Mathematics 3
Vector Calculus and Partial Differential Equations

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1 Riemann Integrals over n-Dimensional Rectangles (Lecture Notes)

1.1 One-Dimensional Case

Consider the interval $[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ and a function $f : [a, b] \mapsto \mathbb{R}$. A set of points $P := \{x_0, \ldots, x_n\}$ such that $a = x_0 < \cdots < x_n = b$ is called a partition of $[a, b]$. We define the mesh of the partition P as $\lambda(P) = \max\{\Delta x_k : 1 \leqslant k \leqslant n\}$, $\Delta x_k = x_k - x_{k-1}$. We consider points $\xi_k \in [x_{k-1}, x_k]$. We can then define the Riemann sum as

$$\sigma(f, P, \xi) := \sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

A function $f : [a, b] \mapsto \mathbb{R}$ is integrable on $[a, b]$ if there exists a limit

$$J = \int_a^b f(x)\, dx := \lim_{\lambda(P)\to 0} \sigma(f, P, \xi)$$

which does not depend on the choice of ξ. That is, for all $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $P = \{x_0, \ldots, x_n\}$ satisfying $\lambda(P) < \delta \quad \forall\, \xi_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$ we have

$$|J - \sigma(f, P, \xi)| < \epsilon$$

This limit is called the Riemann integral of f over $[a, b]$.

1.2 Definition of the Integral

We introduce the set

$$I = I_{a,b} = \{x \in \mathbb{R}^d : a_i \leqslant x \leqslant b_i,\ i = 1, \ldots, d\}$$

which is called a rectangle or an interval in $\mathbb{R}^d$, and the volume or Lebesgue measure of the interval $I_{a,b}$

$$|I_{a,b}| = \prod_{i=1}^{n}(b_i - a_i)$$

Lemma 1.1 *The Lebesgue measure of an interval in $\mathbb{R}^d$ has the following properties.*

1. *It is homogeneous, i.e. $|\lambda I_{a,b}| = \lambda^d |I_{a,b}|$, where $\lambda > 0$ and $\lambda I_{a,b} := I_{\lambda a, \lambda b}$.*
2. *It is additive, i.e. if $I, I_1, \ldots, I_n$ are intervals in $\mathbb{R}^d$ such that $I = \bigcup_{i=1}^{n} I_i$ and no two intervals $I_1, \ldots, I_n$ have common interior points, then $|I| = \sum_{i=1}^{n} |I_i|$*
3. *If $I \subseteq \bigcup_{i=1}^{n} I_i$ where $I, I_1, \ldots, I_n$ are intervals, then $|I| \leqslant \sum_{i=1}^{n} |I_i|$.*

Now we introduce partitions of an interval. Take $I = \{x \in \mathbb{R}^d : a_i \leqslant x_i \leqslant b_i,\ i = 1, \ldots, d\}$. Partitions of the coordinate intervals $[a_i, b_i]$, $i = 1, \ldots, d$ induce a partition of the interval I:

$$I = \bigcup_{j=1}^{n} I_j$$

We write $P = \{I_1, \ldots, I_n\}$. The quantity $\lambda(P) = \max_{j=1,\ldots,n} \mathrm{d}(I_j)$, where $\mathrm{d}(I_j) = \max_{x,y \in I_j} \|x - y\|$, is called the mesh of the partition P.

Definition 1.1 *Let $P = \{I_1, \ldots, I_n\}$ be a partition of the interval I. We consider a function $f : I \mapsto \mathbb{R}$ and points $\xi_i \in I_i$, $i = 1, \ldots, n$. The sum*

$$\sigma(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i)|I_i|$$

is called the Riemann sum of f.

Definition 1.2 *A function $f : I \mapsto \mathbb{R}$ is called Riemann integrable on I if there exists a limit*

$$J = \int\limits_I f(x)\, dx = \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_d}^{b_d} f(x_1, \ldots, x_d)\, dx_1 \ldots dx_d := \lim_{\lambda(P) \to 0} \sigma(f, P, \xi)$$

that is, for all $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $P = \{I_1, \ldots, I_n\}$ of I satisfying $\lambda(P) < \delta \quad \forall \xi_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$ we have

$$|J - \sigma(f, P, \xi)| < \epsilon$$

In this case we write $f \in \mathcal{R}(I)$.

Proposition 1.1 (Necessary Condition of Integrability) *If $f \in \mathcal{R}(I)$, then f is bounded.*

1.3 Darboux Criterion of Integrability

Let the function $f : I \mapsto \mathbb{R}$ and partition $P = \{I_1, \ldots, I_n\}$ of the interval I be given.

Definition 1.3 *The quantities*

$$L(f, P) := \sum_{i=1}^{n} m_i |I_i| \qquad U(f, P) := \sum_{i=1}^{n} M_i |I_i|$$

are called the lower and upper Darboux sums of f, where $m_i = \inf_{x \in I_i} f(x)$ and $M_i = \sup_{x \in I_i} f(x)$.

Remark 1.1 $L(f, P) \leqslant \sigma(f, P, \xi) \leqslant U(f, P)$

Definition 1.4 *The quantities*

$$\underline{J} = \sup_P L(f, P) \qquad \overline{J} = \inf_P U(f, P)$$

are called the lower and upper Darboux integrals of f over the interval I.

Remark 1.2 $L(f, P) \leqslant \underline{J} \leqslant \overline{J} \leqslant U(f, P)$

Theorem 1.1 (Darboux Criterion) *$f \in \mathcal{R}(I)$ if and only if $\underline{J} = \overline{J}$ and if f is bounded on I.*

Proposition 1.2 *A function $f : I \mapsto \mathbb{R}$ is integrable on I if and only if for all $\epsilon > 0$ there exists a partition P of I such that*

$$U(f, P) - L(f, P) < \epsilon$$

1.4 Lebesgue Criterion of Integrability

Definition 1.5 *A set $E \subseteq \mathbb{R}^d$ has Lebesgue measure zero if for every $\epsilon > 0$ there exists at most a countable system $\{I_i\}$ of d-dimensional intervals such that $E \subseteq \bigcup\limits_i I_i$ and $\sum\limits_i |I_i| \leqslant \epsilon$.*

Lemma 1.2 *A union of a finite or countable number of sets of Lebesgue measure zero is a set of Lebesgue measure zero. A subset of a Lebesgue measure zero set is itself of Lebesgue measure zero.*

We say that f is continuous almost everywhere if the set of discontinuities $D_f = \{x \in I : f \text{ is discontinuous at } x\}$ has Lebesgue measure zero.

Theorem 1.2 (Lebesgue Criterion) *f is Riemann integrable if and only if f is bounded and continuous almost everywhere.*

2 Integrals over a Set (Lecture Notes)

2.1 The Measure of a Set

Let $S \subseteq \mathbb{R}^d$ be a bounded set and let I be an interval in $\mathbb{R}^d$ such that $S \subseteq I$. We can then define the Jordan measure of S as

$$\mu(S) = \int\limits_I \mathbb{I}_S(x)\,dx$$

if the integral exists, where

$$\mathbb{I}_S(x) = \begin{cases} 1, \, x \in S \\ 0, \, x \notin S \end{cases}$$

By Th. 1.2, the integral exists if $\mathbb{I}_S$ is continuous almost everywhere. We denote the set of discontinuities of $\mathbb{I}_S$ by $D_{\mathbb{I}_S} = \{x : \mathbb{I}_S \,\text{is discontinuous at}\, x\}$.

Definition 2.1 *The set*

$$\partial S = \{x : \forall\, \epsilon > 0 \quad B_\epsilon(x) \cap S \neq \emptyset,\, B_\epsilon(x) \cap S^c \neq \emptyset\}$$

is called the boundary of S.

Lemma 2.1 *The set $D_{\mathbb{I}_S}$ coincides with ∂S.*

Hence, by Lem. 2.1, the Jordan measure $\mu(S)$ of a set $S \subseteq \mathbb{R}^d$ exists if and only if the boundary ∂S of S has Lebesgue measure zero. If $\mu(S)$ exists, we call S a Jordan-measurable set.

2.2 Integrals over a Set

Definition 2.2 *A set $S \subseteq \mathbb{R}^d$ is admissible if it is bounded in $\mathbb{R}^d$ and ∂S has Lebesgue measure zero.*

Definition 2.3 *The integral of f over S is given by*

$$\int\limits_S f(x)\,dx := \int\limits_I f(x)\mathbb{I}_S(x)\,dx$$

where I is some interval in $\mathbb{R}^d$ and $S \subseteq I$. If the integral exists, then f is said to be Riemann integrable over S.

Lemma 2.2 *For any S, S_1, S_2:*

1. *∂S is closed in $\mathbb{R}^d$*
2. *$\partial(S_1 \cup S_2) \subset \partial S_1 \cup \partial S_2$*
3. *$\partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2$*
4. *$\partial(S_1 \setminus S_2) \subset \partial S_1 \cup \partial S_2$*

Lemma 2.3 *The union or intersection of a finite number of admissible sets is an admissible set. The difference of admissible sets is also an admissible set.*

Theorem 2.1 *A function $f : S \mapsto \mathbb{R}$ is integrable over an admissible set S if and only if it is bounded and continuous almost everywhere.*

3 Fubini's Theorem (Lecture Notes)

3.1 General Properties of the Integral

Proposition 3.1 *If $f, g \in \mathcal{R}(S)$ and $a \in \mathbb{R}$, then $f+g, a\cdot f, f\cdot g \in \mathcal{R}(S)$ and*

$$\int\limits_S (f+g)\,dx = \int\limits_S f\,dx + \int\limits_S g\,dx$$

$$\int\limits_S a\,f\,dx = a\int\limits_S f\,dx$$

Proposition 3.2 *Consider admissible sets S_1, S_2 and a function $f : S_1 \cup S_2 \mapsto \mathbb{R}$. Then $f \in \mathcal{R}(S_1 \cup S_2)$ if and only if $f \in \mathcal{R}(S_1) \cap \mathcal{R}(S_2)$. If additionally $\mu(S_1 \cap S_2) = 0$, then*

$$\int\limits_{S_1\cup S_2} f\,dx = \int\limits_{S_1} f\,dx + \int\limits_{S_2} f\,dx$$

Proposition 3.3 *If $f \in \mathcal{R}(S)$, then $|f| \in \mathcal{R}(S)$ and $\left|\int\limits_S f\,dx\right| \leqslant \int\limits_S |f|\,dx$.*

Proposition 3.4 *If $f \in \mathcal{R}(S)$ and $f \geqslant 0, \forall\, x \in S$, then $\int\limits_S f\,dx \geqslant 0$.*

Corollary 3.1 *If $f, g \in \mathcal{R}(S)$ and $f \leqslant g, \forall\, x \in S$, then $\int\limits_S f\,dx \leqslant \int\limits_S g\,dx$.*

Corollary 3.2 *If $f \in \mathcal{R}(S)$, $m \leqslant f \leqslant M, \forall\, x \in S$, then $m\,\mu(S) \leqslant \int\limits_S f\,dx \leqslant M\,\mu(S)$.*

3.2 Fubini's Theorem

Theorem 3.1 (Fubini's Theorem) *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be intervals and let $f : X \times Y \mapsto \mathbb{R}$ be an integrable function over the interval $X \times Y$. Then*

$$\iint\limits_{X\times Y} f(x,y)\,dx\,dy = \int\limits_X dx \int\limits_Y f(x,y)\,dy = \int\limits_Y dy \int\limits_X f(x,y)\,dx$$

Corollary 3.3 *If $I = [a_1, b_1] \times \cdots \times [a_d, b_d] = I_{a,b}$, then*

$$\int\limits_I f(x)\,dx = \int\limits_{a_d}^{b_d} dx_d \int\limits_{a_{d-1}}^{b_{d-1}} dx_{d-1} \cdots \int\limits_{a_1}^{b_1} f(x_1, \ldots, x_d)\,dx_1$$

Corollary 3.4 *If D is a bounded subset of $\mathbb{R}^{d-1}$, $S = \{(x,y) \in \mathbb{R}^d : x \in D, \varphi_1(x) \leqslant y \leqslant \varphi_2(x)\}$, and $f \in \mathcal{R}(S)$, then*

$$\iint\limits_S f(x,y)\,dy\,dx = \int\limits_D dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

Example *Let S be bounded by $y = x^2$ and $y = x + 2$, and let $f(x, y) = x^2$. The two curves meet when $x^2 = x + 2$. We then have*

$$\iint\limits_S f(x,y)\,dx\,dy = \int\limits_{-1}^{2} dx \int\limits_{x^2}^{x+2} y^2\,dy = \int\limits_{-1}^{2} \left(\frac{(x+2)^3}{3} - \frac{x^6}{3}\right) dx = \frac{423}{28}$$

Example *Let S be bounded by $y = 2x$, $y = \frac{x}{2}$, $y = 6 - x$ and let $f(x, y) = \frac{1}{(1+x+y)^2}$. We split S into*

$$S_1 = \{(x, y) : 0 \leqslant x \leqslant 2,\ \tfrac{x}{2} \leqslant y \leqslant 2x\}$$

$$S_2 = \{(x, y) : 2 \leqslant x \leqslant 4,\ \tfrac{x}{2} \leqslant y \leqslant 6 - x\}$$

We then have

$$\iint\limits_S f(x,y)\,dx\,dy = \int\limits_{0}^{2} dx \int\limits_{\frac{x}{2}}^{2x} \frac{dy}{(1+x+y)^2} + \int\limits_{2}^{4} dx \int\limits_{\frac{x}{2}}^{6-x} \frac{dy}{(1+x+y)^2} = \frac{1}{3}\ln 7 - \frac{2}{7}$$

Example *Let $S = \{(x, y, z) : |x| \leqslant z,\ 0 \leqslant z \leqslant 1,\ z \leqslant y \leqslant \sqrt{4 - x^2 - z^2}\}$ and let $f(x, y, z) = y$. Then*

$$\iiint\limits_S f(x,y,z)\,dx\,dy\,dz = \int\limits_{0}^{1} dz \int\limits_{-z}^{z} dx \int\limits_{z}^{\sqrt{4-x^2-z^2}} y\,dy = \frac{17}{12}$$

4 Change of Variables (Lecture Notes)

4.1 Heuristic Derivation

We consider sets $S \subseteq \mathbb{R}^d$ and $D \subseteq \mathbb{R}^d$, and a bijective map $\varphi : D \mapsto S$. We are interested in finding out if $\int\limits_S f\,dx$ can be rewritten as $\int\limits_D g\,dx$, where g is some function. Let φ be an affine map such that

$$\begin{cases} x(u,v) = a_1 + a_{11}u + a_{12}v \\ y(u,v) = a_2 + a_{21}u + a_{22}v \end{cases}$$

where $(u,v) \in \mathbb{R}^2$. We want to find how the volume of an interval $I \subseteq D$ is changed under the map φ, i.e. what the volume of $\varphi(I)$ is. Let $I = [0, \Delta u] \times [0, \Delta v]$. The interval I is mapped by φ to the parallelogram Γ spanned by the vectors $\vec{r}_1 = (a_{11}\Delta u, a_{21}\Delta u)$ and $\vec{r}_2 = (a_{12}\Delta v, a_{22}\Delta v)$ applied at the point (a_1, a_2). Hence

$$\mu(\Gamma) = \|\vec{r}_1 \times \vec{r}_2\| = \begin{vmatrix} a_{11}\Delta u & a_{21}\Delta u \\ a_{12}\Delta v & a_{22}\Delta v \end{vmatrix} = \Delta u \Delta v \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I|$$

Similarly, for any interval I, we have

$$\mu(\Gamma) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I|$$

where $\Gamma = \varphi(I)$. Now we consider a more general transformation φ. Given

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

we can make use of Taylor's theorem to approximate

$$\mu(\Gamma) \approx \mu(\Gamma') = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I|$$

From there we can formally change variables from x and y to u and v:

$$\iint\limits_S f(x,y)\,dx\,dy \approx \sum_i f\big(\varphi(\xi_i)\big)\mu(\Gamma_i) \approx \sum_i f\big(\varphi(\xi_i)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I_i| \approx \iint\limits_D f\big(\varphi(u,v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

4.2 Change of Variables

Theorem 4.1 (Change of Variables) *Let S, D be admissible sets and let $\varphi : D \mapsto S$ be a continuously differentiable bijection such that its Jacobian is non-zero in D. Then*

$$\int\limits_S f(x)\,dx = \int\limits_D f\big(\varphi(u)\big) \left| \frac{\partial(x_1, \ldots, x_d)}{\partial(u_1, \ldots, u_d)} \right| du$$

Example *Let S be bounded by $z = x^2 + y^2$ and $z = 1$, and let $f(x, y, z) = x^2 + y^2$. Using cylindrical coordinates:*

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad \frac{\partial(x, y, z)}{\partial(r, \varphi, z)} = r$$

we have

$$0 \leqslant \varphi \leqslant 2\pi \qquad 0 \leqslant z \leqslant 1 \qquad 0 \leqslant r \leqslant \sqrt{z}$$

and thus

$$\iiint\limits_S (x^2 + y^2)\, dx\, dy\, dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 dz \int\limits_0^{\sqrt{z}} r^3\, dr = \frac{\pi}{6}$$

Example *Let S be bounded by $x^2 + y^2 + (z-1)^2 = 1$ and let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Using spherical coordinates:*

$$\begin{cases} x = r\cos\varphi\cos\psi \\ y = r\sin\varphi\cos\psi \\ z = r\sin\psi \end{cases} \qquad \frac{\partial(x, y, z)}{\partial(r, \varphi, \psi)} = r^2\cos\psi$$

we have

$$0 \leqslant \varphi \leqslant 2\pi \qquad 0 \leqslant \psi \leqslant \frac{\pi}{2} \qquad 0 \leqslant r \leqslant 2\sin\psi$$

and thus

$$\iiint\limits_S \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} d\psi \int\limits_0^{2\sin\psi} r^3\cos\psi\, dr = \frac{8\pi}{5}$$

5 Improper Integrals (Lecture Notes)

5.1 Improper Integrals

Definition 5.1 *An exhaustion of a set $S \subseteq \mathbb{R}^d$ is a sequence of Jordan-measurable sets $\{S_n\}$ such that $S_n \subseteq S_{n+1} \subseteq S$ for any $n \geqslant 1$, and $\bigcup_{i=1}^{\infty} S_n = S$.*

Lemma 5.1 *If $\{S_n\}$ is an exhaustion of a Jordan-measurable set S, then:*

1. $\lim\limits_{n\to\infty} \mu(S_n) = \mu(S)$
2. *for every $f \in \mathcal{R}(S)$, f also belongs to $\mathcal{R}(S_n)$ and*

$$\lim_{n\to\infty} \int\limits_{S_n} f(x)\,dx = \int\limits_{S} f(x)\,dx$$

Definition 5.2 *Let $\{S_n\}$ be an exhaustion of a set S and let $f : S \mapsto \mathbb{R}$ be integrable over all S_n. Then the limit*

$$\int\limits_{S} f(x)\,dx := \lim_{n\to\infty} \int\limits_{S_n} f(x)\,dx$$

is called the improper integral of f over S if it exists and does not depend on the choice of $\{S_n\}$. In this case, we say that the integral converges.

