}_r\right) + \psi(t,x_0) = \mathbb{E}\left(\psi(t-r,X_r) - \psi(t,x_0)|\mathcal{F}_r\right) = \psi(t-r,X_r),\end{aligned}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by X. Therefore, for every $0 \leq t_1 < t_2$ and a bounden measurable function $g : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{aligned}\mathbb{E}\left(\varphi(X_{t_2})g(X_{t_1})\right) &= \mathbb{E}\left(\mathbb{E}\left[\varphi(X_{t_2})g(X_{t_1})|\mathcal{F}_{t_1}\right]\right) = \mathbb{E}\left(\mathbb{E}\left[\varphi(X_{t_2})|\mathcal{F}_{t_1}\right]g(X_{t_1})\right)\\ &= \mathbb{E}\left(\psi(t_2-t_1,X_{t_1})g(X_{t_1})\right)\\ [X_{t_1} \overset{d}{=} Y_{t_1}] &= \mathbb{E}\left(\psi(t_2-t_1,Y_{t_1})g(Y_{t_1})\right) = \mathbb{E}\left(\varphi(Y_{t_2})g(Y_{t_1})\right).\end{aligned}$$

This implies that the vectors (X_{t_1}, X_{t_2}) and (Y_{t_1}, Y_{t_2}) have the same distribution. Similarly, one can show that n-dimensional distributions of X and Y coincides, that implies the uniqueness of the martingale problem.

The uniqueness in law for equation (13) follows directly from the uniqueness from the martingale problem and Corollary 3.1. □

Definition 3.5. (i) The martingale problem associated with second order differential operator $\mathcal{A}$, defined by (14), is **well posed** if, for every $x_0 \in \mathbb{R}^n$, it has only one solution started from x_0.

(ii) Stochastic differential equation (13) is said to be **well posed** if, for every initial condition $x_0 \in \mathbb{R}^n$, it admits a weak solution which is unique in law.

Corollary 3.3. *Let the coefficients b, σ of SDE (13) be bounded continuous functions and satisfy the assumptions of Theorem 3.6. Then the martingale problem associated with $\mathcal{A}$, defined by (14), and therefore, SDE (13), are well posed.*

Remark 3.2. A sufficient condition for the solvability of Kolmogorov forward PDE (15) in the way required by Theorem 3.6 is that the coefficients b_i, $a_{i,j}$, $i,j \in [n]$, are bounded and Hölder-continuous on $\mathbb{R}^n$, and the matrix a is uniformly positive definite, i.e

$$\sum_{i,j=1}^n a_{i,j}(x)y_iy_j \geq \lambda\|y\|^2, \quad x,y \in \mathbb{R}^n,$$

for some $\lambda > 0$. For the proof of this result see e.g. [SV79, Theorem 3.2.1].

4 Strong Markov property of solutions to SDE

In this section, we will show that solutions to a well posed SDE satisfies the strong Markov property. As preparation, we need to state certain results about regular conditional probabilities.

4.1 Regular conditional probabilities

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{S}$ of $\mathcal{F}$. For a (real-valued) random variable ξ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ recall that the conditional expectation $\mathbb{E}\left(\xi|\mathcal{S}\right)$ is defined as a random variable η which satisfies the following conditions:

1) η is $\mathcal{S}$-measurable;

2) for every $A \in \mathcal{S}$,
$$\mathbb{E}\left(\xi \mathbb{I}_A\right) = \mathbb{E}\left(\eta \mathbb{I}_A\right).$$

Moreover, such a random variable is unique in the following sense. If η' is another random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies conditions 1), 2) above, then $\mathbb{P}\{\eta = \eta'\} = 1$.

We define for every $A \in \mathcal{F}$
$$\tilde{Q}(\omega; A) = \mathbb{P}(A|\mathcal{S})(\omega) := \mathbb{E}(\mathbb{I}_A|\mathcal{S})(\omega), \quad \omega \in \Omega.$$

The dominated convergence theorem for conditional expectations implies that for every disjoint family of events $A_k \in \mathcal{F}$, $k \geq 1$,
$$\tilde{Q}\left(\cdot; \bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \tilde{Q}\left(\cdot; A_k\right) \quad \text{a.s.}$$

However, for $\omega \in \Omega$ the function $\tilde{Q}(\omega; \cdot) : \mathcal{F} \to [0,1]$ is not necessarily a measure on Ω because the definition of $Q(\tilde{\cdot}; A)$ depends on $\mathbb{P}$-null sets[7] for every $A \in \mathcal{F}$. Therefore, some regular choice of $Q(\tilde{\cdot}; A)$ is needed.

Definition 4.1. (i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F}$. A function
$$Q(\omega; A) : \Omega \times \mathcal{F} \to [0,1]$$
is called a **regular conditional probability for $\mathcal{F}$ given $\mathcal{S}$** if

(a) for each $\omega \in \Omega$, $Q(\omega; \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$;

(b) for each $A \in \mathcal{F}$, the mapping $\omega \mapsto Q(\omega; A)$ is $\mathcal{S}$-measurable;

(c) for each $A \in \mathcal{F}$, $Q(\omega; A) = \mathbb{P}\left(A|\mathcal{S}\right)(\omega)$ $\mathbb{P}$-a.e. $\omega \in \Omega$.

(ii) Suppose that, whenever Q' is another function with these properties. If there exists a $\mathbb{P}$-null set such that $Q(\omega; A) = Q'(\omega; A)$ for all $A \in \mathcal{F}$ and $\omega \in \Omega \setminus N$, then we say that the regular conditional probability for $\mathcal{F}$ given $\mathcal{S}$ is **unique**.

For the formulation of the existence result of the regular conditional probability we will need the following definition.

Definition 4.2. Let $(\Omega, \mathcal{F})$ be a measurable space. We say that $\mathcal{F}$ is **countably determined** if there exists a countable collection of sets $\mathcal{M} \subseteq \mathcal{F}$ such that, whenever two probability measures coincide on $\mathcal{M}$, they also coincide on $\mathcal{F}$. We say that $\mathcal{F}$ is **countably generated** if there exists a countable collection of sets $\mathcal{C} \subseteq \mathcal{F}$ such that $\mathcal{F} = \sigma(\mathcal{C})$.

Remark 4.1. If a σ-algebra $\mathcal{F}$ is generated by a countable collection of sets $\mathcal{M}$ which is closed under pairwise intersection, i.e. $\forall A, B \in \mathcal{M} \implies A \cap B \in \mathcal{M}$, then $\mathcal{F}$ is also countably determined. This follows from the Dynkin System Theorem 2.1.3 [KS91]. For instance, the Borel σ-algebra in a separable metric space is countably determined, since we can determine $\mathcal{M}$ as the collection of finite intersection of balls with rational radiuses with centers in points from a countable dense subset.

[7] $N \in \mathcal{F}$ is a $\mathbb{P}$-null set if $\mathbb{P}(N) = 0$.

Exercise 4.1. Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathcal{M} \subseteq \mathcal{F}$ be closed under pairwise intersection and $\sigma(\mathcal{M}) = \mathcal{F}$. Show that any two measures μ, ν on $\mathcal{F}$ which coincide on $\mathcal{M}$ must also coincide on $\mathcal{F}$.

Hint: Define $\mathcal{D} = \{A \in \mathcal{F} : \ \mu(A) = \nu(A)\}$ and show that $\mathcal{D}$ is a Dynkin system[8] containing $\mathcal{M}$. Then use Theorem 2.1.3 [KS91]

The following theorem stays the existence of a regular conditional probability.

Theorem 4.1. *Suppose that Ω is a complete separable metric space, and denote the Borel σ-algebra $\mathcal{B}(\Omega)$ by $\mathcal{F}$. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$, and let $\mathcal{S}$ be a sub-σ-algebra of $\mathcal{F}$. Then a regular conditional probability Q for $\mathcal{F}$ given $\mathcal{S}$ exists and is unique. Furthermore, if $\mathcal{H}$ is a countably determined sub-σ-algebra of $\mathcal{S}$, then there exists a $\mathbb{P}$-null set $N \in \mathcal{S}$ such that*

$$Q(\omega; A) = \mathbb{I}_A(\omega), \quad A \in \mathcal{H}, \quad \omega \in \Omega \setminus N.$$

Exercise 4.2. Let $(\Omega, \mathcal{F})$ and $\mathcal{S}$ be as in Theorem 4.1. Let also Q be a regular conditional probability for $\mathcal{F}$ given $\mathcal{S}$. Show that for an $\mathcal{S}$-measurable random variable ξ taking values in another complete separable metric space one has

$$Q\left(\omega; \{\omega' \in \Omega : \ \xi(\omega') = \xi(\omega)\}\right) = 1, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hint: Take $\mathcal{H} = \sigma(\xi)$ and apply Theorem 4.1

If the σ-algebra is generated by a random variable, then Theorem 4.1 can be reformulated as follows.

