Problem sheet 12

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1111128
until 22:00, July 8.

Here all Hilbert spaces are considered over the scalar field $\mathbb{C}$.

Let E_λ, $\lambda \in \mathbb{R}$, be a spectral family on an interval $[m, M]$ associated with self-adjoint linear operator T on a Hilbert space H. We recall that T admits the spectral representation

$$T = \int_{m-0}^{M} \lambda dE_\lambda. \qquad (1)$$

For a continuous function $f : [m, M] \to \mathbb{R}$ one can define the operator (which is bounded and self-adjoint) $f(T)$ by the equality

$$f(T) := \int_{m-0}^{M} f(\lambda) dE_\lambda.$$

1. **[3 points]** Show that a bounded self-adjoint linear operator is positive if and only if its spectrum consists of non-negative real values only.

2. **[6 points]** Let $Y \neq \{0\}$ be a closed subspace of a Hilbert space H which does not coincide with H, and P be the orthogonal projection of H onto Y. Show that for every $\lambda \notin \{0, 1\}$,

$$(P - \lambda I)^{-1} = -\frac{1}{\lambda} I + \frac{1}{\lambda(1-\lambda)} P.$$

   Find the spectrum of P, spectral family associated with P and writhe the spectral representation (1) for P.

3. **[6 points]** Let an operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ be represented, with respect to a canonical basis, by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Find the corresponding spectral family. Write the spectral representation (1) for T.

4. **[6 points]** Find T_λ^+, $\lambda \in \mathbb{R}$, and the spectral family E_λ, $\lambda \in \mathbb{R}$, associated with operator $T : l^2 \to l^2$ defined by $Tx = \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right)$, $x = (\xi_k)_{k \geq 1}$. Write the spectral representation (1) for T.

5. **[6 bonus points]** For the multiplication operator $T : L^2[0, 1] \to L^2[0, 1]$ defined by

$$(Tx)(t) = tx(t), \quad t \in [0, 1],$$

   compute $\sin T$.