# Problem sheet 6

*Solutions has to be uploaded into Moodle:*
*https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=61456*
*until 20:00, June 29.*

1. **[3 points]** Show that the SDE

$$dX_t = (|X_t|^\alpha \wedge 1)dB_t, \quad X_0 = x_0,$$

has a pathwise unique strong solution for every $\alpha \geq \frac{1}{2}$.

2. **[3+2+3 points]**

(a) Let $W_t$, $t \geq 1$, be a $n$-dimensional Brownian motion. Define $X_t = \|W_t\|^2$, $t \geq 0$. Show that the process $X$ is a weak solution to the SDE

$$dX_t = ndt + 2\sqrt{X_t}dB_t, \quad X_0 = 0. \tag{1}$$

(b) Show that SDE (1) has a pathwise unique strong solution for every $n \in \mathbb{N}$.

(c) Let $Y_t$ be a solution[1] to

$$dY_t = \delta dt + 2\sqrt{Y_t}dB_t, \quad Y_0 = y_0$$

for some $y_0 > 0$ and $\delta \geq 2$. Prove that

$$\mathbb{P}\{\exists t \geq 0 : \ Y_t = 0\} = 0.$$

*Hint:* Use the fact that a two-dimensional Brownian motion never heat a fixed point, i.e

$$\mathbb{P}\{\exists t \geq 0 : \ B_t = a\} = 0$$

for every $a \in \mathbb{R}^2 \setminus \{0\}$.

3. **[5 points]** Let $L_t^{X,0}$ denote the (semimartingale) local time of $X$ at 0, where $X$ is a continuous semimartingale in $\mathbb{R}$. Let also $B_t$, $t \geq 0$, be an one dimensional Brownian motion. Show that

(a) $|B_t|$, $t \geq 0$, is a continuous semimartingale;

(b) $L^{|B|,0} = 2L^{B,0}$.

*Hint:* Use the fact that $\int_0^\infty \mathbb{I}_{\{X_t \neq 0\}} dL_t^{X,0} = 0$ a.s., which we will discuss next time.

[1] This equation has a pathwise unique strong solution for every $\delta \geq 1$. Its solution is called $\delta$**-dimensional squared Bessel process**