Problem sheet 3

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=56488
until 20:00, May 18.

1. **[2+2+3+3 points]** *(Continuous Branching Process)* Consider a family of diffusions $X_t(x)$, $t \geq 0$, $x > 0$ satisfying the SDE[1]

$$dX_t(x) = \alpha X_t(x)dt + \sqrt{\beta X_t(x)}dB_t, \quad X_0(x) = x,$$

where $\alpha \in \mathbb{R}$ and $\beta > 0$. Let $(\tilde{X}, \tilde{B})$ be an independent copy of (X, B) and let $Y_t(x, y) = X_t(x) + \tilde{X}_t(y)$ for $t \geq 0$, $x, y > 0$.

   (a) *(Branching)* Compute the SDE satisfied by Y and prove that $Y_t(x, y)$, $t \geq 0$, has the same law of $X_t(x + y)$, $t \geq 0$.

   Hint: Use martingale caracterization of weak solutions.

   (b) *(Duality)* Show that this implies that there exists a function $u : [0, \infty) \times (0, \infty) \to [0, \infty)$ such that

$$\mathbb{E}\, e^{-\lambda X_t(x)} = e^{-xu(t,\lambda)}, \quad x > 0, \quad \lambda > 0, \tag{1}$$

   if we assume that the map $x \mapsto \mathbb{E}\, e^{-\lambda X_t(x)}$ is continuous.

   (c) Assume that $u : [0, \infty) \times (0, \infty) \to [0, \infty)$ is differentiable with respect to its first parameter. Apply Ito formula to $s \mapsto G_s := e^{-u(t-s,\lambda)X_s(x)}$ and determine which differential equation u should satisfy in order for G to be a local martingale. Prove that in this case equality (1) is satisfied (in particular, if a solution of the equation exists then it is unique).

   (d) *(Extinction probability)* Find the explicit solution u for the differential equation and using equality (1) prove that if $\alpha = 0$ then

$$\mathbb{P}\left\{X_t(x) = 0\right\} = e^{-\frac{2x}{\beta t}}, \quad x, t > 0.$$

2. **[2 points]** Let $\{\xi_k,\ k \geq 1\}$ be a family of random variables in $\mathbb{R}^d$ such that $\sup_{k \in \mathbb{N}} \mathbb{E}\, \|\xi_k\| < \infty$. Show that this family is tight.

3. **[3 points]** Let $\{(X_n^1, \ldots, X_n^d),\ n \geq 1\}$ be a family of random vectors in $\mathbb{R}^d$. Show that it is tight if and only if for each $k \in [d]$ the family $\{X_n^k,\ n \geq 1\}$ is tight in $\mathbb{R}$.

[1] This equation has a unique (pathwise and in law) strong solution