Theorem 4.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be as in Theorem 4.1, and let ξ be a measurable mapping from this space into a complete separable metric space $(S, \mathcal{S})$, where $\mathcal{S}$ is the Borel σ-algebra on S. Let also $\mathbb{P}^{\xi}$ denote the distribution of ξ on $(S, \mathcal{S})$. Then there exists a function $Q(x; A) : S \times \mathcal{F} \to [0, 1]$, called a* **regular conditional probability for $\mathcal{F}$ given ξ**, *such that*

(i) for each $x \in S$, $Q(x; \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$;

(ii) for each $A \in \mathcal{F}$, the mapping $x \mapsto Q(x; A)$ is $\mathcal{S}$-measurable;

(iii) for each $A \in \mathcal{F}$, $Q(x; A) = \mathbb{P}(A|\xi = x)$, $\mathbb{P}^X$-a.e. $x \in S$, (in other words, $Q(\xi; A) = \mathbb{P}(A|\xi) = \mathbb{E}(\mathbb{I}_A|\xi)$).

If Q' is another function with these properties, then there exists a $\mathbb{P}^{\xi}$-null set $N \in \mathcal{S}$ such that $Q(x; A) = Q'(x; A)$ for all $A \in \mathcal{F}$ and $x \in S \setminus N$. Furthermore,

$$Q(x; \{\omega \in \Omega : \ \xi(\omega) \in B\}) = \mathbb{I}_B(x), \quad B \in \mathcal{S}, \quad x \in S \setminus N.$$

In particular,

$$Q(x; \{\omega \in \Omega : \ \xi(\omega) = x\}) = 1, \quad \mathbb{P}^{\xi}\text{-a.e. } x \in S.$$

Proof. For the proof of both theorems see e.g. [IW89, pp.12-16]. □

[8] A collection $\mathcal{D}$ of subsets of a set Ω is called a **Dynkin system** if

(i) $\Omega \in \mathcal{D}$;

(ii) $A, B \in \mathcal{D}$ and $B \subseteq A$ imply $A \setminus B \in \mathcal{D}$;

(iii) $A_k \in \mathcal{D}$, $k \geq 1$, and $A_1 \subseteq A_2 \subseteq \ldots$ imply $\bigcup_{n=1}^{\infty} A_k \in \mathcal{D}$.

4.2 Strong Markov property

In this section, we will consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \tag{17}$$

with $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ locally bounded measurable functions. We will also assume that the equation is well posed. Therefore, for every initial condition $x_0 \in \mathbb{R}^n$ there exists a unique week solution X. Denote its distribution on $(\mathcal{C}([0,\infty),\mathbb{R}^n), \mathcal{B}(\mathcal{C}([0,\infty),\mathbb{R}^n)))$ by P^{x_0}. According to Corollary 3.1 and Remark 3.1, the measure P^{x_0} satisfies the martingale problem associated with

$$\mathcal{A}f(x) := \frac{1}{2}\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad f \in \mathcal{C}^2(\mathbb{R}^n),$$

where $a = \sigma\sigma^T$, which is also well posed. We will further assume that X_t, $t \geq 0$, is the canonical process defined on $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{C}([0,\infty),\mathbb{R}^n), \mathcal{B}(\mathcal{C}([0,\infty),\mathbb{R}^n)), P^{x_0})$. Let also $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by X.

We recall that for any stopping time τ one defines the stopping σ-algebra $\mathcal{F}_\tau$ as follows:

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Exercise 4.3. Show that $\mathcal{F}_\tau$ is a σ-algebra.

The stopped σ-algebra on the canonical space is generated by the stopped process $X_{t\wedge\tau}$, $t \geq 0$. This follows from the following lemma.

Lemma 4.1. *For every bounded stopping time τ, we have*

$$\mathcal{F}_\tau = \sigma(X_{t\wedge\tau},\ t \geq 0).$$

In particular, X_τ is $\mathcal{F}_\tau$-measurable, that is, for every $B \in \mathbb{R}^n$ the event $\{\omega \in \Omega : X_{\tau(\omega)}(\omega) = \omega_{\tau(\omega)} \in B\}$ belongs to $\mathcal{F}_\tau$.

Proof. For the proof of the lemma see [KS91, Lemma 5.4.18]. □

We will fix a bounded stopping time τ and remark that $\mathcal{F}_\tau$ is countably determined due to Lemma 4.1 and Remark 4.1. Therefore, we can define the regular conditional probability Q for $\mathcal{F}$ given $\mathcal{F}_\tau$ (on the space $(\Omega, \mathcal{F}, \mathbb{P})$).

Proposition 4.1. *Let Q be the regular conditional probability for $\mathcal{F}$ given $\mathcal{F}_\tau$, where τ is a bounded stopping time. Then there exists a $\mathbb{P} = P^{x_0}$-null set $N \in \mathcal{F}_\tau$ such that for every $\tilde{\omega} \notin N$ the probability measure*

$$\mathbb{P}_{\tilde{\omega}}(B) := Q\left(\tilde{\omega}; \{X_{\tau+\cdot} \in B\}\right), \quad B \in \mathcal{F} = \mathcal{B}\left(\mathcal{C}([0,\infty),\mathbb{R}^n)\right), \tag{18}$$

solves the martingale problem associated with $\mathcal{A}$ with the initial condition $X_{\tau(\tilde{\omega})}(\tilde{\omega}) = \tilde{\omega}_{\tau(\tilde{\omega})}$.

We note that the statement of Proposition 4.1 means that for $\mathbb{P}$-almost all $\tilde{\omega}$ the canonical process $X_t(\omega) = \omega_t$, $t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{\tilde{\omega}})$ solves the martingale problem:

$$\mathbb{P}_{\tilde{\omega}}\left\{X_0 = \tilde{\omega}_{\tau(\tilde{\omega})}\right\} = 1$$

and for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ the process

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds, \quad t \geq 0,$$

is a continuous $(\mathcal{F}_t)$-martingale, i.e. for any partition $0 \leq t_1 < \dots < t_k < t_{k+1}$, and bounded functions $h_1, \dots, h_k \in \mathcal{C}(\mathbb{R}^n)$ one has

$$\mathbb{E}_{\tilde{\omega}} \left[\left(f(X_{t_{k+1}}) - f(X_{t_k}) - \int_{t_k}^{t_{k+1}} \mathcal{A}f(X_r)dr \right) \prod_{i=1}^k h_i(X_{t_i}) \right] = 0,$$

where the expectation $\mathbb{E}_{\tilde{\omega}}$ is taken with respect to $\mathbb{P}_{\tilde{\omega}}$, and as before $\Omega = \mathcal{C}([0,\infty),\mathbb{R}^n)$ and $\mathcal{F} = \mathcal{B}\left(\mathcal{C}([0,\infty),\mathbb{R}^n)\right)$ (see also Remark 3.1).

Proof of Proposition 4.1. We will first show that for almost all $\tilde{\omega} \in \Omega$ the canonical process X_t, $t \geq 0$, starts at $\tilde{\omega}_{\tau(\tilde{\omega})}$ $\mathbb{P}_{\tilde{\omega}}$-a.s., i.e.

$$\mathbb{P}_{\tilde{\omega}} \left\{ X_0 = \tilde{\omega}_{\tau(\tilde{\omega})} \right\} = \mathbb{P}_{\tilde{\omega}} \left\{ \omega \in \Omega : \ \omega_0 = \tilde{\omega}_{\tau(\tilde{\omega})} \right\} = 1.$$

Since $\mathcal{F}_\tau$ is countably generated, there exists a $\mathbb{P}$-null set $N_1 \in \mathcal{F}_\tau$ such that

$$Q(\tilde{\omega}; A) = \mathbb{I}_A(\tilde{\omega}), \quad A \in \mathcal{F}_\tau, \ \ \tilde{\omega} \in \Omega \setminus N_1,$$

by Theorem 4.1. We also remark that the event

$$\{\omega \in \Omega : \omega_{\tau(\omega)} = \tilde{\omega}_{\tau(\tilde{\omega})}\}$$

belongs to $\mathcal{F}_\tau$, by Lemma 4.1. Hence, for every $\tilde{\omega} \in \Omega \setminus N$

$$\begin{aligned}\mathbb{P}_{\tilde{\omega}} \left\{ X_0 = \tilde{\omega}_{\tau(\tilde{\omega})} \right\} &= Q\left(\tilde{\omega}; \left\{ X_{\tau+0} = \tilde{\omega}_{\tau(\tilde{\omega})} \right\}\right) = Q\left(\tilde{\omega}; \left\{ \omega \in \Omega : \ \omega_{\tau(\omega)} = \tilde{\omega}_{\tau(\tilde{\omega})} \right\}\right) \\ &= \mathbb{I}_{\left\{\omega \in \Omega : \ \omega_{\tau(\omega)} = \tilde{\omega}_{\tau(\tilde{\omega})}\right\}}(\tilde{\omega}) = 1.\end{aligned}$$

