Problem sheet 11

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Solutions will be collected during the lecture on Monday January 20.

1. **[1+2 points]** Using Uniqueness theorem prove the following formulas:
   (a) $\sin^2 z = \frac{1-\cos 2z}{2}, \quad z \in \mathbb{C}$;
   (b) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \quad z_1, z_2 \in \mathbb{C}$.

2. **[1+1 points]** Find the radius of convergence of the following power series:
   (a) $\sum_{n=0}^{\infty} \frac{(z-1)^n}{n^2}$;
   (b) $\sum_{n=0}^{\infty} n z^{2n}$.

3. **[2+3 points]** Expand the function $\frac{z^2}{(z+1)^2}$ in the power series
   (a) $\sum_{n=0}^{\infty} a_n z^n$;
   (b) $\sum_{n=0}^{\infty} a_n (z-1)^n$.

4. **[2 points]** Use Cauchy's integral formula for derivatives in order to compute the integral
$$\frac{1}{2\pi i} \int_\gamma \frac{z e^z}{(z-a)^3} dz,$$
   where γ is a positively oriented simple path surrounding $a \in \mathbb{C}$.

5. **[1+1+2 points]** Does there exist a function f holomorphic at $z = 0$ and such that $f\left(\frac{1}{n}\right)$, $n \geq 1$, equals
   (a) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$;
   (b) $0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, 0, \frac{1}{8}, \ldots$;
   (c) $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \ldots$

   Justify your answers.

6. **[2+4+3 points]** Find the Laurent series for the following functions:
   (a) $\frac{1}{z+3}$ in the annulus $3 < |z| < \infty$;
   (b) $\frac{1}{z(1-z)}$ in the annuli $0 < |z| < 1$ and $0 < |z-1| < 1$;
   (c) $z^2 \sin \frac{1}{z-1}$ in $0 < |z-1| < \infty$.