



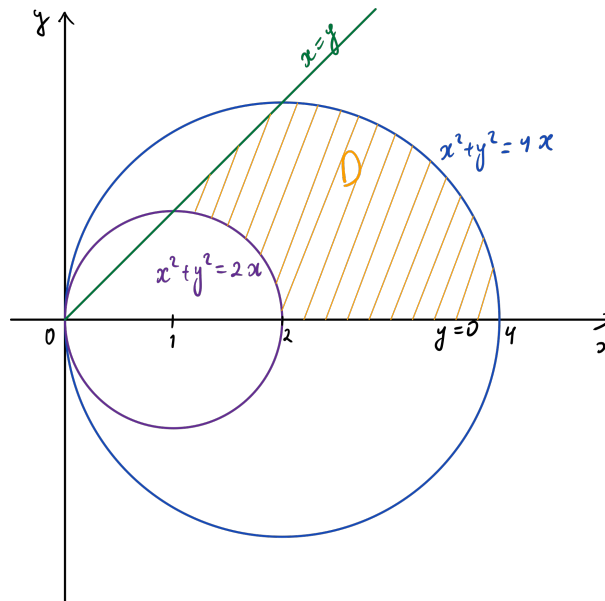
Exam Solutions

Each exercise is graded between 0 and 5 points.

1. Passing to polar coordinates, find the area bounded by the lines

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = x, \quad y = 0.$$

Solution. We first remark that $x^2 + y^2 = 2x$ is a circle with centers $(1, 0)$ and radius 1, since the equation can be rewritten as $(x - 1)^2 + y^2 = 1$. Similarly, $x^2 + y^2 = 4x$ is a circle with center $(2, 0)$ and radius 2.



Passing to the polar coordinate

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad 0 \leq \varphi \leq 2\pi, \quad \rho \geq 0,$$

we can write the equation for the circle $x^2 + y^2 = 2x$ in the new coordinates as follows:

$$\begin{aligned} \rho^2 \cos \varphi + \rho^2 \sin \varphi &= 2\rho \cos \varphi, \\ \rho^2 &= 2\rho \cos \varphi, \\ \rho &= 2 \cos \varphi, \quad 0 \leq \varphi \leq 2\pi. \end{aligned}$$

Similarly, for the second circle we have

$$\rho = 4 \cos \varphi, \quad 0 \leq \varphi \leq 2\pi.$$



Hence, the area $S(D)$ of the region D between the given lines equals

$$\begin{aligned} \text{Area}(D) &= \iint_D dx dy = \int_0^{\frac{\pi}{4}} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} \rho d\rho = \int_0^{\frac{\pi}{4}} \left(\frac{\rho^2}{2} \Big|_{\rho=2\cos\varphi}^{\rho=4\cos\varphi} \right) d\varphi \\ &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (16\cos^2\varphi - 4\cos^2\varphi) d\varphi = 6 \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2\varphi}{2} d\varphi \\ &= 3 \int_0^{\frac{\pi}{4}} (1 + \cos 2\varphi) d\varphi = 3 \left(\varphi + \frac{\sin 2\varphi}{2} \right) \Big|_0^{\frac{\pi}{4}} = 3 \left(\frac{\pi}{4} + \frac{1}{2} \right) = \frac{3}{4}(\pi + 2). \end{aligned}$$

2. Evaluate the line integral

$$\int_{\gamma} xy ds,$$

where γ is a quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

Solution. First we write a parametrisation for the quarter of the ellipse

$$\begin{aligned} x &= a \cos t, \\ y &= b \sin t, \quad 0 \leq t \leq \frac{\pi}{2}. \end{aligned}$$

Then

$$\begin{aligned} \int_{\gamma} xy ds &= \int_0^{\frac{\pi}{2}} ab \cos t \sin t \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt \\ &= ab \int_0^{\frac{\pi}{2}} \sin t \sqrt{a^2 \sin^2 t + b^2(1 - \sin^2 t)} d \sin t \stackrel{\sin t = u}{=} ab \int_0^1 u \sqrt{a^2 u^2 + b^2(1 - u^2)} du \\ &= \frac{ab}{2} \int_0^1 \sqrt{b^2 + (a^2 - b^2)u^2} du \stackrel{u^2 = v}{=} \frac{ab}{2} \int_0^1 \sqrt{b^2 + (a^2 - b^2)v} dv = \\ &= \frac{ab}{2(a^2 - b^2)} \int_0^1 (b^2 + (a^2 - b^2)v)^{\frac{1}{2}} d(b^2 + (a^2 - b^2)v) \\ &= \frac{ab}{2(a^2 - b^2)} \frac{2(b^2 + (a^2 - b^2)v)^{\frac{3}{2}}}{3} \Big|_0^1 = \frac{ab(a^3 - b^3)}{3(a^2 - b^2)}, \end{aligned}$$

if $a \neq b$. If $a = b$, then

$$\int_{\gamma} xy ds = \dots = \frac{ab}{2} \int_0^1 \sqrt{b^2 + (a^2 - b^2)v} dv = \frac{a^2}{2} \int_0^1 \sqrt{a^2} dv = \frac{a^3}{2}$$

3. Evaluate the following surface integral

$$\iint_S x^3 dy dz + y^3 dz dx + z^3 dx dy,$$

where S is the sphere $x^2 + y^2 + z^2 = a^2$ ($a > 0$) oriented inward.