We next check that for every $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds, \quad t \geq 0,$$

is a continuous $(\mathcal{F}_t)$-martingale. We take a partition $0 \leq t_1 < \dots < t_k$ and bounded functions $h_1, \dots, h_k \in \mathcal{C}(\mathbb{R}^n)$ and compute

$$\begin{aligned}&\mathbb{E}_{\tilde{\omega}} \left[\left(f(X_{t_{k+1}}) - f(X_{t_k}) - \int_{t_k}^{t_{k+1}} \mathcal{A}f(X_r)dr \right) \prod_{i=1}^k h_i(X_{t_i}) \right] \\ &\int_\Omega \left[\left(f(X_{t_{k+1}}) - f(X_{t_k}) - \int_{t_k}^{t_{k+1}} \mathcal{A}f(X_r)dr \right) \prod_{i=1}^k h_i(X_{t_i}) \right] d\mathbb{P}_{\tilde{\omega}} \\ &\int_\Omega \left[\left(f(X_{\tau+t_{k+1}}) - f(X_{\tau+t_k}) - \int_{t_k}^{t_{k+1}} \mathcal{A}f(X_{\tau+r})dr \right) \prod_{i=1}^k h_i(X_{\tau+t_i}) \right] dQ(\tilde{\omega}; \cdot) \\ &\mathbb{E} \left[\left(f(X_{\tau+t_{k+1}}) - f(X_{\tau+t_k}) - \int_{\tau+t_k}^{\tau+t_{k+1}} \mathcal{A}f(X_r)dr \right) \prod_{i=1}^k h_i(X_{\tau+t_i}) \Big| \mathcal{F}_\tau \right] (\tilde{\omega}) \\ &= \mathbb{E} \left[\mathbb{E}\left(M^f_{\tau+t_{k+1}} - M^f_{\tau+t_k} \big| \mathcal{F}_{\tau+t_k} \right) \prod_{i=1}^k h(X_{\tau+t_i}) \Big| \mathcal{F}_\tau \right] (\tilde{\omega}),\end{aligned}$$

for $\mathbb{P}$-a.e. $\tilde{\omega}$, by the optional sampling Theorem 1.11 (iii) [Tra20]. This allows to conclude that there exists a $\mathbb{P}$-null set N such that the process M^f is an $(\mathcal{F}_t)$-martingale. This finishes the proof of the proposition. □

We now prove the strong Markov property of weak solutions to a well-posed SDEs.

Theorem 4.3 (Strong Markov property). *Suppose SDE (17) with locally bounded coefficients b and σ are well posed. Then for every bounded $(\mathcal{F}_t)$-stopping time τ and every Borel measurable set $B \subseteq \mathcal{C}([0,\infty),\mathbb{R}^n)$ one has*

$$\mathbb{P}\left\{X_{\tau+\cdot} \in B | \mathcal{F}_\tau\right\} = P^{X_\tau}(B),$$

where P^x is the distribution[9] *of the solution started at $x \in \mathbb{R}^n$.*

Proof. Since the coefficients to SDE (17) are locally bounded and the equation is well posed, the martingale problem associated with the differential operator $\mathcal{A}$ is well posed, by Corollary 3.3. As in the proof of Proposition 4.1 we denote by Q the regular conditional probability for $\mathcal{F}$ given $\mathcal{F}_\tau$. By Proposition 4.1, the there exists a $\mathbb{P}$-null set $N \in \mathcal{F}_\tau$ such that the probability measure $\mathbb{P}_{\tilde{\omega}}$, defined by (18), solves the corresponding martingale problem for every $\tilde{\omega} \in \Omega \setminus N$ with the initial condition $\tilde{\omega}_{\tau(\tilde{\omega})}$. By the uniqueness of solutions to both the martingale problem and to the corresponding SDE, and Theorem 3.2, we get $\mathbb{P}_{\tilde{\omega}} = P^{\tilde{\omega}_{\tau(\tilde{\omega})}}$ for all $\tilde{\omega} \in \Omega \setminus N$. Therefore,

$$\mathbb{P}\left\{X_{\tau+\cdot} \in B | \mathcal{F}_\tau\right\}(\tilde{\omega}) = Q(\tilde{\omega}; \{X_{\tau+\cdot} \in B\}) = \mathbb{P}_{\tilde{\omega}}(B) = P^{\tilde{\omega}_{\tau(\tilde{\omega})}}(B) = \mathbb{P}^{X_{\tau(\tilde{\omega})}(\tilde{\omega})}(B).$$

for $\mathbb{P}$-a.e. $\tilde{\omega}$. This completes the proof of the theorem. □

Using the strong Markov property, the finite dimensional distributions a solution X to SDE (17) can be defined via **transition probabilities for** X

$$P(t,x,\Gamma) = P^x\left\{X_t \in \Gamma\right\}, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad \Gamma \in \mathcal{B}(\mathbb{R}^n).$$

Corollary 4.1. *Under the assumptions of Theorem 4.3, for every $0 < t_1 < \dots < t_k$ and $\Gamma_1, \dots, \Gamma_k \in \mathcal{B}(\mathbb{R}^n)$ one has*

$$\mathbb{P}\left\{X_{t_1} \in \Gamma_1, \dots, X_{t_k} \in \Gamma_k\right\} = \int_{\Gamma_1} P(t_1, x_0, dx_1) \int_{\Gamma_2} P(t_2 - t_1, x_1, dx_2) \int_{\Gamma_3} \dots \int_{\Gamma_k} P(t_k - t_{k-1}, x_{k-1}, dx_k). \tag{19}$$

If for every $t > 0$ and $x \in \mathbb{R}^n$ the transition probability $P(t,x,\cdot)$ has a density $p(t,x,y)$, $y \in \mathbb{R}^n$, with respect to the Lebesgue measure, i.e. $P(t,x,\Gamma) = \int_\Gamma p(t,x,y)dy$, then

$$\mathbb{P}\left\{X_{t_1} \in \Gamma_1, \dots, X_{t_k} \in \Gamma_k\right\} := \int_{\Gamma_1} \cdots \int_{\Gamma_k} p(t_1, x_0, x_1) p(t_2 - t_1, x_1, x_2) \dots p(t_k - t_{k-1}, x_{k-1}, x_k) dx_1 \dots dx_k.$$

[9] The well posedness of SDE (17) implies that the map $x \mapsto P^x(B)$ is Borel measurable for every $B \in \mathcal{B}(\mathcal{C}([0,\infty),\mathbb{R}^n))$. Therefore, the random variable P^{X_τ} is well defined. We will not consider the measurability question here.

Proof. In order to prove the statement we will use the (strong) Markov property and the mathematical induction in k. By the definition of the transition probabilities for X, equality (19) holds for $k = 1$. We assume that the equality holds for $k = l - 1$ and prove it for $k = l$:

$$\begin{aligned}
&\mathbb{P}\left\{X_{t_1} \in \Gamma_1, X_{t_2} \in \Gamma_2, \dots, X_{t_l} \in \Gamma_l\right\} = \mathbb{E}\left[\mathbb{P}\left\{X_{t_1} \in \Gamma_1, X_{t_2} \in \Gamma_2, \dots, X_{t_l} \in \Gamma_l | \mathcal{F}_{t_1}\right\}\right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left(\mathbb{I}_{\{X_{t_1}\in\Gamma_1, X_{t_2}\in\Gamma_2,\dots,X_{t_l}\in\Gamma_l\}} | \mathcal{F}_{t_1}\right)\right] = \mathbb{E}\left[\mathbb{I}_{\left\{X_{t_1}\in\Gamma_1\right\}}\mathbb{E}\left(\mathbb{I}_{\{X_{t_2}\in\Gamma_2,\dots,X_{t_l}\in\Gamma_l\}} | \mathcal{F}_{t_1}\right)\right] \\
&\quad = \mathbb{E}\left[\mathbb{I}_{\left\{X_{t_1}\in\Gamma_1\right\}}\mathbb{P}\left\{X_{t_2} \in \Gamma_2, \dots, X_{t_l} \in \Gamma_l | \mathcal{F}_{t_1}\right)\right] \\
&\quad = \mathbb{E}\left[\mathbb{I}_{\left\{X_{t_1}\in\Gamma_1\right\}} P^{X_{t_1}}\left\{X_{t_2-t_1} \in G_2, \dots, X_{t_l-t_1} \in G_l\right\}\right] \\
&\quad = \mathbb{E}\left[\mathbb{I}_{\left\{X_{t_1}\in\Gamma_1\right\}} \int_{\Gamma_2} P(t_2 - t_1, X_{t_1}, dx_2) \int_{\Gamma_3} P(t_3 - t_2, x_2, dx_3) \int_{\Gamma_4} \cdots \int_{\Gamma_l} P(t_l - t_{l-1}, x_{l-1}, dx_l)\right] \\
&\quad = \int_{\Gamma_1}\left[\int_{\Gamma_2} P(t_2 - t_1, x_1, dx_2) \int_{\Gamma_3} P(t_3 - t_2, x_2, dx_3) \int_{\Gamma_4} \cdots \int_{\Gamma_l} P(t_l - t_{l-1}, x_{l-1}, dx_l)\right] P(t_1, x_0, dx_1).
\end{aligned}$$

□

Corollary 4.2. *Let SDE (17) with locally bounded coefficient b, σ is well posed. Let also the corresponding Kolmogorov forward PDE*

$$\begin{aligned}
\frac{\partial \psi(t,x)}{\partial t} &= \mathcal{A}\psi(t,x), \quad x \in \mathbb{R}^n, \quad t > 0, \\
\psi(0,x) &= \varphi(x), \quad x \in \mathbb{R}^n,
\end{aligned}$$

has a solution defined by

$$\psi(t,x) = \int_{\mathbb{R}^n} \varphi(y) p(t,x,y) dy,$$

for every $t > 0$, $x \in \mathbb{R}^n$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, where $p(t,x,\cdot)$ is a probability density. Then the transition probabilities for the solution X to SDE (17) are defined by

$$P(t,x,\Gamma) = \int_\Gamma p(t,x,y)dy, \quad t > 0, \quad x \in \mathbb{R}^n, \quad \Gamma \in \mathcal{B}(\mathbb{R}^n).$$

Proof. The statement directly follows from equality (16). □

5 Weak and strong solutions

In this section, we will study the relationship between pathwise uniqueness and uniqueness in law. We consider two weak solutions $(X^{(i)}, B^{(i)})$ to the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0, \tag{20}$$

defined on a filtered probability spaces $\left(\Omega^{(i)}, \mathcal{F}^{(i)}, (\mathcal{F}_t^{(i)})_{t\geq 0}, \mathbb{P}^{(i)}\right)$, $i \in [2]$. Our first goal is to define these two solutions on the same probability space. This will allow later to show that the pathwise uniqueness implies the uniqueness in law.