Remark 5.1 *If S is a Jordan-measurable set and $f \in \mathcal{R}(S)$, then the integral of f over S as in Def. 5.2 exists and has the same value as the proper integral of f over S. This follows from Lem. 5.1.*

Proposition 5.1 *If $f : S \mapsto \mathbb{R}$ is non-negative and the limit in Def. 5.2 exists for one exhaustion $\{S_n\}$ of S, then the improper integral of f over S converges.*

Example *Consider the improper integral*

$$\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$

We can define $S_n = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < n^2\}$ and use polar coordinates to evaluate the integral:

$$\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \iint\limits_{S_n} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \int\limits_0^{2\pi} d\varphi \int\limits_0^{n} re^{-r^2}\,dr$$

$$= 2\pi \lim_{n\to\infty} \int\limits_0^{n} re^{-r^2}\,dr = \lim_{n\to\infty} \pi(1-e^{-n^2}) = \pi$$

Remark 5.2 *Improper integrals can arise if S is unbounded or if f is unbounded. Various properties of multiple integrals can be suitably extended to improper integrals.*

Theorem 5.1 (Comparison Test) *Let f and g be functions defined on S. Assume f and g are integrable over exactly the same Jordan-measurable subsets of S, and $|f(x)| \leqslant g(x)$ for all $x \in S$. Then if the improper integral $\int\limits_S g(x)\,dx$ converges, the integrals $\int\limits_S |f(x)|\,dx$ and $\int\limits_S f(x)\,dx$ also converge.*

5.2 Curves in $\mathbb{R}^d$

Definition 5.3 *A curve in $\mathbb{R}^d$ is a continuous map $\gamma : I \mapsto \mathbb{R}^d$, where I is a closed interval consisting of more than one point. The interval I could be $[a, +\infty)$, $(-\infty, b]$, $[a, b] : a < b$, $\mathbb{R}$.*

If $I = [a, b]$, then a is called the initial point of γ and b is called the end point of γ. These two points define a natural orientation of γ from $\gamma(a)$ to $\gamma(b)$. Replacing $\gamma(t)$ with $\gamma(a + b - t)$ will yield the curve with opposite orientation. If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve. If γ is differentiable, the curve is said to be differentiable. If γ has no points of self-intersection, i.e. it is injective on I°, then γ is said to be simple.

Definition 5.4

1. *A simple curve $\gamma : I \mapsto \mathbb{R}^d$ is called regular at t_0 if γ is continuously differentiable on I and $\gamma'(t_0) \neq 0$. γ is regular if γ is regular at any point $t_0 \in I$.*

2. *The vector $\gamma'(t_0)$ is called the tangent vector and $\alpha(t) = \gamma(t_0) + t\gamma'(t_0)$ is called the tangent line to γ at $\gamma(t_0)$.*

6 Line Integrals of Scalar Fields (Lecture Notes)

6.1 Rectifiable Curves

Consider a curve $\gamma : [a, b] \mapsto \mathbb{R}^d$ and a partition $P = \{t_0, \dots, t_n\}$, where $a = t_0 < \dots < t_n = b$. We set

$$l(P, \gamma) := \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|$$

Definition 6.1 *A curve γ is said to be rectifiable if*

$$l(\gamma) := \sup_P l(P, \gamma)$$

is finite, where the supremum is taken over all partitions P of $[a, b]$.

Proposition 6.1 *If γ' is continuous on $[a, b]$, then γ is rectifiable and*

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, dt$$

Remark 6.1 *If $d = 2$ and $\gamma(t) = \big(x(t), y(t)\big)$, $t \in [a, b]$, then*

$$l(\gamma) = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2} \, dt$$

In particular, if γ is the graph of a function f, i.e. $\gamma(t) = \big(t, f(t)\big)$, $t \in [a, b]$, then

$$l(\gamma) = \int_a^b \sqrt{1 + \big(f'(t)\big)^2} \, dt$$

Example *Let $\gamma(t)$ be the graph of the function $f(t) = a \cosh \frac{t}{a}$, $t \in [0, b]$, $b > 0$, $a \neq 0$. Then we compute*

$$l(\gamma) = \int_0^b \sqrt{1 + \left(\sinh \frac{t}{a}\right)^2} \, dt = \int_0^b \cosh \frac{t}{a} \, dt = a \sinh \frac{t}{a}\Big|_0^b = a \sinh \frac{b}{a}$$

Now let $\gamma(\theta) = \big(a(\theta - \sin\theta), a(1 - \cos\theta)\big)$, $\theta \in [0, 2\pi]$. Here, γ parametrizes a cycloid. We can compute

$$l(\gamma) = \int_0^{2\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta} \, d\theta = a \int_0^{2\pi} \sqrt{2 - 2\cos\theta} \, d\theta$$

$$= a\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos\theta} \, d\theta = 2a \int_0^{2\pi} \sin \frac{\theta}{2} \, d\theta = 8a$$

6.2 Natural Parametrization of Rectifiable Curves

Let $\gamma : [a, b] \mapsto \mathbb{R}^d$ be a regular curve, that is, $\gamma'(t) \neq 0 \quad \forall\, t \in [a, b]$. We denote by

$$s(t) = \int_a^t \|\gamma'(r)\|\, dr =: l_t(\gamma)$$

the length of a part of the curve $\gamma(r)$, $r \in [0, t]$. Since $\gamma'(t) \neq 0$, $\|\gamma'(t)\| > 0$, the function $s = s(t)$, $t \in [a, b]$ strictly increases. Consequently, s is invertible with inverse $t = t(s)$, $s \in [0, l]$, and $l = l(\gamma) = s(b)$. We then define the parametrization

$$x(s) = \gamma\big(t(s)\big),\ s \in [0, l] \tag{6.1}$$

of the curve γ.

Lemma 6.1 *The length $l_s(x)$ of the curve given by $x = x(r)$, $r \in [0, s]$ equals s.*

Definition 6.2 *The parametrization x as defined in (6.1) is called the natural parametrization of the curve γ.*

We remark that any regular curve has a natural parametrization.

6.3 Line Integrals of Scalar Fields

Let γ be a rectifiable curve with length L. We assume that γ has a natural parametrization $x(s)$, $s \in [0, L]$. We set

$$\Gamma = \{x(s),\ s \in [0, L]\} = \{\gamma(t),\ t \in [a, b]\}$$

Consider a function $f : \Gamma \mapsto \mathbb{R}$. We take a partition $P = \{s_0, s_1, \ldots, s_n\}$ of $[0, L]$ and define the line integral as

$$\int_\gamma f\, ds := \lim_{\lambda(P)\to 0} \sum_{i=1}^n f\big(x(s_i)\big)(s_i - s_{i-1})$$

if the limit exists.

Remark 6.2 *The line integral $\int_\gamma f\, ds$ coincides with the usual Riemann integral $\int_0^L f\big(x(s)\big)\, ds$, where x is a natural parametrization of γ.*

7 Line Integrals of Scalar Fields and Vector Fields (Lecture Notes)

7.1 Line Integrals of Scalar Fields

Let $\gamma : [a, b] \mapsto \mathbb{R}^d$ be a rectifiable curve with length L. We assume that γ has a natural parametrization $x(s)$, $s \in [0, L]$. Take $\Gamma = \{x(s),\, s \in [0, L]\} = \{\gamma(t),\, t \in [a, b]\}$ and a partition $P = \{s_0, s_1, \ldots, s_n\}$ of $[0, L]$.

Definition 7.1 *The line integral of a function $f : \Gamma \mapsto \mathbb{R}$ along γ is*

$$\int_\gamma f\, ds := \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f\big(x(s_i)\big)(s_i - s_{i-1}) = \int_0^L f\big(x(s)\big)\, ds$$

if the limit exists. If it exists, the line integral is equivalent to a Riemann integral.

Line integrals of scalar fields have the following properties:

1. $\displaystyle\int_\gamma ds = L$

2. If f is bounded and continuous, then the Riemann integral $\displaystyle\int_\gamma f\, ds$ exists.

3. If $\tilde{\gamma}(t)$, $a \leqslant t \leqslant b$ is another regular parametrization of γ, then

$$\int_\gamma f\, ds = \int_a^b f\big(\tilde{\gamma}(t)\big)\|\tilde{\gamma}'(t)\|\, dt$$

4. If γ^R is the time reversal of γ, then

$$\int_{\gamma^R} f\, ds = \int_\gamma f\, ds$$

5. For $a, b \in \mathbb{R}$ and $f, g : \Gamma \mapsto \mathbb{R}$

$$\int_\gamma (af + bg)\, ds = a\int_\gamma f\, ds + b\int_\gamma g\, ds$$

6. Let $l(\gamma)$ be the length of γ. Then

$$\left|\int_\gamma f\, ds\right| \leqslant l(\gamma) \sup_\Gamma |f|$$

7.2 Line Integrals of Vector Fields

Let γ be a regular rectifiable curve on $\mathbb{R}^2$ with parametrization $\gamma(t) = \big(x(t), y(t)\big)$, $a \leqslant t \leqslant b$.

Definition 7.2 *The line integral of a vector field* $\vec{F}(x,y) = \big(P(x,y), Q(x,y)\big)$ *along* γ *is*

$$\int_\gamma \vec{F} \cdot ds := \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \big[P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t)\big]\, dt$$

Example *Take* $\gamma_1(t) = (t, t^2)$, $0 \leqslant t \leqslant 1$ *and* $\vec{F}(x,y) = (y, x)$. *We calculate*

$$\int_{\gamma_1} \vec{F} \cdot ds = \int_0^1 \big(y(t)\, x'(t) + x(t)\, y'(t)\big)\, dt = \int_0^1 (t^2 + 2t^2)\, dt = 1$$

Example *Take* $\gamma_2(t) = (1 - \cos t, \sin t)$, $0 \leqslant t \leqslant \frac{\pi}{2}$ *and* $\vec{F}(x,y) = (y, x)$. *We calculate*

$$\int_{\gamma_2} \vec{F} \cdot ds = \int_0^{\frac{\pi}{2}} \big(y(t)\, x'(t) + x(t)\, y'(t)\big)\, dt = \int_0^{\frac{\pi}{2}} \big(\sin^2 t + (1 - \cos t)\cos t\big)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \cos t\, dt + \int_0^{\frac{\pi}{2}} \cos 2t\, dt = 1$$

Remark 7.1 *Sometimes we write* $\int_\gamma \vec{F} \cdot ds = \int_a^b (P\, dx + Q\, dy)$.

Line integrals of vector fields have the following properties:

1. The definition of $\int_\gamma \vec{F} \cdot ds$ is independent of parametrization.

2. For the time reversal γ^R of γ, we have $\int_{\gamma^R} \vec{F} \cdot ds = -\int_\gamma \vec{F} \cdot ds$.

8 Green's Formula (Lecture Notes)

8.1 Green's Formula

Consider a set S and let γ be such that $\gamma = \partial S$. We say that a curve γ is positively orientated if the set S stays on the left when travelling along γ.

Theorem 8.1 (Green's Formula) *Let $\vec{F}(x,y) = \big(P(x,y), Q(x,y)\big)$ be such that P and Q are continuously differentiable on $\overline{S}$. Then*

$$\iint\limits_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int\limits_\gamma \vec{F} \cdot ds$$

Example *Take the set $S = \{(x,y) : x^2 + y^2 \leqslant 1\}$. Then γ is the positively orientated unit circle. Say that for the field $\vec{F}(x,y) = (x^2 y, xy^2)$ we want to calculate $I = \int\limits_\gamma \vec{F} \cdot ds$. Then we have*

$$I = \int\limits_\gamma (P\,dx + Q\,dy) = \int\limits_\gamma (x^2 y\,dx - xy^2\,dy)$$

$$= \iint\limits_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\iint\limits_S (x^2 + y^2)\,dx\,dy = -\int\limits_0^{2\pi} d\theta \int\limits_0^1 r^3\,dr = -\frac{\pi}{2}$$

8.2 Conservative Vector Fields

Definition 8.1 *A vector field $\vec{F} : S \subseteq \mathbb{R}^d \mapsto \mathbb{R}^d$ is conservative if for any two points $a, b \in S$ and any two curves γ_1, γ_2 connecting a and b we have*

$$\int\limits_{\gamma_1} \vec{F} \cdot ds = \int\limits_{\gamma_2} \vec{F} \cdot ds$$

Definition 8.2 *A vector field $\vec{F} : S \subseteq \mathbb{R}^d \mapsto \mathbb{R}^d$ is called a gradient vector field or a potential field if there exists a continuously differentiable function $\varphi : S \mapsto \mathbb{R}$ such that $\vec{F} = \nabla\varphi$.*

9 Path Independence of Line Integrals (Lecture Notes)

9.1 Work of a Vector Field

Let S be a subset of $\mathbb{R}^d$. We say that S is a domain if S is open and connected. A connected set S has the property that any two points from S can be connected by a curve in S. Let $\vec{F}$ be a continuous force field acting in the domain S. We want to find the work done by $\vec{F}$ when moving along a given trajectory γ. If $\vec{F}$ is constant, the displacement described by a vector $\vec{\xi}$ is associated with an amount of work $\vec{F}\cdot\vec{\xi}$. Let the curve $\gamma = \gamma(t)$, $t \in [a, b]$ be naturally parametrized and differentiable. We take a partition P of $[a, b]$ such that $a = t_0 < \cdots < t_n = b$. Then the work A is given by

$$A \approx \sum_{i=1}^{n} \vec{F}\big(\gamma(t_i)\big)\cdot\gamma'(t_i)\Delta t_i$$

In the limit, the work done by $\vec{F}$ over the curve γ becomes

$$A = \int_a^b \vec{F}\big(\gamma(t)\big)\cdot\gamma'(t)\,dt = \int_\gamma \vec{F}\cdot ds$$

9.2 Conservative and Potential Vector Fields

Proposition 9.1 *Let $\vec{F} : S \mapsto \mathbb{R}^d$ be a continuous vector field, where $S \subseteq \mathbb{R}^d$ is a domain. The following statements are equivalent:*

1. *$\vec{F}$ is a potential vector field in S.*
2. *For any closed curve γ in S, we have $\int_\gamma \vec{F}\cdot ds = 0$.*
3. *$\vec{F}$ is conservative in S.*

Proof: We first prove that 1 implies 2. In this case, $\vec{F} = \nabla\varphi$ is given. We take $\gamma = \gamma(t)$, $t \in [a, b]$ such that $\gamma(a) = \gamma(b)$ and compute

$$\int_\gamma \vec{F}\cdot ds = \int_a^b \vec{F}\big(\gamma(t)\big)\cdot\gamma'(t)\,dt = \int_a^b \nabla\varphi\big(\gamma(t)\big)\cdot\gamma'(t)\,dt = \int_a^b \frac{d}{dt}\big[\varphi\big(\gamma(t)\big)\big]\,dt = \varphi\big(\gamma(b)\big) - \varphi\big(\gamma(a)\big) = 0$$

Next we prove that 2 implies 3. Here we take γ_1, γ_2 and $\gamma = \gamma_1 \cup \gamma_2^R$. Using 2, we have

$$0 = \int_\gamma \vec{F}\cdot ds = \int_{\gamma_1} \vec{F}\cdot ds + \int_{\gamma_2^R} \vec{F}\cdot ds = \int_{\gamma_1} \vec{F}\cdot ds - \int_{\gamma_2} \vec{F}\cdot ds$$

Finally we prove that 3 implies 1. We take points $a, x \in S$, where a is fixed and x is a variable, and define

$$\varphi(x) = \int_\gamma \vec{F}\cdot ds = \int_a^x \vec{F}\cdot ds \tag{9.1}$$

where γ is any curve connecting a and x. Using the definition of the gradient, we want to show that

$$\lim_{h\to 0} \frac{|\varphi(x+h) - \varphi(x) - \vec{F}(x)\cdot h|}{\|h\|} = 0$$

First, using the fact that $\vec{F}$ is conservative, we calculate

$$\varphi(x+h)-\varphi(x)=\int_a^{x+h}\vec{F}\cdot ds-\int_a^{x}\vec{F}\cdot ds=\int_x^{x+h}\vec{F}\cdot ds=\int_0^1\vec{F}(x+th)\cdot h\,dt=\vec{F}(x+\theta h)\cdot h$$

Here we have used the mean value theorem for integrals; in this case $\theta\in[0,1]$. We then have

$$\frac{|\varphi(x+h)-\varphi(x)-\vec{F}(x)\cdot h|}{\|h\|}=\frac{|\vec{F}(x+\theta h)\cdot h-\vec{F}(x)\cdot h|}{\|h\|}\leqslant\|\vec{F}(x+\theta h)-\vec{F}(x)\|$$

We see that indeed

$$\lim_{h\to 0}\frac{|\varphi(x+h)-\varphi(x)-\vec{F}(x)\cdot h|}{\|h\|}\leqslant\lim_{h\to 0}\|\vec{F}(x+\theta h)-\vec{F}(x)\|=0$$

and thus $\vec{F}=\nabla\varphi$. $\square$

Remark 9.1 *(9.1) can be used to find a potential of a potential vector field.*

9.3 Curl-Free Vector Fields

Definition 9.1 *The curl of a vector field $\vec{F}=(P,Q,R)$ in $S\subseteq\mathbb{R}^3$ is the vector field given by*

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

Definition 9.2 *A vector field $\vec{F}$ is called irrotational or curl-free if $\nabla\times\vec{F}=0$ in S.*

A domain S is called simply connected if any closed curve γ in S can be continuously transformed to a point $a\in S$.

