Problem sheet 1

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday October 28.

1. **[2+2 points]** a) Show that a set of Lebesgue measure zero has no interior points.

   b) Construct a set having Lebesgue measure zero whose closure is the entire space $\mathbb{R}^d$.

2. **[3 points]** Show that a union of a finite or countable number of sets of Lebesgue measure zero is a set of Lebesgue measure zero.

3. **[4 points]** Let I be a rectangle in $\mathbb{R}^d$ and $f : I \to \mathbb{R}$ be bounded. Let, for a partition P of I, $L(f, P)$ and $U(f, P)$ denote the lower and upper Darboux sums, respectively. Using the Darboux criterion, show that f is integrable over I if and only if for every $\varepsilon > 0$ there exists a partition P of I such that $U(f, P) - L(f, P) < \varepsilon$.

4. **[4 points]** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that the graph of f

   $$\mathrm{Gr} = \{(x, f(x)) : \ x \in [0, 1]\}$$

   has measure zero in $\mathbb{R}^2$.

5. **[2 points]** Prove that $\partial S = \bar{S} \setminus S^\circ$ for any set $S \subset \mathbb{R}^d$, where $\bar{S}$ and S° denote the closure and the interior of S, respectively.

6. **[2 points]** Let $S_1, S_2 \subset \mathbb{R}^d$. Show that $\partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2$.

7. **[3+1 points]** a) Show that if a set $S \subset \mathbb{R}^d$ is such that $\mu(S) := \int_S dx$ exists and $\mu(S) = 0$, then $\mu(\bar{S})$ also exists and equals zero for the closure $\bar{S}$ of the set S.

   b) Give an example of a bounded set $S \subset \mathbb{R}^d$ of Lebesgue measure zero whose closure $\bar{S}$ is not a set of Lebesgue measure zero.

Problem sheet 2

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday November 4.

1. **[2+2 points]** Change the order of integrations in the following integrals

$$a)\quad \int_0^2 \left(\int_x^{2x} f(x,y)dy \right) dx; \quad b)\quad \int_1^e \left(\int_0^{\ln x} f(x,y)dy \right) dx.$$

2. **[2+3+3 points]** Evaluate the following integrals
   (a) $\iint\limits_S xy^2 dxdy$, where S is bounded by the parabola $y^2 = 4x$ and the line $x = 1$;
   (b) $\iint\limits_S (x^2+y^2)dxdy$, where S is the parallelogram bounded by the lines $y = x$, $y = x+a$, $y = a$ and $y = 3a$ $(a > 0)$;
   (c) $\iiint\limits_S (xy)^2 dxdydz$, where S is given by the inequalities $0 \le x \le y \le z \le 1$.

3. **[3+3 points]** Compute the following integrals
   (a) $\iint_S \sin\sqrt{x^2+y^2}\,dxdy$, where $S = \{(x,y) : \ \pi^2 \le x^2+y^2 \le 4\pi^2\}$;
   (b) $\iiint_S (x^2+y^2)dxdydz$, where $S = \left\{(x,y,z) : \ \frac{x^2+y^2}{2} \le z \le 2\right\}$.

4. **[4 points]** Compute the volume bounded by the surfaces $x^2+y^2+z^2 = 2az$, $x^2+y^2 \le z^2$ $(a > 0)$.

5. **[3 points]** Let

$$B = \left\{ x \in \mathbb{R}^d : \ \sum_{k=1}^d x_k^2 \le 1 \right\}$$

be the unit ball in $\mathbb{R}^d$ and

$$C = \left\{ x \in \mathbb{R}^d : \ \sum_{k=1}^d \frac{x_k^2}{a_k^2} \le 1 \right\}$$

be the d-dimensional ellipsoid $(a_k > 0,\ k = 1,\dots,d)$. Prove that the volume $\mu(C)$ of C equals $a_1 \dots a_d \mu(B)$.

Problem sheet 3

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday November 11.

1. **[3 points]** Find all $\alpha \in \mathbb{R}$ for which the integral
$$\iint\limits_{x^2+y^2\le 1} \frac{dxdy}{(x^2+y^2)^\alpha}$$
converges.