The pair $(X^{(i)}, B^{(i)})$ indicates a measure P_i on the space

$$\Theta = \mathcal{C}([0,\infty), \mathbb{R}^n) \times \mathcal{C}([0,\infty), \mathbb{R}^m)$$

equipped with the Borel σ-algebra $\mathcal{B}(\Theta)$, that is,

$$P_i(A) := \mathbb{P}^{(i)}\left\{(X^{(i)}, B^{(i)}) \in A\right\}, \quad A \in \mathcal{B}(\Theta), \quad i \in [2].$$

We remark that the marginal distribution of P_i on the second coordinate is the distribution of Brownian motion denoted by $\mathbb{P}^B$, i.e.

$$\mathbb{P}^B(A) := P_i\left(\mathcal{C}([0,\infty),\mathbb{R}^n) \times A\right) = \mathbb{P}^{(i)}\left\{B^{(i)} \in A\right\}, \quad A \in \mathcal{B}\left(\mathcal{C}([0,\infty),\mathbb{R}^m)\right).$$

Moreover, (X, W) defined by $X(\theta) = x$, $W(\theta) = w$ for $\theta = (x, w) \in \Theta$ is a weak solutions to SDE (20) on the probability space $(\Theta, \mathcal{B}(\Theta), P_i)$.

According to Theorem 4.2, on $(\Theta, \mathcal{B}(\Theta), P_i)$ there exists a regular conditional probability, denoted by Q_i, given W. We will be interested only in conditional probabilities of sets in $\mathcal{B}(\Theta)$ of the form $F \times \mathcal{C}([0,\infty),\mathbb{R}^m)$, where $F \in \mathcal{B}\left(\mathcal{C}([0,\infty),\mathbb{R}^n)\right)$. Therefore, to abuse of notations, we will write $Q_i(w; F)$ instead of $Q_i\left(w; F \times \mathcal{C}([0,\infty),\mathbb{R}^m)\right)$. Therefore, $Q_i(w; \cdot)$ is a probability measure on $\mathcal{C}([0,\infty),\mathbb{R}^n)$ for every $w \in \mathcal{C}([0,\infty),\mathbb{R}^m)$ and

$$Q_i(W; F) = P_i\left\{F \times \mathcal{C}([0,\infty),\mathbb{R}^m)\right\} = P_i\{X \in F | W\} \quad \mathbb{P}^B - \text{a.e.}.$$

In particular, this implies the equality

$$\int_A Q_i(w; F)\mathbb{P}^B(dw) = P_i(F \times A), \quad F \times A \in \mathcal{B}(\Theta). \tag{21}$$

We define on the measurable space $\Omega = \mathcal{C}([0,\infty),\mathbb{R}^n) \times \Theta$ equipped with Borel σ-algebra $\mathcal{F}$ a new probability measure as follows:

$$\mathbb{P}(A) = \int_A Q_1(w; dz^{(1)})Q_2(w; dz^{(2)})\mathbb{P}^B(dw), \quad A \in \mathcal{F}.$$

The canonical process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is will be denoted by

$$Z_t(\omega) = \left(Z_t^{(1)}(\omega), Z_t^{(2)}(\omega), B_t(\omega)\right) = \left(z_t^{(1)}, z_t^{(2)}, w_t\right), \quad t \geq 0, \quad \omega = (z^{(1)}, z^{(2)}, w) \in \Omega.$$

We also endow the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by Z. Remark that the distribution of $(Z^{(i)}, B)$ coincides with P_i, $i \in [2]$. Indeed,

$$\begin{aligned}
\mathbb{P}\left\{(Z^{(1)}, B) \in F \times A\right\} &= \mathbb{P}\left\{(Z^{(1)}, Z^{(2)}, B) \in F \times \mathcal{C}([0,\infty),\mathbb{R}^n) \times A\right\} \\
&= \int_F \int_{\mathcal{C}([0,\infty),\mathbb{R}^n)} \int_A Q_1(w; dz^{(1)})Q_2(w; dz^{(2)})\mathbb{P}^B(dw) \\
&= \int_F \int_A Q_1(w; dz^{(1)})\mathbb{P}^B(dw) = \int_A Q_1(w; F)\mathbb{P}^B(dw) \\
&= P_1(F \times A), \quad F \times A \in \mathcal{B}(\Theta).
\end{aligned}$$

A similar computation for $(Z^{(2)}, B)$ valid. Therefore, $(Z^{(1)}, B)$ and $(Z^{(2)}, B)$ are two solutions to SDE (20) defined on the same probability space and driven by the same Brownian motion B. We now are ready to prove the follwoing statement.

Theorem 5.1. *Pathwise uniqueness implies uniqueness in law.*

Proof. We consider tow weak solutions $(X^{(i)}, B^{(i)})$ to SDE (20) defined on filtered probability spaces $(\Omega^{(i)}, \mathcal{F}^{(i)}, (\mathcal{F}_t^{(i)})_{t\geq 0}, \mathbb{P}^{(i)})$. We define the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the process $Z_t = (Z_t^{(1)}, Z_t^{(2)}, B_t)$, $t \geq 0$, as above. Since $(Z^{(1)}, B)$ and $(Z^{(2)}, B)$ are two weak solutions to SDE (20), the pathwise uniqueness implies

$$\mathbb{P}\left\{Z^{(1)} = Z^{(2)}\right\} = 1.$$

Therefore, for every $A \in \mathcal{B}(\Theta)$

$$P_1(A) = \mathbb{P}\left\{(Z^{(1)}, B) \in A\right\} = \mathbb{P}\left\{(Z^{(2)}, B) \in A\right\} = P_2(A).$$

This implies that the marginal distributions of P_1 and P_2 coincides, i.e. $\mathrm{Law}(X^{(1)}) = \mathrm{Law}(X^{(2)})$. □

Remark 5.1. In the proof of Theorem 5.1 we have proven more. Namely, that the pathwise uniqueness implies the uniqueness in law for the pair (X, B). More presicely, if $(X^{(i)}, B^{(i)})$, $i \in [2]$, are two solutions to SDE (20) and the pathwise uniqueness holds, then $\mathrm{Law}(X^{(1)}, B) = \mathrm{Law}(X^{(2)}, B)$.

Exercise 5.1. Prove equality (21).

Theorem 5.1 has the remarkable corollary that weak existence and pathwise uniqueness imply strong existence.

Theorem 5.2 (Yamada-Watanabe)**.** *Weak existence and pathwise uniqueness imply strong existence.*

Proof. The idea of the proof is simple. We take a weak solution $(X^{(1)}, B^{(1)})$ defined on a filtered probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\geq 0}, \mathbb{P}^{(1)})$ and take another copy of it which we equip with indices 2 everywhere. We construct Q_1 and Q_2, Q, Z and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as above. Remark that $Q_1 = Q_2$, by Remark 5.1 and the uniqueness of the regular conditional probability (see Theorem 4.2). Let $\Delta := \{(z, z) : z \in \mathcal{C}([0, \infty), \mathbb{R}^n)\}$ be the diagonal in $\mathcal{C}([0, \infty), \mathbb{R}^n)^2$. Since $\mathcal{C}([0, \infty), \mathbb{R}^n)$ is a complete separeble metric space, Δ is measurable subset in $\mathcal{C}([0, \infty), \mathbb{R}^n)^2$. By pathwise uniqueness,

$$1 = \mathbb{P}\left\{Z^{(1)} = Z^{(2)}\right\} = \mathbb{P}\left\{(Z^{(1)}, Z^{(2)}) \in \Delta\right\} = \int_{\Delta \times \mathcal{C}([0,\infty),\mathbb{R}^n)} Q_1(w; dz^{(1)}) Q_2(w; dz^{(2)}) \mathbb{P}^B(dw).$$

Therefore, for $\mathbb{P}^B$-a.e. w, one has

$$\int_\Delta Q_1(w; dz^{(1)}) Q_2(w; dz^{(2)}) = 1.$$

This means, that $Q_1(w; \cdot) = Q_2(w; \cdot)$ is a Dirac measure for $\mathbb{P}^B$-a.e. w., i.e. there exists a function[10] $\Psi : \mathcal{C}([0, \infty), \mathbb{R}^m) \to \mathcal{C}([0, \infty), \mathbb{R}^n)$ such that $Q_i(w; \cdot) = \delta_{\Psi(w)}$. Hence $Z^{(i)} = \Psi(W)$. So, the original process $X^{(1)}$ can be written as a function $\Psi(B^{(1)})$ of the Brownian motion. According to Proposition 1.1, $(X^{(1)}, B^{(1)})$ is a strong solution to SDE (20). □

[10] one can show that the function Ψ is measurable

6 One-dimensional SDEs

In this section, we will discuss the pathwise uniqueness of a solution to SDE in the one-dimensional case. We will also prove the comparison principle. In the one-dimensional case, the Lipschitz condition[11] on the diffusion coefficient can be relaxed considerably.

