Problem sheet 5

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Solutions will be collected during the lecture on Monday November 25.

1. **[1+1+2 points]** Evaluate the following integrals

   (a) $\int\limits_{(-1,2)}^{(2,2)} x dy + y dx$;

   (b) $\int\limits_{(1,-1)}^{(1,1)} (x-y)(dx-dy)$;

   (c) $\int\limits_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}$, where the point (x_1, y_1, z_1) belongs to the sphere $x^2 + y^2 + z^2 = a^2$ and (x_2, y_2, z_2) belongs to $x^2 + y^2 + z^2 = b^2$ ($a > 0$, $b > 0$).

2. **[3 points]** Find a potential of the vector field $\vec{f}(x, y) = \left(x^2 + 2xy - y^2, x^2 - 2xy - y^2\right)$.

3. **[2 points]** Show that the vector field $(e^x(\sin xy + y\cos xy) + 2x - 2z, xe^x \cos xy + 2y, 1 - 2x)$ is conservative.

4. **[3 points]** Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a force field and $\gamma : [a, b] \to \mathbb{R}^3$ be a twice continuously differentiable curve. Use Newton's law $\vec{F}(\gamma(t)) = m\gamma''(t)$, show that the work W done by this force field in moving a particle of mass m along the curve γ is given by

$$W = \frac{m}{2}\left(\|\gamma'(b)\|^2 - \|\gamma'(a)\|^2\right).$$

5. **[4+4+4 points]** Evaluate the following scalar surface integrals

   (a) $\iint\limits_S (x + y + z)dS$, where S is the surface $x^2 + y^2 + z^2 = a^2$, $z \geq 0$ ($a \neq 0$);

   (b) $\iint\limits_S z dS$, where S is given by $x = u\cos v$, $y = u \sin v$, $z = v$ ($0 < u < a$, $0 < v < 2\pi$);

   (c) $\iint\limits_S (x^2 + y^2)dS$, where S is the full surface of the cone $\sqrt{x^2 + y^2} \leq z \leq 1$.