Problem sheet 7

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1071497
until 22:00, June 3.

Let (X, d) denote a metric space.

1. **[2+2 points]** Justify the terms "open ball" and "closed ball" by proving that

   a) any open ball is an open set; b) any closed ball is a closed set.

2. **[2+2 points]** Check if the following sets are open in $C[0, 2]$.

   a) $A = \{x \in C[0, 2] : \ x(0) < 0, \ x(1) > 0\}$; b) $B = \left\{x \in C[0, 2] : \ \int_0^2 |x(t)|dt < 1\right\}$.

3. **[3 points]** Prove that the space l_n^p is separable for every $p \geq 1$.

4. **[3 points]** Using the definition, show that the map $T : l^\infty \to l_2^p$ defined by the equality

   $$Tx = (\xi_1, \xi_3), \quad x = (\xi_k)_{k=1}^\infty \in l^\infty$$

   is continuous for every $p \geq 1$.

5. **[3 points]** If $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X and has a convergent subsequence, say, $x_{n_k} \to x$. Show that $\lim_{n\to\infty} x_n = x$.

6. **[5 points]** Consider the metric space c_0 consisting of all sequences $x = (\xi_k)_{k=1}^\infty$ which converge to 0. A metric on c_0 is defined as $d(x, y) = \max\limits_{k\geq 1} |\xi_k - \eta_k|$, $x = (\xi_k)_{k=1}^\infty$, $y = (\eta_k)_{k=1}^\infty \in c_0$. Prove that c_0 is complete.

7. **[1 points]** Show that the set of all real numbers $\mathbb{R}$ with the metric $d(x, y) = |\arctan x - \arctan y|$, $x, y \in \mathbb{R}$, is not a complete metric space.

8. **[4 bonus points]** We define a map $T : c \to \mathbb{R}$ as follows $Tx = \lim_{k\to\infty} \xi_k$, $x = (\xi_k)_{k=1}^\infty \in c$. Is the map T continuous? Justify your answer.

9. **[5 bonus points]** Consider the metric space $C^1[0, 1]$ of all continuously differentiable functions on $[0, 1]$.[1] Define the metric on $C^1[0, 1]$ as follows

   $$d(x, y) = \max_{t\in[0,1]} |x(t) - y(t)| + \max_{t\in[0,1]} |x'(t) - y'(t)|, \quad x, y \in C^1[0, 1].$$

   Show that $C^1[0, 1]$ is a complete metric space.

[1] Remark that the derivative of a function x can be defined only at inner points of the interval $[0, 1]$. So, we cannot define the derivative at points 0 and 1. Hence, one needs to assume that a function x is continuously differentiable on $[0, 1]$ if x is the restriction of a continuously differentiable function on $\mathbb{R}$.