Theorem 6.1 (Yamada, Watanabe)**.** *Suppose that the coefficiens of the one-dimensional SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0$$

satisfy the conditions

$$|b(t, x) - b(t, y)| \leq L|x - y|, \tag{22}$$

$$|\sigma(t, x) - \sigma(t, y)| \leq h(|x - y|), \tag{23}$$

for every $0 \leq t < \infty$ *and* $x, y \in \mathbb{R}$*, where* L *is a positive constant and* $h : [0, \infty) \to [0, \infty)$ *is a strictly increasing function with* $h(0) = 0$ *and*

$$\int_0^\varepsilon \frac{du}{h^2(u)} = \infty, \quad \varepsilon > 0.$$

Then the pathwise uniqueness holds.

Proof. The proof of the statement can be found in [KS91, Proposition 5.2.13]. □

Remark 6.1. One can take the function h in Theorem 6.1 to be $h(u) = u^\alpha$, $u \geq 0$, for $\alpha \geq \frac{1}{2}$.

Exercise 6.1. Show that the equation

$$dX_t = |X_t|^\alpha dB_s, \quad X_0 = x_0$$

has a pathwise unique strong solution.

Theorem 6.2. *Let* $X^{(i)}$*,* $i \in [2]$*, be continuous processes on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *satisfying the equality*

$$X_t^{(i)} = x_0^{(i)} + \int_0^t b_i(s, X_s^{(i)})ds + \int_0^t \sigma(s, X_s^{(i)})dB_s, \quad t \geq 0,$$

for a Brownian motion B *and some* $x_0^{(1)} \leq x_0^{(2)}$*. We assume that*

(i) the coefficients σ*,* b_i *are continuous functions on* $[0, \infty) \times \mathbb{R}$*,*

(ii) the function σ *satisfies* (23)*,*

(iii) $b_1(t, x) \leq b_2(t, x)$*,* $t \geq 0$*,* $x \in \mathbb{R}$*,*

(iv) either b_1 *or* b_2 *satisfies condtion* (22)*.*

Then

$$\mathbb{P}\left\{X_t^{(1)} \leq X_t^{(2)}, \quad t \geq 0\right\} = 1.$$

[11] see Theorem 2.1 for the case of Lipschitz continuous coefficients

Proof. For concreteness, let us suppose that (22) is satisfied by b_1. By a usual localization argument we may assume that σ and b_i are bounded.

Let $1 = a_0 > a_1 > a_2 > \dots$ be defined by

$$\int_{a_n}^{a_{n-1}} \frac{du}{h^2(u)} = n, \quad n \geq 1.$$

Clearly $a_n \to 0$ as $n \to \infty$. For every $n \geq 1$ let ψ_n be a continuous function such that its support is contained in (a_n, a_{n-1}),

$$0 \leq \psi_n(u) \leq \frac{2}{h^2(u)n}, \quad u > 0,$$

and

$$\int_{a_n}^{a_{n-1}} \psi_n(u)du = 1.$$

Such a function obviously exists. We set

$$\varphi_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \int_0^x \left(\int_0^y \psi_n(u)du\right) dy & \text{if } x > 0. \end{cases} \tag{24}$$

It is easy to see that $\varphi_n \in \mathcal{C}^2(\mathbb{R})$, $\varphi_n(x) = 0$ for $x \leq 0$, $0 \leq \varphi'(x) \leq 1$ and $\varphi_n(x)$ increases to $x^+ := \max\{x, 0\}$ as $n \to \infty$ for every $x \in \mathbb{R}$. An application of Itô's formula yields

$$\varphi_n(X_t^{(1)} - X_t^{(2)}) = I_1(n) + I_2(n) + I_3(n),$$

where

$$I_1(n) = \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\right) dB_s,$$
$$I_2(n) = \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})\right) ds$$

and

$$I_3(n) = \frac{1}{2} \int_0^t \varphi_n''(X_s^{(1)} - X_s^{(2)}) \left(\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\right)^2 ds.$$

It is clear that $\mathbb{E}\, I_1(n) = 0$ and

$$\mathbb{E}\left(I_3(n)\right) \leq \frac{1}{2}\mathbb{E}\left[\int_0^t \varphi_n''(X_s^{(1)} - X_s^{(2)}) h(|X_s^{(1)} - X_s^{(2)}|)^2 ds\right] \leq \frac{t}{n}.$$

Also,

$$\begin{aligned}
I_2(n) &\leq \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(b_1(s, X_s^{(1)}) - b_2(s, X_s^{(2)})\right) ds \\
&= \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(b_1(s, X_s^{(1)}) - b_1(s, X_s^{(2)})\right) ds \\
&+ \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(b_1(s, X_s^{(2)}) - b_2(s, X_s^{(2)})\right) ds \\
&\leq \int_0^t \varphi_n'(X_s^{(1)} - X_s^{(2)}) \left(b_1(s, X_s^{(1)}) - b_1(s, X_s^{(2)})\right) ds \\
&\leq L \int_0^t \mathbb{I}_{\left\{X_s^{(1)} > X_s^{(2)}\right\}} |X_s^{(1)} - X_s^{(2)}| ds = L \int_0^t \left(X_s^{(1)} - X_s^{(2)}\right)^+ ds.
\end{aligned}$$

Hence, by letting $n \to \infty$, we have

$$\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^+\right] \leq L\mathbb{E}\left[\int_0^t \left(X_s^{(1)} - X_s^{(2)}\right)^+ ds\right] = L\int_0^t \mathbb{E}\left[\left(X_s^{(1)} - X_s^{(2)}\right)^+\right] ds.$$

By Gronwall's inequality, $\mathbb{E}\left[\left(X_t^{(1)} - X_t^{(2)}\right)^+\right] = 0$ for every $t \geq 0$. The continuity of $X^{(i)}$, $i \in [2]$, implies the equality

$$\mathbb{P}\left\{X_t^{(1)} \leq X_t^{(2)}, \quad t \geq 0\right\} = 1.$$

□

Exercise 6.2. Show that the family of functions ψ_n, $n \geq 1$, from the proof of Theorem 6.2 exists and the sequence of functions φ_n, $n \geq 1$, defined by (24), satisfy the prescribed properties.

7 Local time

7.1 Motivation and definition

The goal of this section is to define the local time for a continuous semimartingale

$$X_t = x_0 + V_t + M_t, \quad t \geq 0,$$

in $\mathbb{R}$, where V is a continuous process of bounded variation and M is a continuous local martingale. In order to introduce the local time, we will apply Itô's formula to the function $f(x) = |x|$, $x \in \mathbb{R}$. Remark that Itô's formula can not be applied directly, since f does not belong to $\mathcal{C}^2(\mathbb{R})$. Therefore, we will approximate the function f by $f_n \in \mathcal{C}^2(\mathbb{R})$ such that $f_n(x) = -x$ for $x \leq 0$ and $f_n(x) = x - \frac{1}{n}$ for $x \geq \frac{1}{n}$. It is clear that $f_n(x) \to |x|$ and $f_n'(x) \to \operatorname{sgn} x$ as $n \to \infty$ for every $x \in \mathbb{R}$, where

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0. \end{cases}$$

By Itô's formula, we get

$$Y_t^n := f_n(X_t) - f_n(x_0) - \int_0^t f_n'(X_s)dX_s = \frac{1}{2}\int_0^t f_n''(X_s)d\langle M\rangle_s, \quad t \geq 0.$$

Our first goal is to show that for every $T > 0$

$$\max_{t\in[0,T]}\left|\int_0^t f_n'(X_s)dX_s - \int_0^t \operatorname{sgn} X_s dX_s\right| \xrightarrow{\mathbb{P}} 0, \quad n \to \infty. \tag{25}$$

We choose a localization sequence of stopping times τ_m, $m \geq 1$, increasing to infinity such that $|X_{t\wedge\tau_m}| \vee \langle M\rangle_{t\wedge\tau_m} \leq m$, $t \geq 0$, by e.g.

$$\tau_m := \inf\left\{t \geq 0 : \ |X_t| \geq m, \ \langle M\rangle_t \geq m\right\}.$$

Using the Burkholder-Davis-Gundy inequality and the dominated convergence theorem, we get for every $m \geq 1$

$$\mathbb{E}\max_{t\in[0,T\wedge\tau_m]}\left|\int_0^t \left(f_n'(X_s) - \operatorname{sgn} X_s\right) dM_s\right|^2 \leq C_2\mathbb{E}\int_0^{T\wedge\tau_m} \left(f_n'(X_s) - \operatorname{sgn} X_s\right)^2 d\langle M\rangle_s \to 0$$

as $n \to \infty$. This immediately implies

$$\max_{t \in [0,T]} \left| \int_0^t \left(f_n'(X_s) - \operatorname{sgn} X_s \right) dM_s \right| \xrightarrow{\mathbb{P}} 0. \tag{26}$$

Exercise 7.1. Prove the convergence in (26).