Solution. Let V denote the ball with center at $(0, 0, 0)$ and radius a . Using the Gauss-Ostrogradskii formula, we obtain

$$\iint_S x^3 dy dz + y^3 dz dx + z^3 dx dy = - \iiint_V \text{div}(x^3, y^3, z^3) dx dy dz = -3 \iiint_V (x^2 + y^2 + z^2) dx dy dz.$$



Passing to the spherical coordinates

$$x = \rho \cos \varphi \cos \psi, \quad y = \rho \sin \varphi \cos \psi, \quad z = \rho \sin \psi,$$

$$J = \rho^2 \cos \psi,$$

we have

$$\begin{aligned} -3 \iiint_V (x^2 + y^2 + z^2) dx dy dz &= -3 \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^a \rho^2 \rho^2 \cos \psi d\rho \\ &= -3 \cdot 2\pi \cdot \sin \psi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \frac{\rho^5}{5} \Big|_0^a = \frac{12\pi a^5}{5}. \end{aligned}$$

4. Applying Stokes' formula, find the integral

$$\int_{\gamma} (y - z)dx + (z - x)dy + (x - y)dz,$$

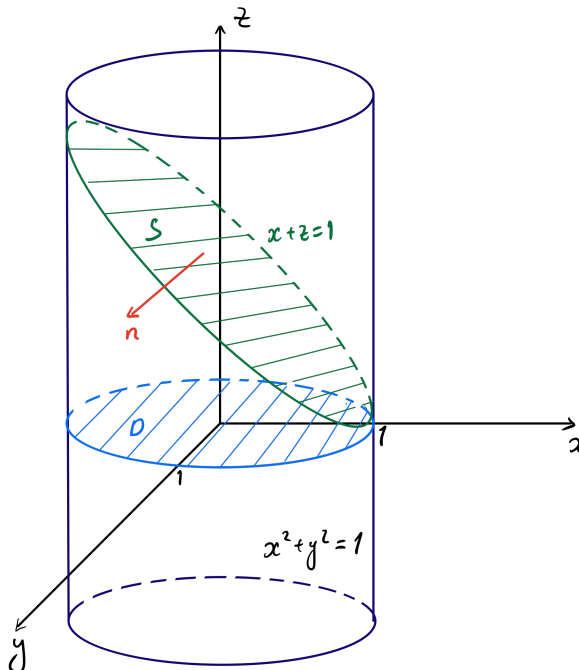
where γ is the ellipse $x^2 + y^2 = 1$, $x + z = 1$, oriented counter clockwise viewed from the point $(0, 0, 0)$.

Solution. By Stokes' formula

$$\int_{\gamma} F \cdot ds = \iint_S \operatorname{curl} F \cdot dS = \iint_S (\operatorname{curl} F \cdot n) dS,$$

where $F = (y - z, z - x, x - y)$, $n = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ and

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} = -(2, 2, 2).$$





So, using the parametrization $z = 1 - x$, $(x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1\}$, for S , we obtain

$$\begin{aligned} \int_{\gamma} (y - z)dx + (z - x)dy + (x - y)dz &= - \iint_S (2, 2, 2) \cdot \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) dS \\ &= 2\sqrt{2} \iint_S dS = 2\sqrt{2} \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2} dx dy \\ &= 2\sqrt{2} \iint_D \sqrt{1 + (-1)^2 + 0} = 2\sqrt{2}^2 \iint_D dx dy = 4 \cdot \text{Area}(D) = 4\pi. \end{aligned}$$

5. Let $f(z) = \bar{z}^2$, $z \in \mathbb{C}$. Is f differentiable at 0? Is f holomorphic at 0? Justify your answer.

Solution. The function $f(z) = \bar{z}^2$ is differentiable at 0 because

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \frac{\bar{z}^2}{z} = \lim_{r \rightarrow 0} \frac{r^2(\cos \varphi - i \sin \varphi)}{r(\cos \varphi + i \sin \varphi)} = \lim_{r \rightarrow 0} \frac{r(\cos \varphi - i \sin \varphi)^2}{\cos^2 \varphi + \sin^2 \varphi} = 0.$$

But the function $f(z) = f(x + iy) = (x - iy)^2 = x^2 - y^2 - 2ixy$ is not holomorphic at 0. Indeed, according to the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial(x^2 - y^2)}{\partial x} = 2x \neq \frac{\partial v}{\partial y}(x, y) = \frac{\partial(-2xy)}{\partial y} = -2x$$

for all $x \neq 0$. So, the function f is not differentiable in a neighbourhood of 0.