Proposition 9.2 *Let $\vec{F}$ be a continuously differentiable vector field in a domain $S\subseteq\mathbb{R}^3$.*

1. *If $\vec{F}$ is conservative in S, it is curl-free in S.*
2. *If $\vec{F}$ is curl-free in S and S is simply connected, then $\vec{F}$ is conservative in S.*

10 Surface Integrals of Scalar Fields (Lecture Notes)

10.1 Surfaces

Definition 10.1 *A surface S in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ that can be parametrized by a continuous vector function r:*

$$S = \{r(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),\ (u,v) \in \overline{D}\}$$

where D is a bounded domain of $\mathbb{R}^2$ and $r(u,v) \neq r(u',v')$ for all $(u,v) \neq (u',v')$ in D (r may not be injective on the boundary of D).

Definition 10.2 *If a surface is parametrized by a continuously differentiable vector function, then it is called a continuously differentiable surface.*

Definition 10.3 *For a surface*

$$S = \{r(u,v) = \big(x(u,v), y(u,v), z(u,v)\big),\ (u,v) \in \overline{D}\}$$

and a point $(u_0, v_0) \in D$, the lines

$$\{r(u, v_0),\ (u, v_0) \in \overline{D}\} \qquad \{r(u_0, v),\ (u_0, v) \in \overline{D}\}$$

are called u- and v-curvilinear coordinates on S at $r(u_0, v_0)$. The tangent vectors to those lines are denoted by

$$r_u = r_u(u_0, v_0) = \frac{\partial r}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$
$$r_v = r_v(u_0, v_0) = \frac{\partial r}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

We will only consider surfaces such that $r_u \times r_v \neq 0$. In this case, r_u and r_v span a plain in $\mathbb{R}^3$ called the tangent plane to S at $r(u_0, v_0)$.

Remark 10.1 *The equation for the tangent plane to S at $r(u_0, v_0) = (x_0, y_0, z_0)$ is*

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_u(u_0, v_0) & y_u(u_0, v_0) & z_u(u_0, v_0) \\ x_v(u_0, v_0) & y_v(u_0, v_0) & z_v(u_0, v_0) \end{vmatrix} = 0$$

Definition 10.4 *The line orthogonal to the tangent plane at $r_0 = (x_0, y_0, z_0) \in S$ is called the normal line to S at r_0. Every non-zero vector parallel to the normal line at r_0 is called a normal vector to S at r_0.*

10.2 Surface Area

Let Γ_i be a rectangle

$$[u_i, u_i + \Delta u_i] \times [v_i, v_i + \Delta v_i]$$

in D and let S_i be its image in S. The area of S_i can be approximated by the area of the parallelogram in $\mathbb{R}^3$ spanned by the vectors $r_u(u_i, v_i)\Delta u_i$ and $r_v(u_i, v_i)\Delta v_i$ as $\Delta u_i, \Delta v_i \to 0$. The area of the parallelogram is given by

$$\|r_u \times r_v\| \Delta u \Delta v$$

Using the Riemann sum approximations, we have

$$\text{Area}(S) = \iint\limits_D \|r_u \times r_v\| \, du \, dv$$

Note that

$$\|r_u \times r_v\|^2 = \|r_u\|^2 \|r_v\|^2 \sin^2\theta = \|r_u\|^2\|r_v\|^2 - \|r_u\|^2\|r_v\|^2 \cos^2\theta = \|r_u\|^2\|r_v\|^2 - (r_u, r_v)^2 = EG - F^2$$

where $E := \|r_u\|^2$, $F := (r_u, r_v)$, and $G := \|r_v\|^2$, so

$$\text{Area}(S) = \iint\limits_D \sqrt{EG - F^2} \, du \, dv$$

Remark 10.2 *The area does not depend on the parametrization.*

10.3 Surface Integrals of Scalar Fields

Let $S = \{r(u, v),\ (u, v) \in \overline{D}\}$ be a continuously differentiable surface in $\mathbb{R}^3$ and let f be a real-valued function defined on S.

Definition 10.5 *The integral of f over S is denoted by and defined as*

$$\iint\limits_S f \, dS = \iint\limits_D f\big(x(u, v), y(u, v), z(u, v)\big) \sqrt{EG - F^2} \, du \, dv$$

Remark 10.3 *A physical interpretation of the integral of f over S for non-negative f is the mass of the surface S with density f.*

Lemma 10.1 *The definition of $\iint\limits_S f \, dS$ is independent of the parametrization of S.*

Example *We compute*

$$I = \iint\limits_S \frac{dS}{\sqrt{x^2 + y^2 + z^2}}$$

where S is the lateral surface of the cylinder

$$\begin{cases} x = a\cos u \\ y = a \sin u \\ z = v \end{cases} \qquad 0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant H$$

We first compute $\sqrt{EG - F^2} = a$ and thus

$$I = \int_0^{2\pi} \int_0^H \frac{a}{\sqrt{a^2 + v^2}} \, du \, dv = 2\pi a \ln\left(\frac{H + \sqrt{a^2 + H^2}}{a}\right)$$

11 Surface Integrals of Vector Fields (Lecture Notes)

11.1 Flux Across a Surface

Suppose there is a steady flow of liquid in a domain G and that $x \mapsto \vec{F}(x)$ is the velocity field of this flow. Let S be a smooth surface in G and let $x \mapsto \vec{n}(x)$ be a field of normal vectors to S. We want to determine the volume of fluid that flows across the surface S per unit time in the direction indicated by the orienting field of normal vectors to the surface.

We remark that if the velocity field is constant, then the flow per unit time across a parallelogram Π is equal to the volume of the parallelepiped defined by the vectors $\vec{F}$, $\vec{\xi}_1$, and $\vec{\xi}_2$. This volume is given by $\vec{F} \cdot (\vec{\xi}_1 \times \vec{\xi}_2) = (\vec{F}, \vec{\xi}_1, \vec{\xi}_2)$, which is the triple product of the vectors $\vec{F}$, $\vec{\xi}_1$, and $\vec{\xi}_2$. If the orientation is opposite to the direction of $\vec{F}$, then the flow is $-\vec{F} \cdot (\vec{\xi}_1 \times \vec{\xi}_2)$.

Now take the surface S with smooth parametrization

$$S = \{r = r(u,v) : (u,v) \in D\}$$

In order to define the flux across S, we fix a partition $\{D_i\}$ of D and approximate the image $r(D_i)$ by the parallelogram spanned by $\vec{\xi}_1 = r_u(u_i, v_i)\Delta u_i$ and $\vec{\xi}_2 = r_v(u_i, v_i)\Delta v_i$. Assume that $\vec{F}(x)$ varies by only small amounts inside $r(D_i)$ such that, replacing $r(D_i)$ by this parallelogram, we may assume that the flux $\Delta\mathcal{F}_i$ across the piece $r(D_i)$ of the surface is approximately equal to the flux of a constant field $\vec{F}(x_i, y_i, z_i) = \vec{F}\big(r(u_i, v_i)\big)$ across this parallelogram spanned by $\vec{\xi}_1$ and $\vec{\xi}_2$. So

$$\Delta\mathcal{F}_i \approx \big(\vec{F}(x_i, y_i, z_i), \vec{\xi}_1, \vec{\xi}_2\big) = \Big(\vec{F}\big(r(u_i, v_i)\big), \vec{r}_u(u_i, v_i), \vec{r}_v(u_i, v_i)\Big)\, \Delta u_i \Delta v_i$$

Summing all these elementary fluxes, we obtain

$$\mathcal{F} = \sum_i \Delta\mathcal{F}_i \approx \sum_i \Big(\vec{F}\big(r(u_i, v_i)\big), \vec{r}_u(u_i, v_i), \vec{r}_v(u_i, v_i)\Big)\, \Delta u_i \Delta v_i$$

Hence we can define

$$\mathcal{F} = \iint_D \vec{F}\big(r(u,v)\big) \cdot \big(\vec{r}_u(u,v) \times \vec{r}_v(u,v)\big)\, du\, dv$$

to be the flux of $\vec{F}$ across S in the direction $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$.

Remark 11.1 *Using the definition of the surface integral of a scalar field, we have*

$$\mathcal{F} = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv = \iint_S \big(\vec{F} \cdot \vec{n}\big)\, dS$$

11.2 Definition of the Surface Integral of a Vector Field

Let $S = \{r = r(u, v), (u, v) \in \overline{D}\}$ be a smooth (differentiable) surface in $\mathbb{R}^3$.

- S is orientable if the unit normal $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ is continuous in D.
- If $\vec{n}$ is a fixed continuous unit normal to S on D, then we say that S is oriented by the normal $\vec{n}$.

So, let S be a smooth surface oriented by a unit normal $\vec{n}$ and let $\vec{F} = (P, Q, R)$ be a vector field defined on S.

Definition 11.1 *The integral of $\vec{F}$ over S is denoted by and defined as*

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S \left(\vec{F} \cdot \vec{n}\right) dS$$

where the right-hand side is the surface integral of the scalar field $\vec{F} \cdot \vec{n}$ over S.

Remark 11.2 *If S is oriented by the normal $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, then by Def. 10.5*

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S \left(\vec{F} \cdot \vec{n}\right) dS = \iint\limits_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \|\vec{r}_u \times \vec{r}_v\| \, du \, dv = \iint\limits_D \vec{F} \cdot \left(\vec{r}_u \times \vec{r}_v\right) du \, dv$$

$$= \iint\limits_D \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du \, dv$$

Remark 11.3 *The identity*

$$\begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = P \frac{\partial(y, z)}{\partial(u, v)} + Q \frac{\partial(z, x)}{\partial(u, v)} + R \frac{\partial(x, y)}{\partial(u, v)}$$

motivates the following alternative notation for the integral of $\vec{F}$ over S when S is oriented by the normal $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$:

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy$$

If S is oriented by the normal $\vec{n} = -\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, then

$$\iint\limits_S \vec{F} \cdot dS = -\iint\limits_S P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy$$

Example *We compute*

$$I = \iint\limits_S z \, dx \, dy$$

where S is the upper part of the lateral surface of the cone $z = \sqrt{x^2 + y^2}$, $0 \leqslant z \leqslant H$ oriented outwards.

We take the parametrization

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{u^2+v^2} \end{cases} \qquad (u,v) \in D = \{(x,y) : x^2+y^2 \leqslant H^2\}$$

We note that the cross product of the vectors

$$\vec{r}_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right) \quad \text{and} \quad \vec{r}_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right)$$

points inwards, which is opposite to the orientation of S. So

$$I = -\iint\limits_D \vec{F} \cdot \left(\vec{r}_u \times \vec{r}_v\right) du\, dv = -\iint\limits_D \begin{vmatrix} 0 & 0 & \sqrt{u^2+v^2} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} du\, dv = -\iint\limits_D \sqrt{u^2+v^2}\, du\, dv$$

$$= -\int\limits_0^{2\pi} d\varphi \int\limits_0^H r^2\, dr = -\frac{2}{3}\pi H^3$$

Example *We compute*

$$I = \iint\limits_S \frac{dy\, dz}{x} + \frac{dz\, dx}{y} + \frac{dx\, dy}{z}$$

where S is part of the ellipsoid

$$\begin{cases} x = a\cos u \cos v \\ y = b \sin u \cos v \\ z = c \sin v \end{cases} \qquad \frac{\pi}{4} \leqslant u \leqslant \frac{\pi}{3}, \frac{\pi}{6} \leqslant v \leqslant \frac{\pi}{4}$$

oriented outwards. First we compute the cross product between the vectors

$$\vec{r}_u = (-a\sin u \cos v, b\cos u \cos v, 0) \quad \text{and} \quad \vec{r}_v = (-a\cos u \sin v, -b \sin u \sin v, c\cos v)$$

and notice that it is also oriented outwards. So

$$I = \iint\limits_D \begin{vmatrix} \frac{1}{a\cos u \cos v} & \frac{1}{b \sin u \cos v} & \frac{1}{c \sin v} \\ -a \sin u \cos v & b\cos u \cos v & 0 \\ -a\cos u \sin v & -b \sin u \sin v & c \cos v \end{vmatrix} du\, dv$$

$$= p \iint\limits_D \cos v\, du\, dv = p \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} du \int\limits_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos v\, dv = p\, \frac{\pi}{12}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

where $p = \dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b}$.

12 Gauss-Ostrogradskii Theorem (Lecture Notes)

Let S be a piecewise smooth surface surrounding a compact domain V in $\mathbb{R}^3$ oriented by the outgoing normal vector (positive orientation). Let $\vec{F} = (P, Q, R)$ be a smooth vector field in the closed domain $\overline{V}$.

Theorem 12.1 (Gauss-Ostrogradskii Theorem)

$$\iint\limits_S \vec{F} \cdot dS = \iiint\limits_V \nabla \cdot \vec{F} \, dx \, dy \, dz \tag{12.1}$$

Remark 12.1 *Let $B_\varepsilon(p)$ and $S_\varepsilon(p)$ denote a ball and sphere respectively, both of center $p \in \mathbb{R}^3$ and radius ε. The Gauss-Ostrogradskii theorem implies*

$$\iiint\limits_{B_\varepsilon(p)} \nabla \cdot \vec{F} \, dx \, dy \, dz = \iint\limits_{S_\varepsilon(p)} \vec{F} \cdot dS$$

Using the mean value theorem for domain integrals, we have

$$\nabla \cdot \vec{F}(\tilde{p}) \, \mathrm{Vol}\big(B_\varepsilon(p)\big) = \iint\limits_{S_\varepsilon(p)} \vec{F} \cdot dS$$

where $\tilde{p}$ is a point from $B_\varepsilon(p)$. Then, by continuity of $\nabla \cdot \vec{F}$, we have

$$\nabla \cdot \vec{F}(p) = \lim_{\epsilon \to 0} \frac{1}{\mathrm{Vol}\big(B_\varepsilon(p)\big)} \iint\limits_{S_\varepsilon(p)} \vec{F} \cdot dS \tag{12.2}$$

In particular, $\nabla \cdot \vec{F}$ is independent of the choice of coordinate system although it is defined as the sum of partial derivatives with respect to a fixed Cartesian coordinate system.

Remark 12.2 *The fraction in the right-hand side of (12.2) can be interpreted as the mean intensity per unit volume of sources in the ball $B_\varepsilon(p)$, that is, $\nabla \cdot \vec{F}(p)$ is the specific intensity per unit volume of the source or sink at the point p.*

Remark 12.3 *If $\nabla \cdot \vec{F}$ is positive for $p \in \mathbb{R}^3$, then p is a source. If $\nabla \cdot \vec{F}$ is negative, then p is a sink. The Gauss-Ostrogradskii theorem states that the flux of $\vec{F}$ across S equals the 'sum' of all flows from sources in V minus the 'sum' of all flows to sinks in V.*

Corollary 12.1 *If V is a connected set whose boundary consists of piecewise smooth surfaces $S, S_1, \ldots, S_k$ (here S is the outer boundary and $S_1, \ldots, S_k$ are boundaries of holes in V) all oriented by outgoing normals, then*

$$\iiint\limits_V \nabla \cdot \vec{F} \, dx \, dy \, dz = \iint\limits_S \vec{F} \cdot dS + \sum_{i=1}^{k} \iint\limits_{S_i} \vec{F} \cdot dS$$

Example *Let S_1 be the lateral surface of the cone $x^2+y^2 \leqslant z^2 \leqslant 1$ and let S_2 be the upper surface such that the whole surface of the cone is $S = S_1 \cup S_2$ which is oriented outwards. We take $\vec{F} = (x^3, y^3, z^3)$. In order to compute*

$$\iint_{S_1} \vec{F} \cdot dS$$

we use (12.1). Here $V = \{(x,y,z) : x^2+y^2 \leqslant z^2 \leqslant 1\}$. Then, using cylindrical coordinates, we calculate

$$\iint_{S} \vec{F} \cdot dS = \iiint_{V} \nabla \cdot \vec{F} \, dx \, dy \, dz = 3 \iiint_{V} (x^2+y^2+z^2) \, dx \, dy \, dz = \frac{9\pi}{10}$$

Hence

$$\iint_{S_1} \vec{F} \cdot dS = \frac{9\pi}{10} - \iint_{S_2} \vec{F} \cdot dS$$

The parametrization of S_2 is

$$\begin{cases} x = u \\ y = v \\ z = 1 \end{cases} \qquad (u,v) \in D = \{(x,y) : x^2+y^2 \leqslant 1\}$$

so the normal vector is $\vec{n} = (0,0,1)$. Therefore

$$\iint_{S_2} \vec{F} \cdot dS = \iint_{S_2} z^3 \, dS = \iint_{S_2} dS = \pi$$

and thus

$$\iint_{S_1} \vec{F} \cdot dS = \frac{9\pi}{10} - \pi = -\frac{\pi}{10}$$

13 Stokes' Theorem (Lecture Notes)

13.1 Stokes' Theorem

Let S be a piecewise smooth surface in $\mathbb{R}^3$ oriented by a unit normal $\vec{n}$ and let γ be the positively oriented boundary of S with respect to the normal $\vec{n}$. Let $\vec{F}$ be a continuously differentiable vector field on S.