We recall that every continuous function $g : [0, \infty) \to \mathbb{R}$ of bounded variation can be uniquely decomposed as $g = g_1 - g_2$, where g_i are nondecreasing continuous continuous functions. Therefore, there exists two continuous nondecreasing processes $V^{(1)}, V^{(2)}$ such that $V = V^{(1)} - V^{(2)}$. By the dominated convergence theorem, we obtain

$$\max_{t \in [0,T]} \left| \int_0^t f_n'(X_s) dV_s - \int_0^t \operatorname{sgn} X_s dV_s \right| \leq \max_{t \in [0,T]} \left| \int_0^t \left(f_n'(X_s) - \operatorname{sgn} X_s \right) dV_s^{(1)} \right|$$
$$+ \max_{t \in [0,T]} \left| \int_0^t \left(f_n'(X_s) - \operatorname{sgn} X_s \right) dV_s^{(2)} \right| \leq \int_0^T \left| f_n'(X_s) - \operatorname{sgn} X_s \right| d(V_s^{(1)} + V_s^{(2)}) \to 0 \quad \text{a.s.}$$

as $n \to \infty$. This implies convergence (25).

From (25) we can conclude that for every $t \geq 0$ there exists the limit in probability

$$\lim_{n \to \infty} Y_t^n = \lim_{n \to \infty} \frac{1}{2} \int_0^t f_n''(X_s) d\langle M \rangle_s = |X_t| - |x_0| - \int_0^t \operatorname{sgn} X_s dX_s. \tag{27}$$

Definition 7.1. Let X_t, $t \geq 0$, be a continuous semimartingale in $\mathbb{R}$. The continuous process

$$L_t^0 := |X_t| - |X_0| - \int_0^t \operatorname{sgn} X_s dX_s, \quad t \geq 0,$$

is called the **semimartingale local time of X at 0**.

Remark 7.1. The local time formally can be defined as

$$L_t^0 = \int_0^t \delta_0(X_s) d\langle M \rangle_s = \int_0^t \delta_0(X_s) d\langle X \rangle_s$$

due to the convergence $\frac{1}{2} f_n'' \to \delta_0$ in distribution, where δ_0 is the δ-function at zero. Indeed, for any smooth function φ with compact support one has

$$\int_{\mathbb{R}} \frac{1}{2} f_n''(x) \varphi(x) dx = -\frac{1}{2} \int_{\mathbb{R}} f_n'(x) \varphi'(x) dx \to -\frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn} x \cdot \varphi'(x) dx$$
$$= \frac{1}{2} \int_{-\infty}^0 \varphi'(x) dx - \frac{1}{2} \int_0^{+\infty} \varphi'(x) dx = \varphi(0) = \int_{\mathbb{R}} \varphi(x) \delta_0(x) dx.$$

We next state one of the most important property of the local time.

Theorem 7.1. *Let L_t^0, $t \geq 0$, be the local time at 0 of a continuous semimartingale X. Then L^0 is a.s. nondecreasing continuous process which increases only on the set $\{t \geq 0 : \ X_t = 0\}$, i.e.*

$$\int_0^\infty \mathbb{I}_{\{X_s \neq 0\}} dL_s^0 = 0. \tag{28}$$

Furthermore, we have a.s.

$$L_t^0 = \left(-|X_0| - \min_{s \in [0,t]} \int_0^s \operatorname{sgn} X_r dX_r \right) \vee 0, \quad t \geq 0.$$

The proof of the theorem is based on the following elementary observation.

Lemma 7.1 (Skorohod)**.** *Let f be a continuous function on $[0,\infty)$ such that $f_0 \geq 0$. Then there exists a unique nondecreasing continuous function g with $g_0 = 0$ and a continuous nonnegative function h such that $f = h - g$ and*

$$\int_0^\infty \mathbb{I}_{\{h_s>0\}} dg_s = 0.$$

Moreover,

$$g_t = \left(- \min_{s\in[0,t]} f_s \right) \vee 0 = \max_{s\in[0,t]} (-f_s) \vee 0$$

Proof. The proof of the lemma can be found in [Kal02, Lemma 22.2] or [IW89, Lemma 4.2]. □

Proof of Theorem 7.1. The fact that L_t^0, $t \geq 0$, is continuous and nondecreasing follows directly from the definition of the local time and convergence (27). We next prove equality (28). We remark that the process

$$\begin{aligned} |X_t| &= |X_0| + \int_0^t \operatorname{sgn} X_s dX_s + L_t^0 \\ &= |X_0| + \underbrace{\int_0^t \operatorname{sgn} X_s dM_s}_{-local\ martingale} + \underbrace{\int_0^t \operatorname{sgn} X_s dV_s + L_t^0}_{-bdd.\ variation}, \quad t \geq 0, \end{aligned}$$

is a continuous semimartingale. Therefore, applying Itô's formula, we get

$$\begin{aligned} X_t^2 = (|X_t|)^2 &= (|X_0|)^2 + 2\int_0^t |X_s| \operatorname{sgn} X_s dM_s + 2\int_0^t |X_s| \operatorname{sgn} X_s dV_s \\ &+ 2\int_0^t |X_s| dL_s^0 + \frac{2}{2}\int_0^t (\operatorname{sgn} X_s)^2 d\langle M\rangle_s \\ &= X_0^2 + 2\int_0^t X_s dM_s + 2\int_0^t X_s dV_s + 2\int_0^t |X_s| dL_s^0 + \langle M\rangle_t, \quad t \geq 0. \end{aligned}$$

On the other hand, by Itô's formula,

$$X_t^2 = X_0^2 + 2\int_0^t X_s dM_s + 2\int_0^t X_s dV_s + \frac{2}{2}\int_0^t d\langle M\rangle_s, \quad t \geq 0.$$

Hence, $\int_0^t |X_s| dL_s^0 = 0$. This directly implies that

$$\int_0^t \mathbb{I}_{\{|X_s|>0\}} dL_s^0, \quad t \geq 0.$$

Consequently, (28) holds.

The last assertion of the theorem is a consequence of Lemma 7.1 applied to $f_t = |X_0| + \int_0^t \operatorname{sgn} X_s dX_s$, $g_t = L_t^0$, and $h_t = |X_t|$, $t \geq 0$. □

As an example of local time, we state a basic relationship between a Brownian motion, its maximum process and its local time at 0.

Corollary 7.1. *Let L^0 be the local time at 0 of a Brownian motion B and define*

$$M_t = \max_{s\in[0,t]} B_s, \quad t \geq 0.$$

Then

$$\mathrm{Law}(L^0, |B|) = \mathrm{Law}(M, M - B).$$

Proof. We define a new process

$$\tilde{B}_t := -\int_0^t \operatorname{sgn} B_s dB_s \quad \text{and} \quad \tilde{M}_t = \max_{s\in[0,t]} \tilde{B}_s, \quad t \geq 0,$$

and conclude that $\tilde{B}$ is a continuous martingale with quadratic variation

$$\langle \tilde{B} \rangle_t = \int_0^t (\operatorname{sgn} B_s)^2 ds = t, \quad t \geq 0.$$

Therefore, $\tilde{B}$ is a Brownian motion, by the Levy characterization theorem. We also remark that M and $\tilde{M}$ are functions of the Brownian motion B and $\tilde{B}$, respectively, i.e. $M = \Psi(B)$ and $\tilde{M} = \Psi(\tilde{B})$, where $\Psi(f)_t = \max_{s\in[0,t]} f_s$, $t \geq 0$. So,

$$\mathrm{Law}(M, B) = \mathrm{Law}(\Psi(B), B) = \mathrm{Law}(\Psi(\tilde{B}), \tilde{B}) = \mathrm{Law}(\tilde{M}, \tilde{B}). \tag{29}$$

Using Theorem 7.1 and Definition 7.1 of local time, we can conclude that for every $t \geq 0$

$$L_t^0 = \left(-\min_{s\in[0,t]} \int_0^s \operatorname{sgn} B_r dB_r \right) \vee 0 = \max_{s\in[0,t]} \left(-\int_0^s \operatorname{sgn} B_r dB_r \right) = \max_{s\in[0,t]} \tilde{B}_s = \tilde{M}_t.$$

and

$$L^0 = |B_t| - \int_0^t \operatorname{sgn} B_s dB_s = |B_t| + \tilde{B}_t.$$

Hence, $|B_t| = L_t^0 - \tilde{B}_t = \tilde{M}_t - \tilde{B}_t$, $t \geq 0$. Consequently, we can conclude that

$$\mathrm{Law}(M, M - B) \overset{(29)}{=} \mathrm{Law}(\tilde{M}, \tilde{M} - \tilde{B}) = \mathrm{Law}(L^0, |B|).$$

