UNIVERSITÄT LEIPZIG

Problem sheet 5

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1049936
until 22:00, May 20.

Let $(X, \mathcal{F})$ denote a measurable space and λ be a measure on $\mathcal{F}$. All functions considered here are $\mathcal{F}$-measurable.

1. **[3 points]** Let $X = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and λ be the Lebesgue measure on $\mathbb{R}$. Let also $f \in L(\mathbb{R}, \lambda)$. Show that the function $\varphi(x) := \int_{(-\infty, x]} f(t)\lambda(dt)$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

2. **[3 points]** Let $f \in L(X, \lambda)$ and $\int_A f d\lambda = 0$ for all $A \in \mathcal{F}$. Show that $f = 0$ λ-a.e.

3. **[2 points]** Let $f_n \to f$ λ-a.e. and $f_n \to g$ λ-a.e. Show that $f = g$ λ-a.e.

4. **[4 points]** Assume that $f : X \to \mathbb{R}$ satisfies the following property: for every $a > 0$

$$\lambda\left(\{x \in X : \ |f(x)| \geq a\}\right) = 0.$$

   Show that $f = 0$ λ-a.e.

   Hint: Prove and use the equality $\{x \in X : \ f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \left\{x \in X : \ |f(x)| \geq \frac{1}{n}\right\}$.

5. **[3+3 points]** Let $f_n \xrightarrow{\lambda} f$.

   (a) Show that $|\lambda_n| \xrightarrow{\lambda} |\lambda|$.

   (b) Let additionally $\lambda(X) < +\infty$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $g(f_n) \xrightarrow{\lambda} g(f)$.

6. **[3+3 points]** Let f_n, $n \geq 1$, be non-negative functions.

   (a) Show that for every $\varepsilon > 0$

$$\varepsilon \lambda\left(\{x \in X : \ f_n(x) \geq \varepsilon\}\right) \leq \int_X f_n d\lambda.$$

   (b) Check that the convergence $\int_X f_n d\lambda \to 0$ implies $f_n \xrightarrow{\lambda} 0$.

7. **[5+2 bonus points]** Let $\lambda(X) < +\infty$.

   (a) Prove that $f_n \to f$ λ-a.e. if and only if

$$\forall \varepsilon > 0 \ \ \lambda\left(\bigcup_{k=n}^{\infty} \{x \in X : \ |f_k(x) - f(x)| \geq \varepsilon\}\right) \to 0, \quad n \to \infty.$$

   Hint: Consider the set $\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \left\{x \in X : \ |f_j(x) - f(x)| \geq \frac{1}{k}\right\}$.

   (b) Show that the convergence of the series $\sum_{n=1}^{\infty} \lambda\left(\{x \in X : \ |f_n(x) - f(x)| \geq \varepsilon\}\right)$ for all $\varepsilon > 0$ implies that $f_n \to f$ λ-a.e.