2. **[3 points]** Check if the following integral converges
$$\iint_{\mathbb{R}^2} \sin(x^2+y^2)dxdy.$$

3. **[4 points]** Compute the integral
$$\iint_{\mathbb{R}^2} \frac{|x|dxdy}{(1+x^2+y^2)^2}.$$

4. **[4 points]** Let the curve γ is given by $\rho = \rho(\varphi)$, $\alpha \le \varphi \le \beta$, in polar coordinates. Prove that the length of γ equals
$$l(\gamma) = \int_\alpha^\beta \sqrt{\rho^2(\varphi) + \dot{\rho}^2(\varphi)}d\varphi.$$

5. **[3+4 points]** Find the length of the curves given by
   (a) $x = a\cos t$, $y = a\sin t$, $z = bt$, $t \in [0, 2\pi]$, where $a, b > 0$;
   (b) $\rho = a\varphi$, $0 \le \varphi \le 2\pi$ (in polar coordinates).

6. **[3 points]** Find a natural parametrisation of the cycloid $\gamma(t) = (a(t - \sin t), a(1 - \cos t))$, $t \in [0, 2\pi]$, where $a > 0$.

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Problem sheet 4

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday November 18.

1. **[2 points]** Compute the line integral $\int_\gamma xyds$, where γ is the part of the circle $x^2 + y^2 = 1$ located in the positive quadrant $\{(x, y) : x \geq 0, y \geq 0\}$.

2. **[3 points]** Compute the line integral $\int_\gamma zds$, where γ is the helix in $\mathbb{R}^3$, $\{(x, y, z) : x = t\cos t,\ y = t\sin t,\ z = t,\ 0 \leq t \leq 2\pi\}$.

3. **[2 points]** Compute $\int_\gamma 2xydx + x^2dy$, where γ is the oriented curve $\left\{(x, y) : y = \frac{x^2}{4},\ 0 \leq x \leq 2\right\}$ with the orientation from $x = 0$ to $x = 2$.

4. **[3 points]** Compute $\int_\gamma (y+z)dx + (z+x)dy + (x+y)dz$, where γ is the oriented curve $\{(x, y, z) : x = \sin^2 t,\ y = 2\sin t\cos t,\ z = \cos^2 t,\ 0 \leq t \leq \pi\}$ with the orientation from $t = 0$ to $t = \pi$.

5. **[3 points]** Using Green's theorem, evaluate $\oint\limits_\gamma (x + y)dx - (x - y)dy$, where γ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oriented counter clockwise.

6. **[5 points]** Evaluate $\oint\limits_\gamma \frac{xdy - ydx}{x^2+y^2}$, where γ is a simple closed curve that does not pass through the origin and is oriented counter clockwise.

   (*Hint:* Let S be the domain surrounded by γ. For the case $(0, 0) \notin S$ just use the Green's theorem for S. If $(0, 0) \in S$ then apply the Green's theorem for the domain $S \setminus B_\varepsilon(0, 0)$ and then make $\varepsilon \to 0$)

7. **[3 points]** Using Green's theorem, compute area of the domain bounded by the astroid $x = a\cos^3 t,\ y = b\sin^3 t$ $(0 \leq t \leq 2\pi)$.

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Problem sheet 5

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday November 25.

1. **[1+1+2 points]** Evaluate the following integrals

   (a) $\int_{(-1,2)}^{(2,2)} xdy + ydx$;

   (b) $\int_{(1,-1)}^{(1,1)} (x-y)(dx-dy)$;

   (c) $\int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}$, where the point (x_1, y_1, z_1) belongs to the sphere $x^2+y^2+z^2=a^2$ and (x_2, y_2, z_2) belongs to $x^2+y^2+z^2=b^2$ $(a>0,\ b>0)$.

2. **[3 points]** Find a potential of the vector field $\vec{f}(x,y) = \left(x^2+2xy-y^2, x^2-2xy-y^2\right)$.

3. **[2 points]** Show that the vector field $(e^x(\sin xy + y\cos xy) + 2x - 2z, xe^x\cos xy + 2y, 1-2x)$ is conservative.