□

7.2 Occupation-time formula

Let as before $X_t = x_0 + V_t + M_t$, $t \geq 0$, be a continuous semimartingale, where V is a continuous process of bounded variation and M is a continuous local martingale. We introduce the **local time L^x at an arbitrary point** $x \in \mathbb{R}$ as the local time of the continuous semimartingale $X_t - x$, $t \geq 0$, at 0. Thus, we define

$$L_t^x = |X_t - x| - |X_0 - x| - \int_0^t \operatorname{sgn}(X_s - x) dX_s, \quad t \geq 0. \tag{30}$$

Once can show that the prcess L_t^x, $t \geq 0$, $x \in \mathbb{R}$, has a version that is continuous in t and càdlàg (right-continuous with left-hand limits) in x. More precisely,

$$L_t^x - L_t^{x-} = 2\int_0^t \mathbb{I}_{\{X_s = x\}} dV_s, \quad x \in \mathbb{R}, \quad t \geq 0.$$

We will omit the discussion of this fact here. However, the proof can be found e.g. in [Kal02, Teorem 22.4].

Our first goal is to prove the following formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^x f''(x)dx \tag{31}$$

for every $f \in \mathcal{C}^2(\mathbb{R})$ with $f'' \in \mathcal{C}_0(\mathbb{R})$.

For $f \in \mathcal{C}^2(\mathbb{R})$ with $f'' \in \mathcal{C}_0(\mathbb{R})$ we introduce the function

$$F(x) = \frac{1}{2}\int_{-\infty}^{+\infty} |x-y| f''(y)dy, \quad x \in \mathbb{R},$$

and remark that

$$F'(x) = f'(x) = \frac{1}{2}\int_{-\infty}^{+\infty} \operatorname{sgn}(x-y) f''(y)dy, \quad x \in \mathbb{R}. \tag{32}$$

Indeed,

$$\begin{aligned} F'(x) &= \left(\frac{1}{2}\int_{-\infty}^{x} (x-y)f''(y)dy + \frac{1}{2}\int_{x}^{+\infty} (y-x)f''(y)dy\right)' \\ &= \frac{1}{2}\int_{-\infty}^{x} f''(y)dy - \frac{1}{2}\int_{x}^{+\infty} f''(y)dy = f'(x) \end{aligned}$$

and

$$F'(x) = \frac{1}{2}\int_{-\infty}^{x} f''(y)dy - \frac{1}{2}\int_{x}^{+\infty} f''(y)dy = \frac{1}{2}\int_{-\infty}^{+\infty} \operatorname{sgn}(x-y)f''(y)dy.$$

This observation implies that there exists a constant C such that

$$f(x) = C + F(x) = C + \frac{1}{2}\int_{-\infty}^{+\infty} |x-y| f''(y)dy, \quad x \in \mathbb{R}.$$

Applying the definition of the local time and a (stochastic) Fubini theorem, we get

$$\begin{aligned} f(X_t) &= C + \frac{1}{2}\int_{-\infty}^{+\infty} |X_t - y| f''(y)dy \overset{(30)}{=} C + \frac{1}{2}\int_{-\infty}^{+\infty} |X_0 - y| f''(y)dy \\ &+ \frac{1}{2}\int_{-\infty}^{+\infty} \left(\int_0^t \operatorname{sgn}(X_s - y)dX_s\right) f''(y)dy + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^y f''(y)dy \\ &= f(X_0) + \int_0^t \left(\frac{1}{2}\int_{-\infty}^{+\infty} \operatorname{sgn}(X_s - y)f''(y)dy\right)dX_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^y f''(y)dy \\ &\overset{(32)}{=} f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^y f''(y)dy \end{aligned}$$

that implies (31)

Theorem 7.2 (Occupation-time formula)**.** *There exists a $\mathbb{P}$-null set outside which for any $t \geq 0$ and any non-negative Borel-measurable function $g : \mathbb{R} \to [0, +\infty)$ we have*

$$\int_0^t g(X_s) d\langle X \rangle_s = \int_{-\infty}^{+\infty} g(x) L_t^x dx. \tag{33}$$

Proof. We will check the occupation-time formula only for $g \in \mathcal{C}_0(\mathbb{R})$. The general case can be proved by the approximation. We introduce the following function

$$f(x) = \int_{-\infty}^{x} \left(\int_{-\infty}^{y} g(y) dy \right), \quad x \in \mathbb{R},$$

Then trivially $f''(x) = g(x)$, $x \in \mathbb{R}$. Applying Itô's formula to $f(X_t)$, we get

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle M \rangle_s \\ &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t g(X_s) d\langle X \rangle_s \\ &\overset{(31)}{=} f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_{-\infty}^{+\infty} g(x) L_t^x dx. \end{aligned}$$

This implies occupation-time formula (33). □

Theorem 7.2 implies that the **occupation measure** at time t

$$\mu_t(A) = \int_0^t \mathbb{I}_A(X_s) d\langle X \rangle_s, \quad A \in \mathcal{B}(\mathbb{R}), \quad t \geq 0,$$

is a.s. absolutely continuous with respect to the Lebesgue measure with density L_t^x, $x \in \mathbb{R}$. This leads to a simple construction of the local time.

Corollary 7.2. *There exists a $\mathbb{P}$-null set outside which for any $t \geq 0$ and $x \in \mathbb{R}$ we have*

$$L_t^x = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \mu_t([x, x + \varepsilon]) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{\{x \leq X_s \leq x+\varepsilon\}} d\langle X \rangle_s.$$

Proof. The statement of the corollary directly follows from the right-continuity of the local time and Theorem 7.2. Indeed,

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_0^t \mathbb{I}_{\{x \leq X_s \leq x+\varepsilon\}} d\langle X \rangle_s = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \mathbb{I}_{[x,x+\varepsilon]}(y) L_t^y dy = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} L_t^y dy = L_t^x,$$

by the mean value theorem. □

7.3 Tanaka's formula

The goal of this section is to generalize Itô's formula to functions which are not twice continuously differentiable. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is **convex** if for every $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$ one has

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

Exercise 7.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Show that there exists the left derivative

$$f'_-(x) = \lim_{y \to x-} \frac{f(y) - f(x)}{y - x}$$

at every point $x \in \mathbb{R}$ which is nondecreasing left-continuous.

For a convex function $f : \mathbb{R} \to \mathbb{R}$ we define the unique measure η_f on $\mathbb{R}$ with

$$\eta_f([a, b)) := f'_-(b) - f'_-(a), \quad a < b.$$

It is easy to see that for a convex function $f \in \mathcal{C}^2(\mathbb{R})$ the measure η_f has the density f'', i.e.

$$\eta_f(A) = \int_A f''(x)dx, \quad A \in \mathcal{B}(\mathbb{R}).$$

Exercise 7.3. Find η_f for the following functions:

1. $f(x) = |x|$, $x \in \mathbb{R}$;
2. $f(x) = x^- = -\min\{x, 0\}$, $x \in \mathbb{R}$.

Theorem 7.3 (Tanaka's formula). *Let X be a continuous semimartingale with right-continuous local time L and f be a convex function. Then*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^x \eta_f(dx), \quad t \geq 0.$$

Proof. The idea of the proof is similar to proof of formula (31). □

8 Sticky-reflected Brownian motion

This section is based on the work [EP14].

The goal of this section is to study the well-possedness of the following SDE

$$dX_t = \lambda \mathbb{I}_{\{X_t=0\}}dt + \mathbb{I}_{\{X_t>0\}}dB_t, \quad X_0 = x_0, \tag{34}$$

where B_t, $t \geq 0$ is a one-dimensional Brownian motion, $x_0 \geq 0$ and $\lambda > 0$. A solution to this equation is called a **sticky-reflected Brownian motion on** $[0, \infty)$.

