Problem sheet 1

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=51047
until 20:00, April 20.

1. Let B_t, $t \geq 0$, be a one dimensional Brownian motion.
   (a) Show that it is a continuous martingale with quadratic variation $\langle B \rangle_t = t$.
   (b) Show that $\int_0^t \mathbb{I}_{\{B_s=0\}} ds = 0$ a.s.

HW1 **[2 points]** Let M_t, $t \geq 0$, be a continuous local martingale and $S \leq T$ be two stopping times. Prove that $\langle M \rangle_T = \langle M \rangle_S < \infty$ a.s. implies $M_t = M_S$ for all $t \in [S, T]$ a.s.

Hint: consider the continuous local martingale $N_t = \int_0^t \mathbb{I}_{(S,T]}(s) dM_s$

HW2 **[2 points]** Let M_t be a non-negative continuous $(\mathcal{F}_t)$-local martingale. Define

$$\tau := \inf \{t \geq 0 : \ M_t = 0\}.$$

Show that $\mathbb{P}\{M_t = 0, \ t \geq \tau\} = 1$.

2. Let B_t, $t \geq 0$, be a one dimensional Brownian motion. Find the SDEs satisfied by the following processes

HW3 **[1 points]** $X_t = \frac{B_t}{1+t}$, $t \geq 0$;

HW4 **[3 points]** $X_t = x_0 e^{-at} + \sigma \int_0^t e^{-a(t-s)} dB_s$, $t \geq 0$, where $x_0 \in \mathbb{R}$, $a > 0$ and $\sigma > 0$ are fixed constants.

   (a) $(X_t, Y_t) = (a \cos B_t, b \sin B_t)$, $t \geq 0$, where $a, b \in \mathbb{R}$ with $ab \neq 0$.

3. Consider the following SDE

$$dX_t = b(t, X_t)dt + \sigma X_t dB_t, \quad X_0 = x_0,$$

where b is a continuous deterministic function and $\sigma > 0$.

   (a) Find an explicit solution Z in the case $b = 0$ and $x_0 = 1$.

HW5 **[3 points]** Use the Ansatz $X_t = C_t Z_t$ to show that X solves the SDE provided C solves an ODE with random coefficients.

HW6 **[3 points]** Apply this method to solve the SDE

$$dX_t = \frac{1}{X_t} dt + \sigma X_t dB_t, \quad X_0 = x,$$

where $\sigma > 0$ is a constant.

4. **[4 bonus points]** Show that the equation

$$dX_t = \mathbb{I}_{\{X_t \neq 0\}} dB_t, \quad X_0 = 0,$$

has a strong solution and a weak solution (which is not a strong solution). Show that the uniqueness in law and the pathwise uniqueness fail.