Theorem 13.1 (Stokes' Theorem)

$$\int_\gamma \vec{F} \cdot ds = \iint_S (\nabla \times \vec{F}) \cdot dS$$

Remark 13.1

1. *Let S be parametrized by $\{r(u,v),(u,v) \in \overline{D}\}$ and let Γ be the positively oriented boundary of $D \subseteq \mathbb{R}^2$. Then $\gamma = r(\Gamma)$ is positively oriented with respect to the normal $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$. If $\vec{F} = (P, Q, R)$, then Stokes' theorem can be equivalently stated as*

$$\int_\gamma P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

2. *If $\vec{n} = (n_x, n_y, n_z)$, then*

$$\iint_S (\nabla \times \vec{F}) \cdot dS = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\,dS = \iint_S \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

Example *Let γ be the curve describing the intersection between the paraboloid $x^2 + y^2 + z = 3$ and the plane $x + y + z = 2$ oriented positively with respect to the vector $(1,1,1)$. Let S be the surface in the plane spanned by γ oriented by the unit normal $\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Note that γ is positively oriented with respect to $\vec{n}$. We want to find*

$$I = \int_\gamma (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz$$

First, for $P = y^2 - z^2$, $Q = z^2 - x^2$, $R = x^2 - y^2$, we compute

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = -2(y+z)$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = -2(x+z)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2(x+y)$$

Thus by Stokes' theorem

$$I = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\,dS = -\frac{4}{\sqrt{3}} \iint_S (x+y+z)\,dS$$

Since S is a subset of the plane $x+y+z=2$, we have

$$I = -\frac{8}{\sqrt{3}} \iint\limits_S dS$$

The surface S can be parametrized as $z = 2-x-y$, $(x,y) \in \overline{D}$. We calculate

$$\sqrt{EG-F^2} = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

Therefore

$$I = -8\,\mathrm{Area}(D)$$

The boundary of D is the projection of γ onto the xy-plane. To find its equation, we eliminate z from the system of equations

$$\begin{cases} x^2+y^2+z=3 \\ x+y+z=2 \end{cases}$$

and obtain $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{3}{2}$ which is a circle of radius $\sqrt{\frac{3}{2}}$. Thus

$$I = -12\pi$$

13.2 Physical Meaning of the Curl

Suppose that the entire space, regarded as a rigid body, is rotating with constant angular velocity ω about the z-axis. Let us find the curl of the vector field $\vec{F}$ of linear velocities of all points in space. In cylindrical coordinates, we have $\vec{F}(r,\varphi,z) = \omega r\,\vec{e}_\varphi$. Calculating the curl, we find $\nabla\times\vec{F} = 2\omega\,\vec{e}_z$, where $\vec{e}_z = (0,0,1)$. That is, $\nabla\times\vec{F}$ is a vector directed along the axis of rotation. The magnitude of $\nabla\times\vec{F}$ is equivalent to the angular velocity up to a factor of 2 and its direction determines the direction of rotation.

Locally, the curl of a vector field at some point characterizes the degree of vorticity of the field in a neighborhood of that point. Let $\vec{n}$ be a unit vector and let γ_ϵ be a circle of radius ϵ centered at $p\in\mathbb{R}^3$, lying in the plane perpendicular to $\vec{n}$ and positively oriented with respect to $\vec{n}$. Then the projection of $\nabla\times\vec{F}$ onto $\vec{n}$ can be computed using Stokes' theorem:

$$\left(\nabla\times\vec{F}(p)\right)\cdot\vec{n} = \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon^2}\int\limits_{\gamma_\epsilon}\vec{F}\cdot ds$$

where $\int\limits_{\gamma_\epsilon}\vec{F}\cdot ds$ is the circulation of $\vec{F}$ along γ. The value of $\left(\nabla\times\vec{F}\right)\cdot\vec{n}$ is maximal in the direction of $\vec{n}$, coinciding with the direction of $\nabla\times\vec{F}$.

13.3 Solenoidal Vector Fields

Definition 13.1 *A vector field $\vec{F}$ in $\mathbb{R}^3$ is solenoidal or divergence-free in $V\subseteq\mathbb{R}^3$ if $\nabla\cdot\vec{F}=0$ in V.*

Proposition 13.1 *Let V be a simply connected domain in $\mathbb{R}^3$ and let $\vec{F}$ be a smooth vector field on $\overline{V}$. Then $\vec{F}$ is solenoidal in V if and only if for any solid $\tilde{V}\subset V$ with smooth boundary $\tilde{S}$, the flux of $\vec{F}$ through $\tilde{S}$ is zero.*

14 Holomorphic Functions (Lecture Notes)

14.1 Basic Notions

For the complex number $z = x + yi \in \mathbb{C}$ we define its

1. Real part: $\operatorname{Re} z = x$
2. Imaginary part: $\operatorname{Im} z = y$
3. Complex conjugate: $\overline{z} = x - yi$
4. Absolute value: $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$

We will denote by

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} = \{z = a + bi : (x - x_0)^2 + (y - y_0)^2 < r\}$$

the open ball in $\mathbb{C}$ with center z_0 and radius r. We call a set $U \subseteq \mathbb{C}$ open if

$$\forall\, z_0 \in U \quad \exists\, r > 0 : B_r(z_0) \subseteq U$$

14.2 Differentiable Functions

We will consider functions from $\mathbb{C}$ to $\mathbb{C}$. Let U be an open subset of $\mathbb{C}$ and let $f : U \mapsto \mathbb{C}$ be a complex function.

Definition 14.1

1. *If the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists, that is

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 : \forall\, z \in B_\delta(z_0),\; z \neq z_0,\; \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

then f is called complex differentiable at $z_0 \in U$ and $f'(z_0)$ is called the derivative of f at z_0.

2. *If f is complex differentiable for every $z_0 \in U$, we say that f is holomorphic in U.*

3. *We say that f is holomorphic at z_0 if f is complex differentiable in a neighborhood of z_0 (some open set U_0 containing z_0).*

Example *We check that $f(z) = z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$, $z = x + yi \in \mathbb{C}$ is differentiable on $\mathbb{C}$:*

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0$$

The limit indeed exists and $f'(z) = 2z$.

Example *We want to show that the function* $f(z) = |z|^2 = x^2 + y^2$, $z = x + yi \in \mathbb{C}$ *is not differentiable at* $z_0 = 1$. *We first consider*

$$\lim_{\epsilon \to 0} \frac{|1+\epsilon|^2 - 1}{\epsilon} = \lim_{\epsilon \to 0} \frac{2\epsilon + \epsilon^2}{\epsilon} = 2$$

Next we consider

$$\lim_{\epsilon \to 0} \frac{|1+i\epsilon|^2 - 1}{i\epsilon} = \lim_{\epsilon \to 0} \frac{1 + \epsilon^2 - 1}{i\epsilon} = 0$$

This shows that $f'(1)$ *does not exist.*

Proposition 14.1 *If* f *is differentiable at* z_0 *then* f *is continuous at* z_0.

Proposition 14.2

1. *If* f *and* g *are holomorphic on* U, *then* $f \pm g$, fg, *and* $\frac{f}{g}$, $g \neq 0$ *are holomorphic on* U *and*
   - $(f \pm g)' = f' \pm g'$
   - $(fg)' = f'g + fg'$
   - $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
2. *If* $f : U \mapsto V$ *and* $g : V \mapsto \mathbb{C}$, *where* U *and* V *are open sets, are holomorphic, then* $g \circ f$ *is holomorphic and* $(g \circ f)'(z) = g'\big(f(z)\big) f'(z)$.

14.3 Cauchy-Riemann Equations

Let every $z \in \mathbb{C}$ correspond to an ordered pair (x, y):

$$\mathbb{C} \ni z = x + yi \leftrightarrow (x, y) \in \mathbb{R}^2$$

Then a complex function $w = f(z)$ similarly corresponds to the functions $u = u(x, y) = \operatorname{Re} f(z)$ and $v = v(x, y) = \operatorname{Im} f(z)$, that is $f(z) = u(x, y) + iv(x, y)$.

Theorem 14.1 (Cauchy-Riemann) *For a function* $f = u + iv : U \mapsto \mathbb{C}$, *where* $U \subseteq \mathbb{C}$ *is open, and a point* $z_0 = x_0 + iy_0 \in U$ *the following statements are equivalent:*

1. f *is complex differentiable at* z_0.
2. u, v *are real differentiable at* (x_0, y_0) *and the Cauchy-Riemann equations are satisfied:*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

If f *is complex differentiable, then* $f'(z_0) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) - i\dfrac{\partial u}{\partial y}(x_0, y_0)$.

15 Properties of Holomorphic Functions (Lecture Notes)

15.1 Properties of Holomorphic Functions

Let U be an open subset of $\mathbb{C}$. If $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy \in U$ is a function from U to $\mathbb{C}$, then it is called locally constant in U if for every $z_0 \in U$ there exists a ball $B_r(z_0) \subset U$ such that f is constant on $B_r(z_0)$. We remark that if f is locally constant, it is constant on each connected component of U.

Lemma 15.1 *Let U be open in $\mathbb{C}$ and let $f : U \mapsto \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$ be holomorphic on U.*

1. *If $f'(z) = 0$ for all $z \in U$, then f is locally constant.*
2. *If f only takes real values, then f is locally constant.*
3. *The functions u, v are harmonic, i.e. $\triangle u = 0$, $\triangle v = 0$ on U.*

Lemma 15.2 *If $u : U \mapsto \mathbb{R}$ is harmonic on a simply connected domain U in $\mathbb{C}$, then there exists a holomorphic function $f : U \mapsto \mathbb{C}$ such that $u = \operatorname{Re} f$.*

15.2 Some Elementary Functions

1. Power Function
   The function
   $$f(z) = z^n,\ z \in \mathbb{C},\ n \in \mathbb{N}$$
   is holomorphic. This follows from Prop. 14.2. Its derivative is
   $$f'(z) = nz^{n-1}$$
   If we write $z = r(\cos\varphi + i\sin\varphi)$ in polar coordinates, then by de Moivre's formula
   $$z^n = r^n(\cos n\varphi + i\sin n\varphi)$$
   Hence, if $z_1, z_2 \in \mathbb{C}$ are such that $|z_1| = |z_2|$ and $\arg z_1 = \arg z_2 + k\frac{2\pi}{n}$, then $z_1^n = z_2^n$. This implies that f is not bijective on $\mathbb{C}$. However, it is bijective from $D = \{z : 0 < \arg z < \frac{2\pi}{n}\}$ to $\mathbb{C} \setminus \{z = x + iy : x > 0\}$.

2. Exponential Function
   We define the function
   $$e^z := \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$$
   To show that the limit exists for any $z = x + iy \in \mathbb{C}$, we calculate
   $$r_n = \left|\left(1 + \frac{z}{n}\right)^n\right| = \left|1 + \frac{z}{n}\right|^n = \left|1 + \frac{x+iy}{n}\right| = \left(\sqrt{\left(1 + \frac{x}{n}\right)^2 + \frac{y^2}{n^2}}\right)^n = \left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}}$$
   $$\lim_{n\to\infty} \left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}} = e^{\lim_{n\to\infty} \frac{n}{2} \ln\left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)}$$
   $$\lim_{n\to\infty} \frac{n}{2} \ln\left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right) = \frac{n}{2}\left(\frac{2x}{n} + \frac{x^2+y^2}{n^2}\right) \frac{\ln\left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)}{\left(\frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)} = x$$

Thus $\lim_{n\to\infty} r_n = e^x$. Next we calculate

$$\varphi_n = \arg\left(1+\frac{z}{n}\right)^n = n\arg\left(1+\frac{z}{n}\right) = n\arg\left(1+\frac{x+iy}{n}\right) = n\arctan\left(\frac{\frac{y}{n}}{1+\frac{x}{n}}\right)$$

$$\lim_{n\to\infty} n\arctan\left(\frac{\frac{y}{n}}{1+\frac{x}{n}}\right) = \lim_{n\to\infty} n\frac{\frac{y}{n}}{1+\frac{x}{n}}\arctan\left(\frac{\frac{y}{n}}{1+\frac{x}{n}}\right)\frac{1+\frac{x}{n}}{\frac{y}{n}} = \lim_{n\to\infty}\frac{y}{1+\frac{x}{n}} = y$$

and find $\lim_{n\to\infty}\varphi_n = y$. Thus

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

In particular, we obtain the Euler formula:

$$e^{iy} = \cos y + i\sin y$$

3. Trigonometric Functions

Using $e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos y - i\sin y$ for all $y \in \mathbb{R}$, we obtain

$$\cos y = \frac{e^{iy}+e^{-iy}}{2} \qquad \sin y = \frac{e^{iy}-e^{-iy}}{2i}$$

with which we can define the trigonometric functions for $z \in \mathbb{C}$:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

These complex trigonometric functions are closely related to the hyperbolic trigonometric functions:

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

So

$$\cosh z = \cos iz$$

$$\sinh z = -i\sin iz$$

$$\cos z = \cosh iz$$

$$\sin z = -i\sinh iz$$

16 Conformal Maps (Lecture Notes)

16.1 Geometric Meaning of $\arg f'(z)$ and $|f'(z)|$

Let $\gamma(t) = x(t) + iy(t)$, $t \in [\alpha, \beta]$ be a continuous path in $\mathbb{C}$ that is also continuously differentiable. We take a function $f : U \mapsto \mathbb{C}$ such that $f'(z_0) \neq 0$. We denote $w_0 = f(z_0)$ and assume $\gamma(t_0) = z_0$. We set

$$l_z = \frac{z - z_0}{|z - z_0|} = \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|}$$

Then

$$l_{z_0} = \lim_{z \to z_0} \frac{z - z_0}{|z - z_0|} = \frac{\gamma'(t)}{|\gamma'(t)|}$$

can be identified as the unit tangent vector to γ at z_0. Next, we consider the image of γ under the map f. We find the tangent vector to $f(\gamma)$ at w_0:

$$L_{z_0} = \lim_{w \to w_0} \frac{w - w_0}{|w - w_0|} = \lim_{t \to t_0} \frac{f\big(\gamma(t)\big) - f\big(\gamma(t_0)\big)}{|f\big(\gamma(t)\big) - f\big(\gamma(t_0)\big)|}$$

$$= \lim_{t \to t_0} \frac{f\big(\gamma(t)\big) - f\big(\gamma(t_0)\big)}{\gamma(t) - \gamma(t_0)} \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|} \frac{|\gamma(t) - \gamma(t_0)|}{|f\big(\gamma(t)\big) - f\big(\gamma(t_0)\big)|}$$

$$= f'(z_0) \cdot l_{z_0} \cdot \frac{1}{|f'(z_0)|} = \frac{f'(z_0)}{|f'(z_0)|} l_{z_0}$$

Next, we compute

$$\arg L_{z_0} = \arg \frac{f'(z_0)}{|f'(z_0)|} l_{z_0} = \arg f'(z_0) + \arg l_{z_0} - \arg |f'(z_0)| = \arg f'(z_0) + \arg l_{z_0}$$

We see that under the map f, a tangent line to any curve at z_0 is rotated on the angle $\arg f'(z_0)$.

Let us now consider two paths γ_1 and γ_2 that pass through z_0. The angle between these two paths at z_0 is defined as the angle φ between their tangent vectors l_1 and l_2 at that point. Then the angle ψ between the tangent vectors L_1 and L_2 of the images of γ_1 and γ_2 is given by

$$\psi = \arg L_2 - \arg L_1 = \arg f'(z_0) + \arg l_2 - \arg f'(z_0) - \arg l_1 = \arg l_2 - \arg l_1 = \varphi$$

Corollary 16.1 *If $f'(z_0) \neq 0$, then f preserves the angles between curves which pass through z_0.*

Definition 16.1

1. *A continuous map $f : U \mapsto \mathbb{C}$ which preserves the angles between curves that pass through $z_0 \in U$ is called conformal at z_0.*

2. *If f is conformal at any point of U, then f is called conformal on U.*

Theorem 16.1 *A holomorphic function f is conformal at any point where its derivative is non-zero.*

Let us now interpret the meaning of $|f'(z_0)|$. We write

$$|f'(z_0)| = \lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \lim_{z \to z_0} \frac{|w - w_0|}{|z - z_0|}$$

So $|f'(z_0)|$ is equal to the dilation coefficient at z_0 under the map f.

