Problem sheet 9

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1087868
until 22:00, June 17.

1. **[1 points]** If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

2. **[3 points]** Let $T : \mathcal{D}(T) \to Y$ be a linear operator. Show that
$$\|T\| := \sup_{x \in \mathcal{D}(T),\ x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in \mathcal{D}(T),\ \|x\|=1} \|Tx\|.$$

3. **[3 points]** Show that the operator $T : l^\infty \to l^\infty$ defined by $Tx = (\eta_k)_{k \geq 1}$, $\eta_k = \frac{\xi_k}{k}$, $k \geq 1$, $x = (\xi_k)_{k \geq 1}$, is linear and bounded. Compute its norm.

4. **[3 points]** Compute the norm of the linear operator $T : C[0,1] \to C[0,1]$
$$(Tx)(t) = \int_0^t s x(s) ds, \quad t \in [0,1].$$

5. **[4 points]** Using the definition, compute the norm of the following functional f on c_0:
$$f(x) = \sum_{k=1}^\infty \frac{\xi_k}{3^k}, \quad x = (\xi_k)_{k \geq 1} \in c_0.$$

6. **[3+3 points]** Compute norms of the following functionals:
   a) $f(x) = \int_0^1 t^3 x(t) dt$ on $L^4[0,1]$; b) $f(x) = \sum_{k=1}^\infty \frac{\xi_k}{\sqrt{k!}}$ on l^2.

7. **[4 bonus points]** Find the norm of the functional defined on $C[-1,1]$ by
$$f(x) = \int_{-1}^0 x(t) dt - 2 \int_0^1 x(t) dt.$$

8. **[5 bonus points]** Show that a linear functional is continuous on a normed space if and only if its kernel is closed.