Problem sheet 2

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=54153
until 20:00, May 4.

1. **[2+4+1 points]** Let B_t, $t \geq 0$, be a one dimensional Brownian motion.

   (a) Define the process
   $$X_t = a(t)\left(x_0 + \int_0^t b(s)dB_s\right),$$
   where $a, b : [0, \infty) \to \mathbb{R}$ are differentiable functions with $a(0) = 1$ and $a(t) > 0$. Compute the SDE satisfied by the process.

   (b) Use (a) to find an explicit solution for the following SDEs:
   $$\begin{aligned} dX_t &= -\alpha X_t dt + \sigma dB_t, \quad t \in [0, T], \\ X_0 &= x_0, \end{aligned}$$
   where a, σ, T are positive constants;
   $$\begin{aligned} dX_t &= -\frac{X_t}{1-t}dt + dB_t, \quad t \in [0, 1), \\ X_0 &= 0. \end{aligned}$$

   (c) Are the solutions to the SDEs above strong and pathwise unique?

2. **[2 points]** Let X_t, $t \geq 0$, be a continuous process on $\mathbb{R}$ and $X_0 = 0$. Assume that for every $f \in \mathcal{C}^2(\mathbb{R})$
   $$M_t^f := f(X_t) - f(0) - \int_0^t \frac{1}{2} f''(X_s)ds, \quad t \geq 0,$$
   is a continuous local martingale. Show that X_t, $t \geq 0$, is a Brownian motion.

3. **[4 points] (Feynman-Kac formula for Ito diffusions)**
   Consider the solution (X, B) to the SDE in $\mathbb{R}^n$
   $$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x_0,$$
   where B is an m-dimensional Brownian motion and $b : [0, \infty) \times \mathbb{R}^n$, $\sigma : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ bounded continuous functions. Let $\mathcal{A}_t$ be the associated second order differential operator define in Section 2.3 of the skript. Fix $t > 0$ and assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $V : [0, t] \times \mathbb{R}^n \to [0, \infty)$ are continuous functions. Show that any bounded solution $u \in \mathcal{C}^{1,2}([0, t] \times \mathbb{R}^n)$ to the PDE
   $$\begin{aligned} \frac{\partial}{\partial s} u(s, x) &= \mathcal{A}_s u(s, x) - V(s, x)u(s, x), \quad (s, x) \in (0, t] \times \mathbb{R}^n, \\ u(0, x) &= \varphi(x), \quad x \in \mathbb{R}^n, \end{aligned}$$
   has the stochastic representation
   $$u(t, x) = \mathbb{E}\left[\varphi(X_t) \exp\left(-\int_0^t V(t - s, X_s)ds\right)\right].$$
   In particular, there is at most only one solution of the PDE.
   Hint: Show that $M_r = \exp\left(-\int_0^r V(t - s, X_s)ds\right) u(t - r, X_r)$, $r \in [0, t]$, is a martingale