Problem sheet 10

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday January 13.

1. **[1+2+1 points]** Let γ be a positively oriented closed path in $\mathbb{C}$. Use Cauchy's integral formula to compute $\int_\gamma \frac{dz}{z^2+9}$ if
   (a) γ surrounds the point $3i$, but does not surround the point $-3i$;
   (b) γ surrounds the points $3i$ and $-3i$;
   (c) γ surrounds neither the point $3i$ not $-3i$.

2. **[3 points]** Use Cauchy's integral formula to compute the integral $\int_\gamma \frac{zdz}{z^4-1}$, where γ is a positively oriented circle $|z| = a$ and $a > 1$ is a real number.

3. **[4 points]** Let $f_n : U \to \mathbb{C}$ be continuous function on an open subset U of $\mathbb{C}$ for all $n \geq 0$. Let the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on U. Show that for every $z_0 \in U$
$$\lim_{z\to z_0} \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} f_n(z_0).$$

4. **[4 points]** Show that the series
$$\sum_{n=0}^{\infty} \frac{nz^n}{1-z^n}$$
converges uniformly on each closed disc $|z| \leq R$ for every $R \in (0, 1)$.

5. **[2+3 points]** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (holomorphic on $\mathbb{C}$). Show that
   (a) if $|f(z)| \geq 1$ for all $z \in \mathbb{C}$, then f is constant in $\mathbb{C}$;
   (*Hint:* Apply the Liouville theorem to the function $\frac{1}{f(z)}$)
   (b) if
$$\lim_{z\to\infty} \frac{f(z)}{1+|z|^{\frac{7}{2}}} = 0,$$
   then f is a polynomial of degree less or equal than 3.
   (*Hint:* Use the Cauchy inequality)