Problem sheet 11

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1104170
until 22:00, July 1.

Here all Hilbert spaces are considered over the scalar field $\mathbb{C}$.

1. **[3 points]** Let T be a bounded linear operator on a Hilbert space H. Show that $(\operatorname{Im} T)^{\perp} = \ker T^*$, where $\operatorname{Im} T = \{Tx,\ x \in H\}$.

2. **[3 points]** Let $k(t,s)$, $t,s \in [0,1]$, be a continuous function and let the operator T act on $L^2[0,1]$ by the formula
$$(Tx)(t) = \int_0^1 k(t,s)x(s)ds, \quad t \in [0,1].$$
Find the adjoint operator T^*.

3. **[3 points]** Let S, T be bounded linear operators on a normed space X. Show that for every $\lambda \in \rho(S) \cap \rho(T)$ one has
$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S - T)R_\lambda(S).$$

4. **[3 points]** Let T be a linear operator on l^2 defined by $Tx = (\xi_2, \xi_1, \xi_3, \xi_4, \xi_5, \dots)$ (permutation of first two components). Find and classify the spectrum of T.

5. **[4 points]** Let $X = C[0,\pi]$ and define $T : \mathcal{D}(T) \to X$ by $Tx = x''$, where
$$\mathcal{D}(T) = \left\{x \in X :\ x', x'' \in X,\ x(0) = x(\pi) = 0\right\}.$$
Show that $\sigma(T)$ is not compact.

6. **[5 points]** Let
$$a(t) = \begin{cases} t & \text{if } t \in [0,1], \\ 1 & \text{if } t \in (1,2]. \end{cases}$$
Find and classify the spectrum of the operator $(Tx)(t) = a(t)x(t)$ acting on $C[0,2]$.

7. **[6 bonus points]** Let T be the left-shift operator on l^2 defined as follows
$$Tx = (\xi_2, \xi_3, \dots), \quad x = (\xi_k)_{k \ge 1} \in l^2.$$
Find the spectrum of T.