16.2 Fractional Linear Transformations

Fractional linear transformations are functions of the form

$$w = \frac{az+b}{cz+d},\ ad-bc \neq 0$$

where a, b, c, d are fixed complex numbers and z is the complex variable. The condition $ad - bc \neq 0$ is imposed to exclude the degenerate case where w is constant. This function is defined for all $z \neq -\frac{d}{c}$ if $c \neq 0$. We set $w = \infty$ at $z = -\frac{d}{c}$.

Theorem 16.2 *A fractional linear transformation is a homeomorphism, that is, it is a continuous bijective map, from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*

Definition 16.2 *Let γ_1 and γ_2 be two paths that pass through the point $z = \infty$. The angle between γ_1 and γ_2 at $z = \infty$ is the angle between their images Γ_1 and Γ_2 under the map $z \mapsto \frac{1}{z}$ at the point 0.*

Theorem 16.3 *A fractional linear map is conformal on $\overline{\mathbb{C}}$.*

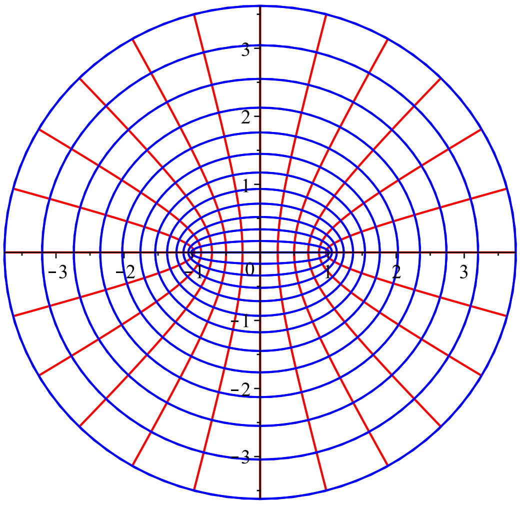
16.3 Geometric Properties

We first introduce the convention that a circle in $\overline{\mathbb{C}}$ is either a circle or a straight line on the complex plane $\mathbb{C}$.

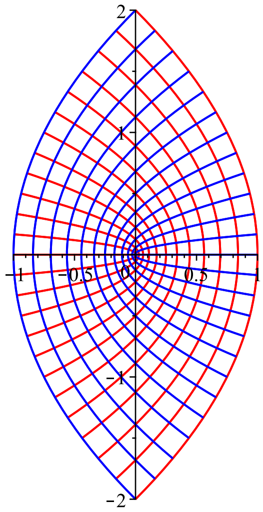
Theorem 16.4 *Fractional linear transformations map a circle in $\overline{\mathbb{C}}$ to a circle in $\overline{\mathbb{C}}$.*

Remark 16.1 *Let l be a circle in $\overline{\mathbb{C}}$ and let L be its image under a fractional linear transformation.*

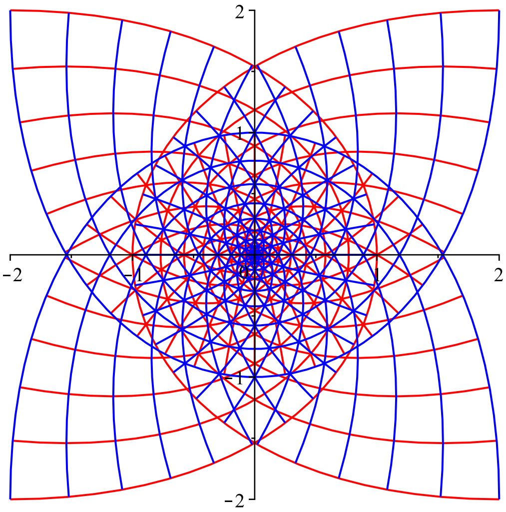
1. $-\dfrac{d}{c} \in l$ *is equivalent to L being a straight line.*
2. $\dfrac{a}{c} \in L$ *is equivalent to l being a straight line.*



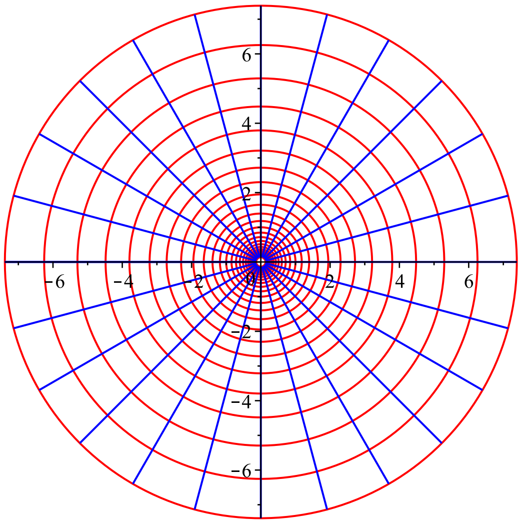
$f(z) = \sin z$



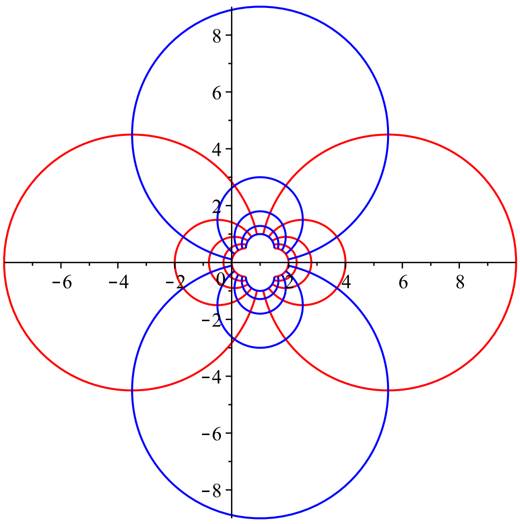
$f(z) = z^2$



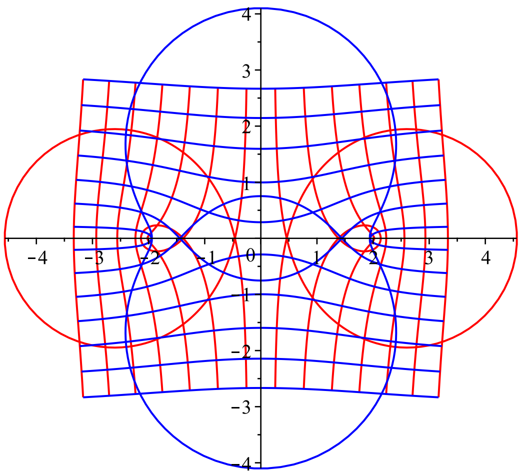
$f(z) = z^3$



$f(z) = e^z$



$f(z) = \dfrac{z}{z-1}$



$f(z) = z + \dfrac{1}{z}$

17 Cauchy's Theorem (Lecture Notes)

17.1 Integration

Let U be an open set in $\mathbb{C}$. Let γ be a piecewise continuously differentiable path in U and take a continuous function $f : \gamma \mapsto \mathbb{C}$.

Definition 17.1 *If there exists*

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k) \Delta z_k =: \int\limits_{\gamma} f(z)\, dz$$

where $\lambda = \max\limits_{k} |\Delta z_k|$ *and* $\Delta z_k = z_k - z_{k-1}$*, that does not depend on the choice of the points* $\{\xi_k\}$ *and partition* $\{z_k\}$*, then this limit is called the integral of* f *along* γ*.*

To make a connection between this integral and the known line integral, we rewrite the integral sum. First, we rewrite $\Delta z_k = \Delta x_k + i\Delta y_k$ and take $f(z) = u(x, y) + iv(x, y)$. Let $\xi_k = \eta_k + i\zeta_k$. Then

$$\sum_{k=1}^{n} f(\xi_k)\Delta z_k = \sum_{k=1}^{n} \big(u(\eta_k, \zeta_k) + iv(\eta_k, \zeta_k)\big)\big(\Delta x_k + i\Delta y_k\big)$$

$$= \sum_{k=1}^{n} \big(u(\eta_k, \zeta_k)\Delta x_k - v(\eta_k, \zeta_k)\Delta y_k\big) + i\sum_{k=1}^{n} \big(v(\eta_k, \zeta_k)\Delta x_k + u(\eta_k, \zeta_k)\Delta y_k\big)$$

This immediately implies

$$\int\limits_{\gamma} f(z)\, dz = \int\limits_{\gamma} u(x, y)\, dx - v(x, y)\, dy + i\int\limits_{\gamma} v(x, y)\, dx + u(x, y)\, dy$$

17.2 Properties of the Integral

1. For $\alpha, \beta \in \mathbb{C}$
$$\int\limits_{\gamma} (\alpha f + \beta g)\, dz = \alpha\int\limits_{\gamma} f\, dz + \beta\int\limits_{\gamma} g\, dz$$

2. If γ^- is obtained from γ by a change in orientation, then
$$\int\limits_{\gamma} f\, dz = -\int\limits_{\gamma^-} f\, dz$$

3. If $\gamma_1 \cup \gamma_2$ is a path such that the end point of γ_1 is the initial point of γ_2, then
$$\int\limits_{\gamma_1\cup\gamma_2} f\, dz = \int\limits_{\gamma_1} f\, dz + \int\limits_{\gamma_2} f\, dz$$

4. From the inequality

$$\left|\sum_{k=1}^{n} f(\xi_k)\Delta z_k\right| \leqslant \sum_{k=1}^{n} |f(\xi_k)||\Delta z_k| \leqslant \sum_{k=1}^{n} |f(\xi_k)|\Delta s_k$$

where Δs_k is the length of γ between z_{k-1} and z_k, follows

$$\left|\int_{\gamma} f(z)\,dz\right| \leqslant \int_{\gamma} |f(z)|\,ds$$

5. Let $L(\gamma)$ be the length of γ. Then

$$\left|\int_{\gamma} f(z)\,dz\right| \leqslant \max_{z\in\gamma} |f(z)|\,L(\gamma)$$

follows from the previous result

Now let $\gamma(t) = x(t) + iy(t)$, $t \in [\alpha, \beta]$. Then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma} u(x,y)\,dx - v(x,y)\,dy + i\int_{\gamma} v(x,y)\,dx + u(x,y)\,dy$$

$$= \int_{\alpha}^{\beta} \big[u\big(x(t),y(t)\big)x'(t) - v\big(x(t),y(t)\big)y'(t)\big]\,dt + i\int_{\alpha}^{\beta} \big[v\big(x(t),y(t)\big)x'(t) + u\big(x(t),y(t)\big)y'(t)\big]\,dt$$

$$= \int_{\alpha}^{\beta} \big[u\big(x(t),y(t)\big) + iv\big(x(t),y(t)\big)\big]\big[x'(t) + iy'(t)\big]\,dt$$

$$= \int_{\alpha}^{\beta} f\big(\gamma(t)\big)\gamma'(t)\,dt$$

Consequently, we have obtained

$$\int_{\gamma} f(z)\,dz = \int_{\alpha}^{\beta} f\big(\gamma(t)\big)\gamma'(t)\,dt$$

where $\gamma = \gamma(t)$, $t \in [\alpha, \beta]$.

Example *Let* $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$. *We compute*

$$\int\limits_{|z-a|=R} \frac{dz}{(z-a)^n} = \int\limits_{\gamma} \frac{dz}{(z-a)^n}$$

where $n \in \mathbb{Z}$. *If* $n \neq 1$, *then*

$$\int\limits_{|z-a|=R} \frac{dz}{(z-a)^n} = \int\limits_0^{2\pi} \frac{iRe^{it}}{R^n e^{int}}\,dt = \frac{i}{R^{n-1}} \int\limits_0^{2\pi} e^{-i(n-1)t}\,dt = -\frac{1}{R^{n-1}}\frac{1}{(n-1)} e^{-i(n-1)t}\Big|_0^{2\pi} = 0$$

using $e^{-i(n-1)t} = \cos\big((n-1)t\big) - i\sin\big((n-1)t\big)$. *If* $n = 1$, *then*

$$\int\limits_{|z-a|=R} \frac{dz}{z-a} = i\int\limits_0^{2\pi} dt = 2\pi i$$

Hence

$$\int\limits_{|z-a|=R} \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

17.3 Cauchy's Theorem

Proposition 17.1 *Suppose that a function* $F : U \mapsto \mathbb{C}$ *is holomorphic and is an antiderivative of a continuous function* $f : U \mapsto \mathbb{C}$ *such that* $F'(z) = f(z)$. *Then for any piecewise continuously differentiable path* γ *joining* z_1 *and* z_2 *in* U

$$\int\limits_{\gamma} f(z)\,dz = F(z_2) - F(z_1)$$

Moreover, if γ *is closed in* U, *then*

$$\oint\limits_{\gamma} f(z)\,dz = 0$$

Example

$$\int\limits_{2+3i}^{1-i} z^3\,dz = \frac{z^4}{4}\Big|_{2+3i}^{1-i} = \frac{(1-i)^4 - (2+3i)^4}{4}$$

Theorem 17.1 (Cauchy's Theorem) *Let* U *be a simply connected domain in* $\mathbb{C}$ *and let* $f : U \mapsto \mathbb{C}$ *be a holomorphic function in* U. *Assume that the path* γ *joining* z_1 *and* z_2 *in* U *is piecewise continuously differentiable in* U. *Then*

$$\int\limits_{\gamma} f(z)\,dz$$

depends only on z_1 *and* z_2 *and not the choice of the path* γ. *In particular, if* γ *is closed, then*

$$\oint\limits_{\gamma} f(z)\,dz = 0$$

18 The Cauchy Integral Formula (Lecture Notes)

18.1 Consequences of Cauchy's Theorem

Proposition 18.1 *Any holomorphic function f in a simply connected domain U has an antiderivative in this domain.*

We say that f is holomorphic on $\overline{U}$ if there exists an open set G such that $\overline{U} \subseteq G$, and if f can be extended into this domain G, that is, there exists a holomorphic function $\tilde{f}$ on G such that $\tilde{f} = f$ on U.

Proposition 18.2 (Generalization of Cauchy's Theorem) *Let f be holomorphic on $\overline{U}$, where U is simply connected and ∂U is a piecewise continuously differentiable curve. Then*

$$\int_{\partial U} f(z)\, dz = 0$$

Definition 18.1 *Let the boundary of a bounded domain U consist of a finite number of closed curves γ_k, $k = 0, 1, \ldots, n$ which are piecewise continuously differentiable. The boundary of U for which the orientations of γ_k are positive is called the oriented boundary of U and is denoted by ∂U.*

Proposition 18.3 *Let U be a bounded domain with oriented boundary and let f be a holomorphic function on $\overline{U}$. Then*

$$\int_{\partial U} f(z)\, dz = \int_{\gamma_0} f(z)\, dz + \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz = 0$$

18.2 The Cauchy Integral Formula

Theorem 18.1 *Let f be a holomorphic function on $\overline{U}$, where U is bounded by a finite number of piecewise continuously differentiable curves. Then for every $z \in U$*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\xi)}{\xi - z} d\xi$$

Consequence 18.1 *Let f be holomorphic in U, where U is an open set. Let γ be a simple continuously differentiable curve in U surrounding a set D contained in U. Then for any $z \in D$*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi$$

18.3 Series

Let a_n, $n \geqslant 1$ be complex numbers. We say that a series $\sum\limits_{n=0}^{\infty} a_n$ is convergent if the sequence of its partial sums $S_n = \sum\limits_{k=0}^{n} a_k$ has a finite limit S. This limit is called the sum of the series. A functional series $\sum\limits_{n=0}^{\infty} f_n(z)$, where the functions f_n are defined on a set $M \subseteq \overline{\mathbb{C}}$, converges uniformly on M if

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} : \forall n \geqslant N, \forall z \in M$$

$$\left| \sum_{k=n+1}^{\infty} f_k(z) \right| = \left| f(z) - \sum_{k=0}^{n} f_k(z) \right| < \epsilon,\ f(z) = \sum_{n=0}^{\infty} f_n(z)$$

19 The Taylor Series (Lecture Notes)

19.1 Uniform Convergence of Series

Recall the definitions from 18.3.

Example *Consider the series*

$$\sum_{n=0}^{\infty} z^n$$

We remark that this series converges for every $z \in M = \{z : |z| < 1\}$. We show this by first calculating

$$S_n = \sum_{k=0}^{n} z^k = 1 + z + z^2 + \cdots + z^n$$

then

$$zS_n = z + z^2 + \cdots + z^{n+1}$$

Subtraction of these two equations yields

$$S_n = \frac{1 - z^{n+1}}{1 - z} = \frac{1 - r^{n+1}\big[\cos\big((n+1)\varphi\big) + i \sin\big((n+1)\varphi\big)\big]}{1 - z} \to S = \frac{1}{1 - z},\ n \to \infty$$

However, the series converges uniformly only on $M_\delta = \{z : |z| < 1 - \delta\}$ for any $\delta > 0$:

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \left| \frac{1}{1 - z} - \frac{1 - z^{n+1}}{1 - z} \right| = \frac{|z^{n+1}|}{|1 - z|} = \frac{|z|^{n+1}}{|1 - z|} \leqslant \frac{(1 - \delta)^{n+1}}{\delta} \to 0,\ n \to \infty \quad \forall\, z \in M_\delta$$

Assume that the series converges uniformly on M, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ and for all $|z| < 1$

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \frac{|z|^{n+1}}{|1 - z|} < \varepsilon$$

Take $z = x + 0 \cdot i = x$, $x > 0$. Then

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \frac{x^{n+1}}{1 - x} < \varepsilon$$

Notice that the above inequality does not hold for x close to 1. Consequently, the series does not uniformly converge on M.

