Problem sheet 4

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=58192
until 20:00, June 1.

1. (a) **[3+2 points]** Let the coefficients b, σ be bounded on $\mathbb{R}^n$ and assume that the martingale problem associated with a second order differential operator $\mathcal{A}$ is well posed. Suppose that there exists a bounded function $f : \mathbb{R}^n \to [0, \infty)$ from $\mathcal{C}^2(\mathbb{R}^n)$ such that

$$\mathcal{A}f(x) + \lambda f(x) \leq c, \quad x \in \mathbb{R}^n,$$

holds for some $\lambda > 0$ and $c \geq 0$. Show that for any solution X_t, $t \geq 0$, to the corresponding martingale problem started at $x_0 \in \mathbb{R}^n$ one has

$$\mathbb{E}\, f(X_t) \leq f(x_0)e^{-\lambda t} + \frac{c}{\lambda}(1 - e^{-\lambda t}), \quad t \geq 0.$$

(b) Get the same estimate in the case b, σ are locally bounded and f is any function (not necessarily bounded) from $\mathcal{C}^2(\mathbb{R}^n)$.

2. **[2+2 points]**

(a) Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], \mathcal{B}([0, 1]), \lambda)$, where λ denotes the Lebesgue measure on $[0, 1]$. Find the regular conditional probability for $\mathcal{F}$ given $\mathcal{S} = \sigma\left(\left\{\left[0, \frac{1}{3}\right), \left[\frac{1}{3}, 1\right]\right\}\right)$.

(b) Let B_t, $t \geq 0$, be an one-dimensional Brownian motion and ξ a random variable on $\mathbb{R}$ independent of B. We define the Brownian motion

$$X_t := \xi + B_t, \quad t \geq 0,$$

started at ξ. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) = \left(\mathcal{C}([0, \infty), \mathbb{R}), \mathcal{B}(\mathcal{C}([0, \infty), \mathbb{R})), \mathbb{P}^X\right)$, where $\mathbb{P}^X$ denotes the distribution of X. Find the regular conditional probability of $\tilde{\mathcal{F}}$ given ξ.

Hint: Use the analog of Fubini's theorem for conditional expectations: for two independent random variables η and ξ taking values in complete separable metric spaces E and S, respectively, and any bounded measurable function $f : E \times S \to \mathbb{R}$ one has

$$\mathbb{E}\left(f(\eta, \xi)|\xi\right) = \mathbb{E}\left(f(\eta, y)\right)\big|_{y=\xi},$$

.

3. **[2+3 points]** Let B_t, $t \geq 0$, be a one-dimensional Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P}) = \left(\mathcal{C}([0, 1], \mathbb{R}), \mathcal{B}(\mathcal{C}([0, 1], \mathbb{R})), \mathbb{P}^B\right)$, where $\mathbb{P}^B$ denotes the distribution of B.

(a) Show that for every $y \in \mathbb{R}$, the process

$$B_t^y := B_t - t(B_1 - y), \quad t \in [0, 1],$$

is independent of B_1.

Hint: Use the fact that two Gaussian vectors are independent if and only if their covariation equals zero

(b) Let $Q(y, \cdot)$ denotes the law of B^y on $\mathcal{C}([0,1], \mathbb{R})$, i.e.

$$Q(y, A) := \mathbb{P}\left\{B^y \in A\right\}, \quad A \in \mathcal{B}(\mathcal{C}([0,1], \mathbb{R})).$$

Show that Q is a regular conditional probability for $\mathcal{F}$ given B_1. In particular, this result implies that

$$\mathbb{P}\left\{(B_t)_{t\in[0,1]} \in A | B_1 = y\right\} = \mathbb{P}\left\{(B^y_t)_{t\in[0,1]} \in A\right\}, \quad A \in \mathcal{B}(\mathcal{C}([0,1], \mathbb{R})).$$

4. **[3 bonus points]** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and τ be a stopping time. Show that

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : \ A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \text{ for all } \ t \geq 0\}$$

is a σ-algebra.