Problem sheet 8

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1078688
until 22:00, June 10.

Let X denote a normed space with norm $\|\cdot\|$.

1. **[2 points]** Prove that $\|x\|_p = (\sum_{k=1}^n |\xi_k|^p)^{\frac{1}{p}}$, $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, is not a norm in $\mathbb{R}^n$ for $0 < p < 1$ and $n \geq 2$.

2. **[2 points]** Show that a closed ball

$$B_r(x_0) = \{x \in X : \ \|x - x_0\| \leq r\}$$

in X is convex[1] for any $x_0 \in X$ and $r > 0$.

3. **[3 points]** Show that the convergences $x_n \to x$, $y_n \to y$ in X and $\alpha_n \to \alpha$ in the field K imply that $x_n + y_n \to x + y$ and $\alpha_n x_n \to \alpha x$ in X.

4. **[3 points]** Show that the closure $\bar{Y}$ of a subspace Y of X is again a vector subspace.

5. **[5 bonus points]** Show that X must be complete, if absolute convergence of any series always implies convergence of that series in X.

6. **[3 points]** Show that in a Banach space, an absolutely convergent series is convergent.

7. **[3 points]** Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Show that the product vector space $X = Y \times Z$ becomes a normed space if we define

$$\|x\| = \max\{\|y\|_Y, \|z\|_Z\}, \quad x = (y, z) \in X.$$

Check also that a sequence $x_n = (y_n, z_n)$, $n \geq 1$, converges to $x = (y, z)$ in X if and only if $y_n \to y$ in Y and $z_n \to z$ in Z.

8. Let X be a Banach space and B_n be a family of closed balls in X such that $B_{n+1} \subset B_n$, $n \geq 1$. Show that

   (a) **[5 points]** there exists $x \in X$ such that $\bigcap_{n=1}^\infty B_n = \{x\}$, if radii r_n of the balls B_n converges to zero;

   (b) **[4 bonus points]** $\bigcap_{n=1}^\infty B_n \neq \emptyset$ without the assumption that $r_n \to 0$.

[1] A subset A of a vector space V is said to be *convex* if for every $x, y \in A$ it implies that

$$\alpha x + (1 - \alpha) y \in A$$

for all $\alpha \in [0, 1]$.