19.2 The Taylor Series

Theorem 19.1 *Let f be a holomorphic function in U and take $z_0 \in U$. Then f can be represented as the following sum:*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

inside any disk $B_R = \{z : |z - z_0| < R\} \subset U$.

Proof: Let $z \in B_R$ be an arbitrary point. Choose $r > 0$ such that $|z - z_0| < r < R$. We denote

$$\gamma_r = \{\xi : |\xi - z_0| = r\}$$

The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\xi)}{\xi - z} d\xi$$

We write

$$\frac{1}{\xi - z} = \left[(\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)\right]^{-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}$$

then multiply both sides by $\frac{1}{2\pi i} f(\xi)$ and integrate term-wise along γ_r:

$$\int\limits_{\gamma_r} \frac{1}{2\pi i} \frac{f(\xi)\, d\xi}{\xi - z} = \int\limits_{\gamma_r} \sum_{n=0}^{\infty} \frac{1}{2\pi i} \frac{f(\xi)(z - z_0)^n\, d\xi}{(\xi - z_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

The above series converges uniformly since

$$\left| \frac{z - z_0}{\xi - z_0} \right| = \frac{|z - z_0|}{r} = q < 1$$

and

$$\sum_{n=0}^{\infty} q^n < \infty$$

Consequently, the term-wise integration is legitimate and we obtain

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}},\ n = 0, 1, 2, \ldots$$

□

Definition 19.1 *The power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$$

is the Taylor series of the function f at the point z_0.

If the function f is holomorphic in a closed disk

$$\overline{B}_r = \{z : |z - z_0| \leqslant r\}$$

and its absolute value on the circle $\gamma_r = \partial \overline{B}_r$ is bounded by a constant M, then we have the Cauchy inequality:

$$|c_n| \leqslant \frac{M}{r^n},\ n = 0, 1, \ldots$$

Theorem 19.2 (Liouville's Theorem) *If the function f is holomorphic and bounded in the whole complex plane, then it is constant.*

20 Taylor Series and Further Properties of Holomorphic Functions (Lecture Notes)

20.1 Differentiability of the Taylor Series

We recall that any function f that is holomorphic on U can be expanded into a Taylor series, that is, it can be expressed as a sum:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in any disk $B_R = \{z : |z - z_0| < R\} \subset U$ for some $z_0 \in U$. The coefficients are given by

$$c_n = \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$$

where $\gamma_r = \{z : |z - z_0| = r\}$.

Remark 20.1 *Let γ be any simple and positively oriented path around the point z_0. Then*

$$c_n = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$$

Rem. 20.1 follows from Prop. 18.3. We will now discuss the radius of convergence of the power series. We will assume further that c_n, $n \geqslant 0$ are any complex numbers.

Theorem 20.1 (Cauchy-Hadamard Formula) *Let the coefficients of the power series*

$$\sum_{n=0}^{\infty} c_n (z - a)^n \tag{20.1}$$

satisfy

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \frac{1}{R}$$

where $0 \leqslant R \leqslant +\infty$. Then (20.1) converges at all z such that $|z - a| < R$ and diverges at all z such that $|z - a| > R$.

Th. 20.1 implies that the set $B_R = \{z : |z - a| < R\}$ is the domain of convergence of (20.1).

Theorem 20.2 *The sum of a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

is holomorphic in its domain of convergence. Moreover

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z - a)^n$$

20.2 Properties of Holomorphic Functions

Theorem 20.3 *If f is holomorphic in an open subset $U \subseteq \mathbb{C}$, then f' is also holomorphic in U.*

Theorem 20.4 *Any holomorphic function f in U has derivatives of all orders in U which are also holomorphic in U.*

Theorem 20.5 *Assume that a function f can be represented by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in a disk $B_R = \{z : |z - z_0| < R\}$. Then the coefficients c_n are uniquely determined:

$$c_n = \frac{f^{(n)}(z_0)}{n!},\ n = 0, 1, \ldots$$

The Cauchy integral formula for derivatives of a holomorphic function f in U is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)\, d\xi}{(\xi - z)^{n+1}},\ n = 1, 2, \ldots$$

where γ is a simple and positively oriented path in U around z. This follows from Th. 18.1 and Th. 20.5.

20.3 Zeros of Holomorphic Functions

Definition 20.1 *A zero of a function f is a point $a \in \mathbb{C}$ such that $f(a) = 0$.*

Theorem 20.6 *Let a point $a \in \mathbb{C}$ be a zero of a function f that is holomorphic at a. Assume that f is not equal to zero in a neighborhood of a. Then there exists $n \in \mathbb{N}$ such that*

$$f(z) = (z - a)^n \varphi(z)$$

where φ is holomorphic at a and $\varphi(z) \neq 0$ for all z in a neighborhood of a.

21 The Laurent Series (Lecture Notes)

21.1 Uniqueness of Holomorphic Functions

Theorem 21.1 (Uniqueness) *Let f_1 and f_2 be holomorphic in a connected open set $U \subset \mathbb{C}$. Then if $f_1(z) = f_2(z)$ for all $z \in E \subseteq U$, where E has a limit point in U, then $f_1(z) = f_2(z)$ for all $z \in U$.*

Theorem 21.2 (Morera) *If a function f is continuous in U and*

$$\int\limits_{\partial\Delta} f(z)\, dz = 0$$

for any triangle $\Delta \subseteq U$, then f is holomorphic.

Theorem 21.3 (Weierstrass) *If the series*

$$f(z) = \sum_{n=0}^{\infty} f_n(z)$$

of holomorphic functions f_n in U converges uniformly on any compact subset of U, then the function f is also holomorphic and

$$f^{(m)}(z) = \sum_{n=0}^{\infty} f_n^{(m)}(z)$$

for any $m \in \mathbb{N}$.

21.2 The Laurent Series

Theorem 21.4 (Laurent) *Any holomorphic function f in an annulus*

$$V = \{z \in \mathbb{C} : r < |z - a| < R\}$$

may be represented in V as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{21.1}$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\xi)\, d\xi}{(\xi - a)^{n+1}},\ n = 0,\ \pm1,\ \pm2,\ \ldots \tag{21.2}$$

$$\gamma_\rho = \{z : |z-a| = \rho\},\ r < \rho < R$$

Definition 21.1 *The series (21.1) with coefficients (21.2) is called the Laurent series of the function f in the annulus V. The term*

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

is called the regular part and

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n$$

is called the principal part.

Let us understand how the domain of convergence of

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

can be defined. By Th. 20.1, the series

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

converges in the disk $\{z : |z-a| < R\}$, where $\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$. Next we consider the series

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n$$

We replace $w := \frac{1}{z-a}$ and obtain

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n = c_{-1}w + c_{-2}w^2 + c_{-3}w^3 + \dots$$

This series converges for all $|w| < \rho$, where $\frac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_{-n}|}$, or for $|z-a| > \frac{1}{\rho} =: r$. Consequently, the domain of convergence of

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

is the annulus $V = \{z : r < |z-a| < R\}$, where

$$r = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_{-n}|}$$

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$$

Example *The function* $f(z) = \frac{1}{(z-1)(z-2)}$ *is holomorphic in the disk* $V_1 = \{z : |z| < 1\}$ *and annuli* $V_2 = \{z : 1 < |z| < 2\}$ *and* $V_3 = \{z : 2 < |z| < \infty\}$. *In order to obtain its Laurent (or Taylor) series, we represent* f *as*

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

Consider the domain V_1. *The following:*

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n$$

converges for $|z| < 2$, *and*

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

converges for $|z| < 1$. *Therefore*

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n,\ |z| < 1$$

Now consider V_2. The following:

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

again converges for $|z| < 2$, and

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=-\infty}^{-1} z^n$$

converges for $|z| > 1$. Therefore

$$f(z) = -\sum_{n=-1}^{-\infty} z^n - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n$$

Similarly, for V_3

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{2}\sum_{n=-1}^{-\infty}\frac{1}{2^n} z^n$$

converges for $|z| > 2$, so

$$f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n$$

Example *We write the expansion of $f(z) = \frac{1}{(1-z)(2+z)}$ in the annulus $V = \{z : 0 < |z-1| < 3\}$. We rewrite*

$$\begin{aligned} f(z) = \frac{1}{(1-z)(2+z)} &= -\frac{1}{(z-1)(z-1+3)} = -\frac{1}{z-1}\cdot\frac{1}{3}\cdot\frac{1}{1+\frac{z-1}{3}} \\ &= -\frac{1}{z-1}\cdot\frac{1}{3}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n} \\ &= \sum_{n=-1}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}}(z-1)^n,\ z \in V \end{aligned}$$

Example *Let $f(z) = \frac{1}{(z-i)^3}$. We want to write the Laurent series for f with $a = 0$. We will use the formula*

$$(1+z)^{\alpha} = 1 + \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}z^n,\ |z| < 1$$

Then

$$\begin{aligned} \frac{1}{(z-i)^3} = (z-i)^{-3} = (-i)^{-3}\left(\frac{z}{-i}+1\right)^{-3} &= \frac{1}{i}\left(\frac{z}{-i}+1\right)^{-3} \\ &= \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\frac{(-3)(-4)\dots(-3-n+1)}{n!}\left(\frac{z}{-i}\right)^n\right) \\ &= \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\frac{(3)(4)\dots(n+2)}{n!}\frac{z^n}{i^n}\right) \\ &= \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}\frac{z^n}{i^n}\right) \\ &\frac{1}{i}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}\frac{z^n}{i^n} \\ &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2i^{n+1}}z^n,\ |z| < 1 \end{aligned}$$

22 Residues (Lecture Notes)

22.1 Isolated Singular Points

Definition 22.1 *A point $a \in \overline{\mathbb{C}}$ is an isolated singular point of a function f if there exists a punctured neighborhood $\{z : 0 < |z-a| < r\}$ if $a \neq \infty$, or $\{z : R < |z| < \infty\}$ if $a = \infty$, on which f is holomorphic.*

Definition 22.2 *An isolated singular point a of a function f is said to be*

1. *removable if $\lim_{z\to a} f(z)$ exists and is finite*
2. *a pole if $\lim_{z\to a} f(z) = \infty$ exists*
3. *an essential singularity of f if f has neither a finite nor infinite limit as $z \to a$*

Example

1. *The function $f(z) = \dfrac{\sin z}{z}$ has a removable singularity since*

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \Rightarrow \lim_{z\to 0} f(z) = 1$$

2. *The function $f(z) = \dfrac{1}{z^n}$, where $n \in \mathbb{N}$, has a pole at $z = 0$.*
3. *The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. If $z = x \in \mathbb{R}$, then indeed*

$$0 = \lim_{x\to 0^-} e^{\frac{1}{x}} \neq \lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$$

It also has no limit along the imaginary axis:

$$\lim_{y\to 0} e^{iy} = \lim_{y\to 0}\left(\cos\frac{1}{y} + i\sin\frac{1}{y}\right)$$

Theorem 22.1 *An isolated singular point $a \in \mathbb{C}$ of a function f is a removable singularity if and only if its Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

has no principal part.

Theorem 22.2 *An isolated singular point a of a function f is removable if and only if f is bounded in a neighborhood of the point a.*

Theorem 22.3 *An isolated singular point $a \in \mathbb{C}$ is a pole of f if its Laurent expansion near a has the form*

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n$$

for some $N \in \mathbb{N}$ and $c_N \neq 0$.

Definition 22.3 *The number N in Th. 22.3 is called the order of a pole of f.*

Theorem 22.4 *$a \in \mathbb{C}$ is a pole of the function f if and only if the function $\varphi = \dfrac{1}{f}$ is holomorphic in a neighborhood of a and $\varphi(a) = 0$.*

Definition 22.4 *The order of a zero $a \in \mathbb{C}$ of a function φ that is holomorphic at this point is the order k of the first non-zero derivative $\varphi^{(k)}(a) \neq 0$.*

Proposition 22.1 *The order of a pole a of a function f is the order of a as a zero of the function $\varphi = \dfrac{1}{f}$.*

Theorem 22.5 *An isolated singular point a of f is an essential singularity if and only if the principal part of the Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

of f near a contains infinitely many non-zero terms.

Theorem 22.6 *If a is an essential singularity of a function f, then for any $A \in \overline{\mathbb{C}}$ we may find a sequence $\{z_n\}_{n \geqslant 1}$ such that $\lim\limits_{n\to\infty} z_n = a$ and $\lim\limits_{n\to\infty} f(z_n) = A$.*

22.2 Residues

Definition 22.5 *Let $a \in \mathbb{C}$ be an isolated singular point of f. The number*

$$\mathrm{res}_a f = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} f(z)\, dz$$

is called the residue of f at a. Here we define $\gamma_\rho = \{z : |z-a| = \rho\}$, $0 < \rho < R$ and assume f is holomorphic in $\{z : 0 < |z-a| < R\}$.

Proposition 22.2 *The residue of a function f at an isolated singular point $a \in \mathbb{C}$ is equal to the coefficient c_{-1} of the term $(z-a)^{-1}$ in the Laurent expansion of f around a.*

Theorem 22.7 *Let the function f be holomorphic everywhere in a domain U, which is an open and connected subset of $\mathbb{C}$, except at an isolated set of singular points $a_1, \ldots, a_n$. Let γ be a positively oriented, simply connected path in U surrounding $a_1, \ldots, a_n$. Then*

$$\int\limits_{\gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{a_k} f$$

22.3 Computation of Residues

1. If a is removable then
$$\mathrm{res}_a f = 0$$
2. If a is a pole of order 1 then
$$\mathrm{res}_a f = \lim_{z\to a} (z-a) f(z)$$
3. If $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, where φ and ψ are holomorphic, $\psi(a) = 0$, $\psi'(a) \neq 0$, $\varphi(a) \neq 0$, then
$$\mathrm{res}_a f = \frac{\varphi(a)}{\psi'(a)}$$
4. If a is a pole of order n, then
$$\mathrm{res}_a f = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \big((z-a)^n f(z) \big)$$

23 Application of Residues to Computations of Integrals (Lecture Notes)

23.1 Computation of Residues

Recall that

1. If a is removable then

$$\mathrm{res}_a f = 0$$

2. If a is a pole of order 1 then

$$\mathrm{res}_a f = \lim_{z\to a}(z-a)f(z)$$

3. If $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, where φ and ψ are holomorphic, $\psi(a) = 0$, $\psi'(a) \neq 0$, $\varphi(a) \neq 0$, then

$$\mathrm{res}_a f = \frac{\varphi(a)}{\psi'(a)}$$

4. If a is a pole of order n, then

$$\mathrm{res}_a f = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\big((z-a)^n f(z)\big)$$

Example

1. *Consider* $f(z) = \dfrac{z}{(z-1)(z-2)^2}$. *Then* $a_1 = 1$ *and* $a_2 = 2$ *are poles of order 1 and 2 respectively.*

$$\mathrm{res}_1 f = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z-2)^2} = 1$$

$$\mathrm{res}_2 f = \lim_{z\to 2}\frac{d}{dz}\big((z-2)^2 f(z)\big) = \lim_{z\to 2}\frac{d}{dz}\frac{z}{z-1} = \lim_{z\to 2}\frac{-1}{(z-1)^2} = -1$$

2. *Consider* $f(z) = \dfrac{\sin z}{\cos z}$. *Then* $a = \dfrac{\pi}{2}$ *is a pole of order 1.*

$$\mathrm{res}_{\frac{\pi}{2}} f = \frac{\sin\frac{\pi}{2}}{\frac{d}{dz}\cos z\big|_{z=\frac{\pi}{2}}} = -1$$

3. *Consider* $f(z) = \dfrac{1}{z+2}\cos\dfrac{1}{z}$. *Then* $a = 0$ *is an essential singularity. We rewrite*

$$\cos\frac{1}{z} = 1 - \frac{1}{2!\cdot z^2} + \frac{1}{4!\cdot z^4} - \dots$$

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \dots\right)$$

Multiplying these series together, we find the coefficients of $\dfrac{1}{z}$*:*

$$f(z) = \frac{1}{2}\frac{1}{z}\left(\frac{1}{2^1\cdot 2!} - \frac{1}{2^3\cdot 4!} + \frac{1}{2^5\cdot 6!} - \frac{1}{2^7\cdot 8!} + \dots\right) + \dots$$

which we use to obtain $c_{-1} = \mathrm{res}_a f$.

23.2 ∞ as an Isolated Singular Point

We recall that ∞ is an isolated singularity of f if f is holomorphic in $\{z : |z| > R\}$ for some large $R > 0$. Similarly

- ∞ is removable if $\lim_{z\to\infty} f(z)$ exists and is finite
- ∞ is a pole if $\lim_{z\to\infty} f(z) = \infty$
- ∞ is an essential singularity if $\lim_{z\to\infty} f(z)$ does not exist

We remark that ∞ is an isolated singular point of f if and only if 0 is an isolated singular point of $\tilde{f}(z) = f\left(\frac{1}{z}\right)$. We define the Laurent expansion of f at ∞ as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{23.1}$$

where the series converges for $R < |z| < \infty$. Next we characterize the type of singularity at ∞ via the Laurent expansion. We write the Laurent series of $\tilde{f}$ at 0:

$$\tilde{f}(z) = \sum_{n=-\infty}^{\infty} \tilde{c}_n z^n$$

Hence

$$f(z) = \tilde{f}\left(\tfrac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \tilde{c}_n z^{-n} = \sum_{n=-\infty}^{\infty} \tilde{c}_{-n} z^n = \sum_{n=-\infty}^{\infty} c_n z^n \tag{23.2}$$

where $c_n = \tilde{c}_{-n}$. We will call $\sum_{n=-\infty}^{0} c_n z^n$ the regular part of the Laurent series, and $\sum_{n=1}^{\infty} c_n z^n$ its principal part. The equality (23.2) immediately implies that ∞ is

- removable if the principal part of (23.1) equals zero
- a pole if the principal part has a finite number of non-zero terms
- an essential singularity if the principal part consists of an infinite number of non-zero terms

Definition 23.1 *If ∞ is an isolated singular point of the function f, then*

$$\mathrm{res}_\infty f = \frac{1}{2\pi i} \int_{\gamma_\rho^-} f(z)\, dz$$

where $\gamma_\rho^- = \{z : |z| = \rho\}$ is a circle of a sufficiently large radius oriented clockwise. From Th. 21.4, we obtain $\mathrm{res}_\infty f = -c_{-1}$.