Theorem 8.1. *There exists a weak solution to SDE (34) which is unique in law and non-negative.*

Proof. Step I. We will show that any weak solution to equation (34) is non-negative. Let (X, B) be a weak solution to (34) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, i.e.

$$X_t = x_0 + \int_0^t \lambda \mathbb{I}_{\{X_s=0\}}ds + \int_0^t \mathbb{I}_{\{X_s>0\}}dB_s, \quad t \geq 0.$$

We consider for a fixed $z < 0$ the function $f(x) = (x - z)^- = -\min\{x - z, 0\}$, $x \in \mathbb{R}$, and apply Tanaka's formula. Since $f'_-(x) = -\mathbb{I}_{(-\infty,z]}(x)$ and $\nu_f = \delta_z$, we get

$$(X_t - z)^- = (x_0 - z)^- + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int_{-\infty}^{+\infty} L_t^x(X)\eta_f(dx) = 0 - \int_0^t \mathbb{I}_{\{X_s\leq z\}}dX_s + \frac{1}{2}L_t^z(X)$$
$$= -\int_0^t \lambda \mathbb{I}_{\{X_s\leq z\}}\mathbb{I}_{\{X_s=0\}}ds - \int_0^t \mathbb{I}_{\{X_s\leq z\}}\mathbb{I}_{\{X_s>0\}}dB_s + \frac{1}{2}L_t^z(X) = \frac{1}{2}L_t^z(X),$$

where $L_t^z(X)$ denotes the local time of the semimartingale X at 0. Using Corollary 7.2, we can conclude

$$L_t^z(X) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_0^t \mathbb{I}_{\{z\le X_s\le z+\varepsilon\}} d\langle X\rangle_s = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_0^t \mathbb{I}_{\{z\le X_s\le z+\varepsilon\}}\mathbb{I}^2_{\{X_s>0\}} ds = 0.$$

Therefore, for every $z<0$, one has $(X_t+z)^-=0$, $t\ge 0$. Passing to the limit as $z\to 0-$, we get $X_t^-=0$, $t\ge 0$. This implies that X takes non-negative values.

Step II. We compute $L_t^0(X)$. Applying Tanaka's formula to $f(x)=x^-$ similarly as above, we obtain

$$\begin{aligned} 0 = X_t^- &= -\int_0^t \mathbb{I}_{\{X_s\le 0\}} dX_s + \frac{1}{2}L_t^0(X) = -\int_0^t \mathbb{I}_{\{X_s\le 0\}}\cdot\lambda\mathbb{I}_{\{X_s=0\}} ds \\ &- \int_0^t \mathbb{I}_{\{X_s=0\}}\mathbb{I}_{\{X_s>0\}} dB_s + \frac{1}{2}L_t^0(X) = -\lambda\int_0^t \mathbb{I}_{\{X_s=0\}} ds + \frac{1}{2}L_t^0(X), \quad t\ge 0. \end{aligned}$$

Hence,

$$\frac{1}{2}L_t^0(X) = \lambda\int_0^t \mathbb{I}_{\{X_s=0\}} ds, \quad t\ge 0.$$

We remark that any weak solution to SDE (34) solves the system

$$\begin{aligned} X_t &= x_0 + \frac{1}{2}L_t^0(X) + \int_0^t \mathbb{I}_{\{X_s>0\}} dB_s, \quad t\ge 0, \\ \frac{1}{2}L_t^0(X) &= \lambda\int_0^t \mathbb{I}_{\{X_s=0\}} ds, \quad t\ge 0, \end{aligned} \tag{35}$$

where $L^0(X)$ is the local time of X at zero. Inversely, if (X,B) satisfies system (35), then it is trivially a solution to (34). Consequently, (34) and (35) are equivalent.

Step III. We will construct a pair (X,B) which satisfies (35). Let $\tilde{B}$ be a Brownian motion on a filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},(\tilde{\mathcal{F}}_t)_{t\ge 0},\tilde{\mathbb{P}})$. We set $\tilde{W}_t := x_0+\tilde{B}_t$, $t\ge 0$, and consider the strictly increasing process

$$A_t := t + \frac{1}{\lambda}L_t^0(\tilde{W}), \quad t\ge 0,$$

where $L^0(\tilde{W})$ is the local time of $\tilde{W}$ at zero. Since A is continuous and strictly increasing, one can define its inverse

$$T_t = A_t^{-1} = \inf\{s\ge 0:\ A_s\ge t\}, \quad t\ge 0.$$

We remark that for every $t\ge 0$ T_t is an $(\tilde{\mathcal{F}}_t)$-stopping time. Moreover T_t, $t\ge 0$, is a continuous and strictly increasing process. We consider the time-change continuous process

$$Y_t := \tilde{W}_{T_t} = x_0 + \tilde{B}_{T_t}, \quad t\ge 0. \tag{36}$$

By the optional sampling theorem, the process Y_t, $t\ge 0$, is a continuous $(\tilde{\mathcal{F}}_{T_t})$-martingale. Note further that

$$\tilde{B}_{T_t} = \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s\ne 0\}} d\tilde{B}_s = \int_0^t \mathbb{I}_{\{\tilde{W}_{T_s}\ne 0\}} d\tilde{B}_{T_s} = \int_0^t \mathbb{I}_{\{Y_s\ne 0\}} d\tilde{B}_{T_s}, \quad t\ge 0. \tag{37}$$

Moreover, we have

$$\langle \tilde{B}_T \rangle_t = T_t = \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s \neq 0\}} ds = \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s \neq 0\}} ds + \underbrace{\frac{1}{\lambda} \int_0^t \mathbb{I}_{\{\tilde{W}_s \neq 0\}} dL_s^0(\tilde{W})}_{\overset{(28)}{=} 0}$$
$$= \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s \neq 0\}} dA_s = \int_0^t \mathbb{I}_{\{\tilde{W}_{T_s} \neq 0\}} dA_{T_s} = \int_0^t \mathbb{I}_{\{Y_s \neq 0\}} ds, \quad t \geq 0.$$

We take a Brownian motion B_t^0, $t \geq 0$, independent of B_{T_t}, $t \geq 0$, and defined on an 1-extension of $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_{T_t})_{t \geq 0}, \tilde{\mathbb{P}})$ which we will denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Set

$$B_t = \tilde{B}_{T_t} + \int_0^t \mathbb{I}_{\{Y_s = 0\}} dB_s^0, \quad t \geq 0.$$

Then B_t, $t \geq 0$, is a Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, by Levy's characterisation theorem. Indeed, it is a continuous $(\mathcal{F}_t)$-martingale as a sum of continuous martingales and its quadratic variation equals

$$\langle B \rangle_t = \langle \tilde{B}_T \rangle_t + \int_0^t \mathbb{I}_{\{Y_s = 0\}} ds = \int_0^t \mathbb{I}_{\{Y_s \neq 0\}} ds + \int_0^t \mathbb{I}_{\{Y_s = 0\}} ds = t, \quad t \geq 0.$$

We also remark that

$$x_0 + \int_0^t \mathbb{I}_{\{Y_s \neq 0\}} dB_s = x_0 + \int_0^t \mathbb{I}_{\{Y_s \neq 0\}} d\tilde{B}_{T_s} \overset{(37)}{=} x_0 + B_{T_t} = Y_t, \quad t \geq 0. \tag{38}$$

We define $X_t = |Y_t|$, $t \geq 0$, and, using the definition of local time at zero, compute

$$X_t = |Y_t| = |x_0| + \int_0^t \operatorname{sgn} Y_s dY_s + L_t^0(Y) \overset{(38)}{=} x_0 + \int_0^t \operatorname{sgn} Y_s \mathbb{I}_{\{Y_s \neq 0\}} dB_s + \frac{1}{2} L_t^0(|Y|)$$
$$= x_0 + \int_0^t \mathbb{I}_{\{X_s > 0\}} d\hat{B}_s + \frac{1}{2} L_t^0(X),$$

where $\hat{B}_t = \int_0^t \operatorname{sgn} Y_s dB_s$, $t \geq 0$, is a Brownian motion, by Levy's characterisation theorem.

We also compute

$$\int_0^t \mathbb{I}_{\{X_s = 0\}} ds = \int_0^t \mathbb{I}_{\{Y_s = 0\}} ds = \int_0^t \mathbb{I}_{\{\tilde{W}_{T_s} = 0\}} d\underbrace{A_{T_s}}_{=s} = \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s = 0\}} dA_s$$
$$= \underbrace{\int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s = 0\}} ds}_{=0} + \frac{1}{\lambda} \int_0^{T_t} \mathbb{I}_{\{\tilde{W}_s = 0\}} dL_s^0(\tilde{W}) \overset{(28)}{=} \frac{1}{\lambda} L_{T_t}^0(\tilde{W})$$

By Definition 7.1,

$$L_{T_t}^0(\tilde{W}) = |\tilde{W}_{T_t}| - |x_0| - \int_0^{T_t} \operatorname{sgn} \tilde{W}_s d\tilde{W}_s = |Y_t| - |x_0| - \int_0^t \operatorname{sgn} \tilde{W}_{T_s} d\tilde{W}_{T_s}$$
$$= |Y_t| - |x_0| - \int_0^t \operatorname{sgn} Y_s dY_s = L_t^0(Y), \quad t \geq 0.$$

This implies that

$$\int_0^t \mathbb{I}_{\{X_s=0\}} ds = \frac{1}{\lambda} L_t^0(Y) = \frac{1}{2\lambda} L_t^0(|Y|) = \frac{1}{2\lambda} L_t^0(X).$$

Consequently, $(X, \hat{B})$ satisfies (35). Hence $(X, \hat{B})$ is a weak solution to SDE (34) on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. □

Exercise 8.1. Let Y_t, $t \geq 0$, be the semimartingale defined by (36). Show that $L_t^0(|Y|) = 2L_t^0(Y)$.

Hint: Use the definition of local time and the fact that

$$\int_0^t \mathbb{I}_{\{\tilde{W}_s=0\}} d\tilde{W}_s = 0.$$

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