6. Find the Laurent series for the function $f(z) = \frac{1}{(z-2)(z+1)^2}$ in $0 < |z + 1| < 3$.

Solution. We will use the expansion $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$.

$$\begin{aligned} f(z) &= \frac{1}{(z-2)(z+1)^2} = \frac{1}{(z+1)^2} \cdot \frac{1}{-3+(z+1)} = -\frac{1}{3(z+1)^2} \cdot \frac{1}{1-\frac{z+1}{3}} \\ &= -\frac{1}{3(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{3^n} = -\sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{3^{n+1}}, \quad 0 < |z+1| < 3. \end{aligned}$$

7. Compute the integral

$$\int_{|z-2|=2} z^2 \left(\cos^2 \frac{1}{z-1} + \frac{1}{(4-z^2)^2} \right) dz.$$

Solution. Consider separately $\int_{|z-2|=2} z^2 \cos^2 \frac{1}{z-1} dz$ and $\int_{|z-2|=2} \frac{z^2}{(4-z^2)^2} dz$. By the residue theorem

$$\int_{|z-2|=2} z^2 \cos^2 \frac{1}{z-1} dz = 2\pi i \operatorname{res}_1 z^2 \cos^2 \frac{1}{z-1}.$$

The point $z = 1$ is an essential isolated singularity for the function $z^2 \cos^2 \frac{1}{z-1}$. We find its residue at $z = 1$ by computing c_{-1} in the Laurent series of $z^2 \cos^2 \frac{1}{z-1}$ around the point $z = 1$. So,

$$\begin{aligned} z^2 \cos^2 \frac{1}{z-1} &= ((z-1) + 1)^2 \left(1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} + \dots \right)^2 \\ &= ((z-1)^2 + 2(z-1) + 1) \left(1 - \frac{2}{2!(z-1)^2} + \left(\frac{2}{4!} + \frac{1}{(2!)^2} \right) \frac{1}{(z-1)^4} + \dots \right) \end{aligned}$$



Hence $c_{-1} = 2 \cdot \frac{-2}{2!} = -2$.

Similarly,

$$\int_{|z-2|=2} \frac{z^2}{(4-z^2)^2} dz = 2\pi i \operatorname{res}_2 \frac{z^2}{(4-z^2)^2}.$$

Here, the point $z = 2$ is a pole of order 2 for the function $\frac{z^2}{(4-z^2)^2} = \frac{z^2}{(2-z)^2(2+z)^2}$. Hence

$$\begin{aligned} \operatorname{res}_2 \frac{z^2}{(4-z^2)^2} &= \lim_{z \rightarrow 2} \frac{d}{dz} \left((z-2)^2 \frac{z^2}{(2-z)^2(2+z)^2} \right) = \lim_{z \rightarrow 2} \frac{d}{dz} \frac{z^2}{(2+z)^2} \\ &= \lim_{z \rightarrow 2} \frac{2z(2+z)^2 - 2z^2(2+z)}{(2+z)^4} = \frac{1}{8}. \end{aligned}$$

Thus,

$$\int_{|z-2|=2} z^2 \left(\cos^2 \frac{1}{z-1} + \frac{1}{(4-z^2)^2} \right) dz = 2\pi i \left(-2 + \frac{1}{8} \right) = -\frac{15\pi i}{4}.$$

8. Solve the heat equation

$$\begin{aligned} u_t(t, x) &= \frac{1}{4} u_{xx}(t, x), \quad x \in \mathbb{R}, \quad t > 0, \\ u(0, x) &= x^2, \quad x \in \mathbb{R}. \end{aligned}$$

Solution. The solution to the heat equation is given by the formula

$$u(t, x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}} y^2 dy.$$

We change the variables $\frac{y-x}{\sqrt{t}} = \alpha$. Then $y = \sqrt{t}\alpha + x$ and $dy = \sqrt{t}d\alpha$. So,

$$\begin{aligned} u(t, x) &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\alpha^2} (\sqrt{t}\alpha + x)^2 \sqrt{t} d\alpha = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} (t\alpha^2 + 2\sqrt{t}x\alpha + x^2) d\alpha \\ &= \frac{t}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} \alpha^2 d\alpha + \frac{2\sqrt{t}x}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} \alpha d\alpha + \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha \\ &= -\frac{t}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \alpha de^{-\alpha^2} + 0 + x^2 = -\frac{t}{2\sqrt{\pi}} \left(\alpha e^{-\alpha^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha \right) + x^2 = \frac{t}{2} + x^2. \end{aligned}$$

We have used here $\int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha = \sqrt{\pi}$, and $\int_{-\infty}^{\infty} e^{-\alpha^2} \alpha d\alpha = 0$ because the function $e^{-\alpha^2} \alpha$ is odd.