Theorem 23.1 *If the function f is holomorphic in $\mathbb{C} \setminus \{a_1, \dots, a_n\}$, then*

$$\sum_{k=1}^{n} \mathrm{res}_{a_k} f + \mathrm{res}_\infty f = 0$$

Example *We compute*

$$\int\limits_{|z|=2} \frac{dz}{(z^8+1)^2} = 2\pi i \sum_{k=1}^{8} \operatorname{res}_{a_k} f = -2\pi i \operatorname{res}_{\infty} f$$

We first find c_{-1} in the Laurent series at ∞:

$$\frac{1}{(z^8+1)^2} = \frac{1}{z^{16}} \frac{1}{\left(1+\frac{1}{z^8}\right)^2} = \frac{1}{z^{16}} \frac{1}{1+\frac{1}{z^8}} \frac{1}{1+\frac{1}{z^8}} = \frac{1}{z^{16}} \left(1 - \frac{1}{z^8} + \dots\right) \left(1 - \frac{1}{z^8} + \dots\right)$$

Hence $c_{-1} = 0$, and

$$\int\limits_{|z|=2} \frac{dz}{(z^8+1)^2} = 0$$

23.3 Application to Riemann Integrals

We first consider

$$\int_0^{2\pi} R(\cos\varphi, \sin\varphi)\, d\varphi$$

where $R(t,s) = \dfrac{P(t,s)}{Q(t,s)}$, and P and Q are polynomials of t and s. We recall that

$$z = e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$\overline{z} = e^{-i\varphi} = \cos\varphi - i\sin\varphi$$

$$\cos\varphi = \frac{z+\overline{z}}{2}$$

$$\sin\varphi = \frac{z-\overline{z}}{2i}$$

If $\varphi \in [0, 2\pi]$, then $z = e^{i\varphi}$ defines the circle $|z| = 1$. With

$$dz = ie^{i\varphi}\, d\varphi = iz\, d\varphi \Rightarrow d\varphi = \frac{dz}{iz}$$

we can compute

$$\int_0^{2\pi} R(\cos\varphi, \sin\varphi)\, d\varphi = \int\limits_{|z|=1} R\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \frac{dz}{iz}$$

Example *We compute*

$$\int_0^{2\pi} \frac{d\varphi}{5-4\cos\varphi} = \int\limits_{|z|=1} \frac{dz}{iz\left(5-4\cdot\frac{z+\overline{z}}{2}\right)} = \frac{1}{i} \int\limits_{|z|=1} \frac{dz}{5z-2z^2-2z\overline{z}} = -\frac{1}{i} \int\limits_{|z|=1} \frac{dz}{2z^2-5z+2}$$

$$= -\frac{1}{i} \cdot 2\pi i \operatorname{res}_{\frac{1}{2}} \frac{1}{2z^2-5z+2} = -2\pi \frac{1}{\frac{d}{dz}(2z^2-5z+2)\big|_{z=\frac{1}{2}}} = \frac{2\pi}{(4z-5)\big|_{z=\frac{1}{2}}} = \frac{-2\pi}{2-5} = \frac{2}{3}\pi$$

Next consider

$$\int_{-\infty}^{\infty} R(x)\,dx$$

where $R(t) = \dfrac{P(t)}{Q(t)}$, and P and Q are polynomials such that $\deg P \leqslant \deg Q - 2$ and $Q(t) \neq 0$, $\forall\, t \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{a_k} R$$

where a_k are zeros of Q such that $\operatorname{Im} a_k > 0$.

Example *We compute*

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \operatorname{res}_i \frac{1}{(1+z^2)^3} = 2\pi i \cdot \frac{1}{2!} \lim_{z\to i} \frac{d^2}{dz^2}\left(\frac{(z-i)^3}{(z^2+i)^3}\right) = \pi i \lim_{z\to i} \frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)$$

$$= \pi i \lim_{z\to i}(-3)(-4)(z+i)^{-5} = \frac{12\pi i}{(2i)^5} = \frac{3}{8}\pi$$

24 Introduction to Partial Differential Equations (Lecture Notes)

24.1 Transport Equation

Let us consider a wave moving with a constant speed c. Let $u(t,x)$ be the wave profile at point x and time t. The lines $x - ct = x_0$, on which u is constant, are called characteristic lines. This implies that the directional derivative of u in the direction of $x - ct = x_0$ is zero. So for $l = (1, c)$ we have

$$\frac{\partial u}{\partial l} = (1, c) \cdot \nabla u = u_t + cu_x = 0$$

where $u_t = \dfrac{\partial u}{\partial t}$ and $u_x = \dfrac{\partial u}{\partial x}$. With the initial condition $u(0,x) = f(x)$, we obtain

$$\begin{cases} u_t + cu_x = 0,\ t > 0,\ x \in \mathbb{R} \\ u(0,x) = f(x),\ x \in \mathbb{R} \end{cases} \tag{24.1}$$

which is a transport equation with constant coefficients. Next we will find a function $u : [0,\infty) \times \mathbb{R} \mapsto \mathbb{R}$ which is differentiable in t and x, and satisfies (24.1).

Method of Characteristics

We assume that $x = x(t)$, which we can interpret as the coordinate of a moving observer. Then $u\big(t, x(t)\big)$ is the point which the observer sees at time t. We compute

$$\frac{d}{dt} u\big(t, x(t)\big) = u_t + \frac{dx}{dt} u_x$$

Then u satisfies (24.1) if $\dfrac{dx}{dt} = c$ and $\dfrac{d}{dt} u\big(t, x(t)\big) = 0$. This yields $x = ct + x_0$ and hence

$$u\big(t, x(t)\big) = u\big(0, x(0)\big) = f(x_0)$$

$$u(t,x) = f(x - ct)$$

is a solution to (24.1). Now we show that the equation has no other solutions. Let u be a solution to (24.1). We consider a new function

$$v(t,x) = u(t, x + ct)$$

Then

$$v_t(t,x) = u_t(t, x+ct) + cu_x(t, x+ct) = 0 \Rightarrow v(t,x) = F(x)$$

But from the initial condition

$$v(0,x) = u(0,x) = f(x)$$

so $F(x) = f(x)$ which implies $v(t,x) = f(x)$ and $u(t,x) = v(t, x - ct) = f(x - ct)$.

Example *We will solve*

$$\begin{cases} u_t + 2u_x = 0 \\ u(0,x) = \cos x \end{cases}$$

Using the method of characteristics we obtain $\dfrac{dx}{dt} = 2 \Rightarrow x = 2t + x_0$ *and consequently* $u(t,x) = \cos(x - 2t)$.

Remark 24.1 *The same method works in the case of the equation*

$$a(t,x)u_t + b(t,x)u_x = 0$$

or, dividing by $a(t,x)$, we can rewrite this as

$$u_t + c(t,x)u_x = 0$$

Example *We will solve*

$$\begin{cases} u_t - (x+1)u_x = 0 \\ u(0,x) = f(x) \end{cases}$$

We rewrite

$$\frac{dx}{dt} = -(x+1) \Rightarrow \frac{dx}{x+1} = -dt$$

and solve the differential equation with the initial condition $x(0) = x_0$ to obtain

$$x = ce^{-t} - 1 = (x_0+1)e^{-t} - 1$$

Solving for $x_0 = (x+1)e^t - 1$, we can substitute it into

$$u(t,x) = f(x_0) = f\big((x+1)e^t - 1\big)$$

24.2 Partial Differential Equations and Fundamental Examples

Definition 24.1 *A partial differential equation (PDE) of a single unknown u is an equation involving u and its partial derivatives. All such equations can be written as*

$$F\big(u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, \ldots, u_{x_{i_1} \cdot x_{i_N}}, x_1, \ldots, x_n\big) = 0$$

for some function F. Here N is called the order of the PDE and is the maximum order of the derivatives appearing in the equation.

Example (Heat Equation)

$$u_t = a^2 u_{xx}$$

Here t and x are temporal and spatial coordinates respectively, while $u(t,x)$ is the temperature at point x and time t. The equation describes the conductance of temperature through a metal wire.

Example (Wave Equation)

$$u_{tt} = a^2 u_{xx} = 0$$

Again t and x are temporal and spatial coordinates respectively, while $u(t,x)$ describes a wave profile at point x and time t.

Example (Laplace Equation)

$$u_{xx} + u_{yy} = 0$$

Here x and y are spatial variables. This equation can describe mechanical or temperature equilibrium.

25 Heat Equation (Lecture Notes)

25.1 Fourier Transform on $\mathbb{R}^d$

Definition 25.1 *The Fourier transform of a continuous, absolutely integrable function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is defined by*

$$\hat{f}(\sigma) = \mathcal{F}[f](\sigma) = \frac{1}{\sqrt{(2\pi)^d}} \int\limits_{\mathbb{R}^d} e^{-i\sigma\cdot x} f(x)\, dx$$

where $\sigma \cdot x = \sigma_1 x_1 + \cdots + \sigma_d x_d$.

Theorem 25.1 *Let f and $\hat{f}$ be absolutely integrable. Then*

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{(2\pi)^d}} \int\limits_{\mathbb{R}^d} e^{i\sigma\cdot x} \hat{f}(\sigma)\, d\sigma$$

Remark 25.1 *From Th. 25.1 it follows that*

$$\mathcal{F}^{-1}\big[\mathcal{F}[f]\big] = f$$

$$\mathcal{F}\big[\mathcal{F}^{-1}[g]\big] = g$$

Next, if we assume that f is differentiable, then

$$\frac{\partial}{\partial x_1} f(x) = \frac{\partial}{\partial x_1}\mathcal{F}^{-1}[\hat{f}](x) = \frac{\partial}{\partial x_1}\frac{1}{\sqrt{(2\pi)^d}} \int\limits_{\mathbb{R}^d} e^{i\sigma\cdot x} \hat{f}(\sigma)\, d\sigma = \frac{1}{\sqrt{(2\pi)^d}} \int\limits_{\mathbb{R}^d} i\sigma_1 e^{i\sigma\cdot x} \hat{f}(\sigma)\, d\sigma = \mathcal{F}^{-1}[i\sigma_1 \hat{f}(\sigma)]$$

Hence

$$\mathcal{F}\left[\frac{\partial f}{\partial x_1}\right](\sigma) = i\sigma_1 \hat{f}(\sigma)$$

A similar computation gives

$$\mathcal{F}[D^\alpha f] = (i\sigma)^\alpha \mathcal{F}[f] \tag{25.1}$$

$$D^\alpha \mathcal{F}[f] = \mathcal{F}\big[(-ix)^\alpha f\big]$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ and $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$.

For two functions f, g we define the convolution as

$$(f * g)(x) = \int\limits_{\mathbb{R}^d} f(x-y) g(y)\, dy$$

25.2 Heat Equation on $\mathbb{R}$

Here we will solve

$$\begin{cases} u_t = a^2 u_{xx} + f(t,x),\ t > 0,\ x \in \mathbb{R} \\ u(0,x) = \varphi(x),\ x \in \mathbb{R} \end{cases} \tag{25.2}$$

In order to find a solution, we first need to do formal computations. We take the Fourier transform of both sides of the equation and obtain

$$\mathcal{F}[u_t] = \mathcal{F}[a^2 u_{xx} + f(t,x)]$$

By (25.1) we obtain

$$\frac{d}{dt}\mathcal{F}[u(t,\cdot)](\sigma) = a^2(-i\sigma)^2\mathcal{F}[u(t,\cdot)](\sigma) + \hat{f}(t,\sigma)$$

If we denote $v(t,\sigma) := \mathcal{F}[u(t,\cdot)](\sigma)$, then we have obtained an equation for v:

$$\frac{d}{dt}v(t,\sigma) = -a^2\sigma^2 v(t,\sigma) + \hat{f}(t,\sigma) \tag{25.3}$$

where $\sigma \in \mathbb{R}$ is a parameter. We note that (25.3) is a linear ordinary differential equation. Next, we take the Fourier transform of the initial condition:

$$v(0,\sigma) = \hat{\varphi}(\sigma) \tag{25.4}$$

Solving (25.3) with (25.4), we have

$$v(t,\sigma) = e^{-a^2\sigma^2 t}\hat{\varphi}(\sigma) + \int_0^t e^{-a^2\sigma^2(t-s)}\hat{f}(s,\sigma)\,ds \tag{25.5}$$

Since $v(t,\sigma) = \mathcal{F}[u(t,\cdot)](\sigma)$, we can take the inverse Fourier transform of (25.5):

$$u(t,\cdot) = \mathcal{F}^{-1}\big[e^{-a^2\sigma^2 t}\hat{\varphi}\big] + \int_0^t \mathcal{F}^{-1}\big[e^{-a^2\sigma^2(t-s)}\hat{f}(s,\cdot)\big]\,ds$$

$$= \mathcal{F}^{-1}\big[e^{-a^2\sigma^2 t}\big] * \mathcal{F}^{-1}[\hat{\varphi}] + \int_0^t \mathcal{F}^{-1}\big[e^{-a^2\sigma^2(t-s)}\big]\mathcal{F}^{-1}[\hat{f}(s,\cdot)]\,ds$$

$$= \mathcal{F}^{-1}\big[e^{-a^2\sigma^2 t}\big] * \varphi + \int_0^t \mathcal{F}^{-1}\big[e^{-a^2\sigma^2(t-s)}\big] * f(s,\cdot)\,ds$$

With

$$\mathcal{F}^{-1}\big[e^{-a^2\sigma^2 t}\big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\sigma x}e^{-a^2\sigma^2 t}\,d\sigma = \frac{1}{\sqrt{4\pi a^2 t}}e^{-\frac{x^2}{4a^2 t}} =: G(t,x)$$

we obtain the following solution to (25.2):

$$u(t,x) = \int_{-\infty}^{\infty} G(t,x-y)\varphi(y)\,dy + \int_0^t\int_{-\infty}^{\infty} G(t-s,x-y)f(s,y)\,dy\,ds$$

25.3 Heat Equation on an Interval $[0, l]$

Here we will consider

$$\begin{cases} u_t(t,x) = a^2 u_{xx}(t,x) + f(t,x),\ t > 0,\ x \in (0,l) \\ u(0,x) = \varphi(x),\ x \in (0,l) \end{cases}$$

We also need either Dirichlet boundary conditions:

$$\begin{cases} u(t,0) = \nu_1(t) \\ u(t,l) = \nu_2(t) \end{cases}$$

or Neumann boundary conditions:

$$\begin{cases} u_x(t,0) = \mu_1(t) \\ u_x(t,l) = \mu_2(t) \end{cases}$$

There could also be mixed boundary conditions. For a specific example, we now consider the equation

$$u_t = a^2 u_{xx} + \cos\frac{3\pi}{2l}x,\ t > 0,\ x \in (0,l) \tag{25.6}$$

with boundary conditions

$$\begin{cases} u_x(t,0) = 0 \\ u(t,l) = 0,\ t \geqslant 0 \end{cases} \tag{25.7}$$

and initial condition

$$u(0,x) = A(l-x),\ x \in [0,l] \tag{25.8}$$

We first find a solution to (25.6) in the form

$$u(x,t) = X(x)T(t)$$

with $f = 0$. We obtain

$$T'(t)X(x) = a^2 T(t)X''(x)$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

and find an equation for X:

$$X''(x) + \lambda X(x) = 0 \tag{25.9}$$

Next we substitute $u(x,t)$ into (25.7) which must be zero boundary conditions:

$$T(t)X'(0) = 0$$

$$T(t)X(l) = 0$$

We now obtain boundary conditions for (25.9):

$$X'(0) = 0,\ X(l) = 0 \tag{25.10}$$

Now we find non-zero solutions to (25.9), (25.10), which is called the Sturm-Liouville problem. (25.9) is a linear second order differential equation. To find its solution, we need to find roots of the characteristic polynomial.

1. If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. From (25.10), we obtain the following system:

$$\begin{cases} X'(0) = c_1 - c_2 = 0 \\ X(l) = c_1 e^{\sqrt{-\lambda}l} + c_2 e^{-\sqrt{-\lambda}l} = 0 \end{cases}$$

   which only has $c_1 = c_2 = 0$ as solution.

