Problem sheet 5

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=59752
until 20:00, June 15.

1. **[4 points]** A solution to the following one-dimensional SDE

$$dX_t = -\theta X_t dt + \sigma dB_t, \quad X_0 = x_0$$

is called the **Ornstein-Uhlenbeck process**, where $\theta > 0$, $\sigma > 0$ are constants and B is an one-dimensional Brownian motion. Show that the Ornstein-Uhlenbeck process X is a strong Markov process whose transition distributions $P(t, x_0, \cdot)$ coincides with normal distribution with expectation $x_0 e^{-\theta t}$ and variance $\frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$.

Hint: Show that the transition density $p(t, x, y)$, $y \in \mathbb{R}$, satisfies the corresponding Kolmogorov forward PDE in t, x.

2. **[4+3 points]** *(Disintegration of probability measures)*

(a) Let P be a probability measure on the product $E \times S$ of complete separable metric spaces endowed with the Borel σ-algebra $\mathcal{B}(E \times S)$. Let μ be the marginal distribution of P on the first coordinate, that is,

$$\mu(A) = P(A \times S), \quad A \in \mathcal{B}(E).$$

Show that there exists a family $Q(x; A)$, $x \in E$, $A \in \mathcal{B}(S)$, such that

i. for every $A \in \mathcal{B}(S)$ the map $x \mapsto Q(x; A)$ is measurable;
ii. for every $x \in E$ $Q(x; \cdot)$ is a measure on S;
iii. for every $F \in \mathcal{B}(E)$ and $A \in \mathcal{B}(S)$

$$P(F \times A) = \int_F Q(x; A)\mu(dx).$$

Hint: Use the theorem about the existence of regular conditional probability

(b) Let E, S be complete separable metric spaces and ξ, η be random variables taking values in E, S, respectively. Let also P be the distribution of (ξ, η). Construct the family Q from (a) in the following cases:

(i) ξ, η are independent;
(ii) $\eta = f(\xi)$, where $f : E \to S$ is a Borel measurable map;
(iii) $\eta = f(\xi, \xi_1)$, where ξ_1 is an E-valued random variable independent of ξ and $f : E^2 \to S$ is a Borel measurable map.

3. **[4 points]** *(Coupling of random vectors)* Let E and S be complete separable metric spaces. Consider $E \times S$-valued random vectors (ξ_1, η_1) and (ξ_2, η_2), defined probably on different probability spaces, such that $\operatorname{Law} \eta_1 = \operatorname{Law} \eta_2$. Construct a coupling (ζ_1, ζ_2, η) (on a probability space) such that $\operatorname{Law}(\zeta_1, \eta) = \operatorname{Law}(\xi_1, \eta_1)$ and $\operatorname{Law}(\zeta_2, \eta) = \operatorname{Law}(\xi_2, \eta_2)$.

Hint: Construct the coupling (ζ_1, ζ_2, η) on the canonical probability space using the disintegration approach from Exercise 2.