2. If $\lambda = 0$, then $X(x) = c_1 x + c_2$. Similarly, from (25.10) we find $c_1 = c_2 = 0$.

3. If $\lambda > 0$, then $X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. From (25.10), we find

$$X'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$X'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

   Then

$$X(l) = c_1 \cos\sqrt{\lambda}l = 0$$

   which implies

$$\sqrt{\lambda_n} = \frac{\pi(2n+1)}{2l},\ n = 0, 1, 2, \ldots$$

   if we want non-zero solutions. We have thus obtained non-zero solutions to (25.9), (25.10):

$$X_n(x) = \cos\frac{\pi(2n+1)}{2l}x,\ n = 0, 1, 2, \ldots$$

26 Wave Equation (Lecture Notes)

26.1 Heat Equation on $[0, l]$

We will revisit the example considered in the previous lecture. We consider the equation

$$u_t = a^2 u_{xx} + \cos\frac{3\pi}{2l}x,\ t > 0,\ x \in (0, l) \tag{26.1}$$

with boundary conditions

$$\begin{cases} u_x(t, 0) = 0 \\ u(t, l) = 0,\ t \geqslant 0 \end{cases} \tag{26.2}$$

and initial condition

$$u(0, x) = A(l - x),\ x \in [0, l] \tag{26.3}$$

1. We first found a solution to (26.1) in the form

$$u(x, t) = X(x)T(t)$$

and obtained

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

which gives the equation

$$X''(x) + \lambda X(x) = 0 \tag{26.4}$$

From the boundary conditions in (26.2) we get

$$X'(0) = 0,\ X(l) = 0 \tag{26.5}$$

We then obtained non-zero solutions to (26.4), (26.5) as

$$X_n(x) = \cos\frac{\pi(2n+1)}{2l}x,\ n = 0, 1, 2, \ldots$$

and

$$\sqrt{\lambda_n} = \frac{\pi(2n+1)}{2l},\ n = 0, 1, 2, \ldots$$

2. Now we find solutions to (26.1)-(26.3) in the form

$$u(t, x) = \sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} T_n(t) \cos\frac{\pi(2n+1)}{2l}x$$

Substituting $u(t, x)$ into (26.1), we get

$$\sum_{n=0}^{\infty} T_n'(t) X_n(x) = \sum_{n=0}^{\infty} a^2 T_n(t) X_n''(x) + f(t, x)$$

If we can write

$$f(t,x)=\sum_{n=0}^{\infty} f_n(t)X_n(x)$$

then

$$\sum_{n=0}^{\infty} T_n'(t)X_n(x)=\sum_{n=0}^{\infty} a^2T_n(t)(-\lambda_n)X_n(x)+\sum_{n=0}^{\infty} f_n(t)X_n(x)$$

We then get an equation for T_n:

$$T_n'(t)+a^2\lambda_nT_n(t)=f_n(t) \tag{26.6}$$

Next we plug in $u(t,x)$ into (26.3):

$$\sum_{n=0}^{\infty} T_n(0)X_n(x)=A(l-x)=\sum_{n=0}^{\infty} b_nX_n(x)$$

and obtain

$$T_n(0)=b_n \tag{26.7}$$

3. Now we need to find the coefficients b_n and functions $f_n(t)$ using the formula

$$b_n=\frac{1}{\|X_n\|^2}\int\limits_0^l \varphi(x)X_n(x)\,dx$$

where $\|X_n\|^2=\int\limits_0^l X_n^2(x)\,dx=\int\limits_0^l \cos^2\frac{\pi(2n+1)}{2l}x\,dx=\frac{l}{2}$. So

$$b_n=\frac{2A}{l}\int\limits_0^l (l-x)\cos\frac{\pi(2n+1)}{2l}x\,dx=\frac{8A}{\pi^2(2n+1)^2}$$

For f_n we remark that

$$f(t,x)=\cos\frac{3\pi}{2l}x=X_1(x)$$

This means that $f_1(t)=1$ and $f_n(t)=0$ for $n\neq 1$.

4. Finally, we find T_n from (26.6), (26.7).

   (a) For $n\neq 1$ we have

$$T_n'(t)+a^2\lambda_nT_n(t)=0,\, T_n(0)=b_n\Rightarrow T_n(t)=b_ne^{-a^2\lambda_nt}=\frac{8A}{\pi^2(2n+1)^2}e^{-\frac{a^2\pi^2(2n+1)^2}{4l^2}t}$$

   (b) For $n=1$ we have

$$T_1'(t)+a^2\lambda_1T_1(t)=1,\, T_1(0)=b_1\Rightarrow T_1(t)=b_1e^{-a^2\lambda_1t}+\frac{1}{a^2\lambda_1}=\frac{8A}{9\pi^2}e^{-\frac{9a^2\pi^2}{4l^2}t}+\frac{4l^2}{9a^2\pi^2}$$

We finally obtain a solution to (26.1)-(26.3):

$$u(t,x)=\left(\frac{8A}{9\pi^2}e^{-\frac{9a^2\pi^2}{4l^2}t}+\frac{4l^2}{9a^2\pi^2}\right)\cos\frac{3\pi}{2l}x+\sum_{n=0,\,n\neq1}^{\infty}\frac{8A}{\pi^2(2n+1)^2}e^{-\frac{a^2\pi^2(2n+1)^2}{4l^2}t}\cos\frac{\pi(1+2n)}{2l}x$$

26.2 Wave Equation on $\mathbb{R}$, D'Alembert's Formula

Here we will solve the wave equation on $\mathbb{R}$:

$$u_{tt} = a^2 u_{xx} \tag{26.8}$$

with initial position

$$u(x, 0) = f(x) \tag{26.9}$$

and initial velocity

$$u_t(x, 0) = g(x) \tag{26.10}$$

In order to derive a formula for the solution to (26.8)-(26.10), we first need to find a general solution to (26.8).

1. Let u be a solution to (26.8). We consider a new function

$$w = u_t + au_x$$

and show that w solves the transport equation:

$$w_t - aw_x = u_{tt} + au_{xt} - au_{tx} - a^2 u_{xx} = 0$$

Moreover, (26.8) is equivalent to

$$\begin{cases} u_t + au_x = w \\ w_t - aw_x = 0 \end{cases} \tag{26.11}$$

That is, if w and u satisfy (26.11), then u solves (26.8). An example of a solution to (26.11) is $w = 0$, which leads to

$$u_t + au_x = 0$$

In this case, we know that

$$u(t, x) = p(x - at)$$

Similarly, (26.8) is equivalent to

$$\begin{cases} u_t - au_x = v \\ v_t + av_x = 0 \end{cases}$$

For $v = 0$, this gives

$$u(t, x) = q(x + at)$$

Adding these two solutions, we have

$$u(t, x) = p(x - at) + q(x + at) \tag{26.12}$$

where p and q are twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

2. We will now find the functions p and q from the initial conditions (26.9), (26.10). We calculate

$$u_t(x,t) = -ap'(x-at) + aq'(x+at)$$

Then

$$u(x,0) = p(x) + q(x) = f(x) \tag{26.13}$$

$$u_t(x,0) = -ap'(x) + aq'(x) = g(x)$$

Integrating the second equation gives

$$-ap(x) + aq(x) = G(x) \tag{26.14}$$

where $G'(x) = g(x)$. Combining (26.13), (26.14) gives

$$p(x) = \frac{1}{2}f(x) - \frac{1}{2a}G(x)$$

$$q(x) = \frac{1}{2}f(x) + \frac{1}{2a}G(x)$$

Hence, we obtain D'Alembert's formula:

$$u(t,x) = \frac{1}{2}\big(f(x-at) + f(x+at)\big) + \frac{1}{2a}\int\limits_{x-at}^{x+at} g(y)\,dy$$

Example *We will solve*

$$u_{tt} = u_{xx}$$

$$u(0,x) = \sin x$$

$$u_t(0,x) = x + \cos x$$

Using D'Alembert's formula:

$$u(t,x) = \frac{1}{2}\big(\sin(x-t) + \sin(x+t)\big) + \frac{1}{2}\int\limits_{x-at}^{x+at} (y+\cos y)\,dy$$

$$= \sin\frac{x+t+x-t}{2}\cos\frac{x+t-(x-t)}{2} + \frac{1}{2}\left(\frac{y^2}{2} + \sin y\right)\Bigg|_{x-at}^{x+at}$$

$$= \sin x\cos t + \frac{1}{4}\big((x+t)^2 - (x-t)^2\big) + \frac{1}{2}\big(\sin(x+t) - \sin(x-t)\big)$$

$$= xt + 2\sin t\cos x$$

27 Laplace Equation (Lecture Notes)

We consider a domain $D \subseteq \mathbb{R}^d$. The equation

$$\triangle u = \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2} = 0$$

$$u_{\partial D} = f$$

is called the Laplace equation. Here we will consider the case

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

So we consider the equation

$$\triangle u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ x \in \mathbb{D} \tag{27.1}$$

$$u(x,y) = f(x,y),\ (x,y) \in \partial D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \tag{27.2}$$

We now want to rewrite this Laplace equation in polar coordinates and then use the method of separation of variables. We take

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad 0 \leqslant r \leqslant 1,\ -\pi \leqslant \varphi \leqslant \pi$$

$$U(r,\varphi) := u(r\cos\varphi, r\sin\varphi)$$

Then

$$\frac{\partial U}{\partial r} = u_x \cos\varphi + u_y \sin\varphi$$

$$\frac{\partial U}{\partial \varphi} = -u_x r\sin\varphi + u_y r\cos\varphi$$

$$\begin{aligned}\frac{\partial^2 U}{\partial r^2} &= (u_{xx}\cos\varphi + u_{xy}\sin\varphi)\cos\varphi + (u_{xy}\cos\varphi + u_{yy}\sin\varphi)\sin\varphi \\ &= u_{xx}\cos^2\varphi + 2u_{xy}\cos\varphi\sin\varphi + u_{yy}\sin^2\varphi\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 U}{\partial \varphi^2} &= (-u_{xx} r\sin\varphi + u_{xy} r\cos\varphi)(-r\sin\varphi) + (-u_{xy} r\sin\varphi + u_{yy} r\cos\varphi) r\cos\varphi \\ &\quad -u_x r\cos\varphi - u_y r\sin\varphi \\ &= r^2(u_{xx}\sin^2\varphi - 2u_{xy}\cos\varphi\sin\varphi + u_{yy}\cos^2\varphi) - r(u_x\cos\varphi + u_y\sin\varphi)\end{aligned}$$

Hence

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} = u_{xx} + u_{yy} = \triangle u$$

So (27.1) now has the form

$$U_{rr}(r,\varphi) + \frac{1}{r}U_r(r,\varphi) + \frac{1}{r^2}U_{\varphi\varphi}(r,\varphi) = 0,\ r \in (0,1),\ \varphi \in (-\pi,\pi) \tag{27.3}$$

Let $F(\varphi) := f(\cos\varphi, \sin\varphi)$, $\varphi \in [-\pi,\pi]$. Then

$$U(1,\varphi) = F(\varphi),\ \varphi \in [-\pi,\pi] \tag{27.4}$$

is the boundary condition for U.

We note that by continuity, we must meet the following conditions:

$$\begin{cases} U(r, \pi - 0) = U(r, -\pi + 0) \\ U_\varphi(r, \pi - 0) = U_\varphi(r, -\pi + 0) \\ \lim\limits_{r \to 0^+} U(r, \varphi) \text{ exists} \end{cases} \tag{27.5}$$

Next we find a solution to (27.3)-(27.5) using the method of separation of variables.

1. We want to find a solution in the form

$$U(r, \varphi) = v(r)w(\varphi)$$

Substituting this into (27.3) gives

$$v''(r)w(\varphi) + \frac{1}{r}v'(r)w(\varphi) + \frac{1}{r^2}v(r)w''(\varphi) = 0$$

$$-r^2\frac{v''(r)}{v(r)} - r\frac{v'(r)}{v(r)} = \frac{w''(\varphi)}{w(\varphi)} = -\lambda$$

Hence

$$w''(\varphi) + \lambda w(\varphi) = 0,\ \varphi \in (-\pi, \pi) \tag{27.6}$$

From (27.5) we get

$$\begin{cases} w(-\pi) = w(\pi) \\ w'(-\pi) = w'(\pi) \end{cases} \tag{27.7}$$

2. Now we want to find non-zero solutions to the Sturm-Liouville problem (27.6), (27.7).

   (a) If $\lambda < 0$, then $w(\varphi) = c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi}$. From (27.7) we get

$$c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = c_1 e^{-\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi}$$

$$\Rightarrow (c_1 - c_2)e^{\sqrt{-\lambda}\pi} = (c_1 - c_2)e^{-\sqrt{-\lambda}\pi} \Rightarrow c_1 = c_2$$

   and

$$c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} = c_1\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} - c_2\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi}$$

$$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = c_2 = 0$$

   We obtain only zero solutions.

   (b) If $\lambda = 0$, then $w(\varphi) = c_1\varphi + c_2$. From (27.7) we get

$$c_1\pi + c_2 = -c_1\pi + c_2 \Rightarrow c_1 = 0$$

   and

$$w'(\varphi) = c_1 = 0$$

   Hence

$$w_0(\varphi) = \frac{a_0}{2},\ \lambda_0 = 0$$

   is a non-zero solution to (27.6), (27.7).

(c) If $\lambda > 0$, then $w(\varphi) = c_1 \cos \sqrt{\lambda}\varphi + c_2 \sin \sqrt{\lambda}\varphi$. From (27.7) we get

$$c_1 \cos \sqrt{\lambda}\pi + c_2 \sin \sqrt{\lambda}\pi = c_1 \cos \sqrt{\lambda}\pi - c_2 \sin \sqrt{\lambda}\pi \Rightarrow 2c_2 \sin \sqrt{\lambda}\pi = 0$$

and

$$-c_1\sqrt{\lambda} \sin \sqrt{\lambda}\pi + c_2\sqrt{\lambda} \cos \sqrt{\lambda}\pi = c_1\sqrt{\lambda} \sin \sqrt{\lambda}\pi + c_2\sqrt{\lambda} \cos \sqrt{\lambda}\pi \Rightarrow 2c_1\sqrt{\lambda} \sin \sqrt{\lambda}\pi = 0$$

This implies that $\sin \sqrt{\lambda}\pi = 0 \Rightarrow \sqrt{\pi} = n = 1, 2, 3, \ldots$, hence

$$w_n(\varphi) = a_n \cos n\varphi + b_n \sin n\varphi$$

3. Now we find a solution to (27.6) in the form

$$U(r, \varphi) = \sum_{n=0}^{\infty} v_n(r) w_n(\varphi)$$

Substituting into (27.6) gives

$$\sum_{n=0}^{\infty} v_n''(r) w_n(\varphi) + \frac{1}{r} \sum_{n=0}^{\infty} v_n'(r) w_n(\varphi) + \frac{1}{r^2} \sum_{n=0}^{\infty} v_n(r) w_n''(\varphi) = 0$$

So we obtain

$$v_n''(r) + \frac{1}{r} v_n'(r) - \frac{n^2}{r^n} v_n(r) = 0,\ r \in (0, 1) \tag{27.8}$$

Now we find a general solution to (27.8).

(a) If $n = 0$, then we have

$$v_0''(r) + \frac{1}{r} v_0'(r) = 0$$

This is an ordinary differential equation with separable variables for $v_0'(r)$. This yields

$$v_0(r) = c \ln r + \tilde{c}$$

From the third equality of (27.5), we must have $c = 0$ and thus

$$v_0(r) = 1$$

(b) If $n = 1, 2, \ldots$, then $v(r) = r^n$ and $v(r) = r^{-n}$ are solutions to (27.8). However, only $v(r) = r^n$ satisfies (27.5), hence

$$v_n(r) = r^n,\ n = 1, 2, \ldots$$

Consequently

$$U(r, \varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi) r^n$$

4. Finally, we need to find the coefficients a_n, b_n from (27.4).

$$U(1, \varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi) = F(\varphi)$$

Using the orthogonality of $\{\sin n\varphi, \cos n\varphi\}$ for different n, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\psi)\cos n\psi\, d\psi,\ n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\psi)\sin n\psi\, d\psi,\ n = 1, 2, \ldots$$

We then obtain

$$U(r,\varphi) = \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{\pi} F(\psi)\, d\psi + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} F(\psi)(\cos n\psi\cos n\varphi + \sin n\psi \sin n\varphi)r^n\, d\psi\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} F(\psi)\left(\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\varphi - \psi)\right) d\psi\right]$$

We simplify the term

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\varphi-\psi) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} r^n\left(e^{in(\varphi-\psi)} + e^{-in(\varphi-\psi)}\right)$$

$$= \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\left(re^{i(\varphi-\psi)}\right)^n + \sum_{n=1}^{\infty}\left(re^{-i(\varphi-\psi)}\right)^n\right]$$

$$= \frac{1}{2}\left[1 + \frac{re^{i(\varphi-\psi)}}{1-re^{i(\varphi-\psi)}} + \frac{re^{-i(\varphi-\psi)}}{1-re^{-i(\varphi-\psi)}}\right]$$

$$= \frac{1}{2}\frac{1-r^2}{1-2r\cos(\varphi-\psi)+r^2}$$

We have obtained

$$U(r,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\varphi)\frac{1-r^2}{1-2r\cos(\varphi-\psi)+r^2}\, d\psi$$

To return to the old variables (x, y), let $z := (x, y) = (r\cos\varphi, r\sin\varphi)$ and $\zeta := (\xi, \eta) = (\cos\psi, \sin\psi)$. Then

$$\|z\|^2 = r^2$$

$$\|z-\zeta\|^2 = 1 - 2r\cos(\varphi-\psi) + r^2$$

So

$$u(x,y) = u(z) = \frac{1}{2\pi}\int_{\|\zeta\|=1} \frac{1-\|z\|^2}{\|z-\zeta\|^2} f(\zeta)\, ds(\zeta)$$