4. **[3 points]** Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a force field and $\gamma : [a,b] \to \mathbb{R}^3$ be a twice continuously differentiable curve. Use Newton's law $\vec{F}(\gamma(t)) = m\gamma''(t)$, show that the work W done by this force field in moving a particle of mass m along the curve γ is given by

$$W = \frac{m}{2}\left(\|\gamma'(b)\|^2 - \|\gamma'(a)\|^2\right).$$

5. **[4+4+4 points]** Evaluate the following scalar surface integrals

   (a) $\iint_S (x+y+z)dS$, where S is the surface $x^2+y^2+z^2=a^2$, $z \ge 0$ $(a \ne 0)$;

   (b) $\iint_S zdS$, where S is given by $x = u\cos v$, $y = u\sin v$, $z = v$ $(0<u<a,\ 0<v<2\pi)$;

   (c) $\iint_S (x^2+y^2)dS$, where S is the full surface of the cone $\sqrt{x^2+y^2} \le z \le 1$.

Problem sheet 6

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Tuesday December 3.

1. **[3+4+4 points]** Evaluate the following surface integrals

   (a) $\iint_S (2z - x)dydz + (x + 2z)dzdx + 3zdxdy$, where S is the upper side (oriented up) of the triangle $x + 4y + z = 4$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

   (b) $\iint_S \left(\frac{dydz}{x} + \frac{dzdx}{y} + \frac{dxdy}{z}\right)$, where S is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ oriented outward;

   (c) $\iint_S (y - z)dydz + (z - x)dzdx + (x - y)dxdy$, where S is the surface $x^2 + y^2 = z^2$ $(0 \leq z \leq h)$ oriented outward.

2. **[3+3+5 points]** Using the Gauss-Ostrogradskii divergence theorem evaluate the following integrals

   (a) $\iint_S x^2dydz + y^2dzdx + z^2dxdy$, where S is the boundary of the cube $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$ oriented outward.

   (b) $\iint_S xy^2dydz + yz^2dzdx + zx^2dxdy$, where S is the sphere $x^2 + y^2 + z^2 = R^2$ oriented outward.

   (c) $\iint_S x^2dydz + y^2dzdx + z^2dxdy$, where S is the part of the cone $x^2 + y^2 = z^2$ $(0 \leq z \leq h)$ oriented outward.

   (*Hint:* Add the part of plane $z = h$, $x^2 + y^2 \leq h^2$.)

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Problem sheet 7

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday December 9.

1. **[4x3 points]** Using Stokes' theorem to compute the following line integrals
   (a) $\int_\gamma ydx + zdy + xdz$, where γ is the circle $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$ with counter clockwise orientation when viewed from the positive side of axes x;
   (b) $\int_\gamma xydx + yzdy + zxdz$, where γ is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$ with counter clockwise orientation when viewed above;
   (c) $\int_\gamma (z^2 - x^2)dx + (x^2 - y^2)dy + (y^2 - z^2)dz$, where γ is the intersection of the half sphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, with the cone $x^2 + y^2 = z^2$ with counter clockwise orientation when viewed above;
   (d) $\int_\gamma z^2dy + x^2dz$, where γ is the curve $y^2 + z^2 = 9$, $4x + 3z = 5$ oriented clockwise viewed form the point $(0, 0, 0)$.
2. **[2 points]** For which $a \in \mathbb{C}$ the following function is continuous at 0?
$$f(z) = \begin{cases} \frac{\operatorname{Re} z}{z} & \text{if } z \neq 0, \\ a & \text{if } z = 0. \end{cases}$$
3. **[2+3 points]** For which real numbers a and b the function f is holomorphic:
   (a) $f(z) = x + ay + i(bx + cy)$, $z = x + iy$;
   (b) $f(z) = \cos x\,(\cosh y + a \sinh y) + i \sin x\,(\cosh y + b \sinh y)$, $z = x + iy$?
4. **[4 points]** Let $z = re^{i\varphi}$ and $f(z) = u(r, \varphi) + iv(r, \varphi)$. Obtain Cauchy-Riemann equations in polar coordinates.
5. **[2 points]** Prove that the function $f(z) = \overline{z}$ is not complex differentiable.

Problem sheet 8

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday December 16.

1. **[3 points]** Let u is a harmonic function. For which twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ the function $f(u)$ is also harmonic?

2. **[3+4 points]** In the following situations, find a holomorphic function f whose real part is u.
   (a) $u = x^2 - y^2 + y$;
   (b) $u = x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$;

3. **[2+3 points]** For which φ the following functions are harmonic:
   (a) $u = \varphi(xy)$;
   (b) $u = \varphi(x^2 + y^2)$.

4. **[3 points]** Show that the functions e^z, $\cos z$ and $\sin z$ are holomorphic in the whole complex plane and compute their derivatives.

Problem sheet 9

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Tuesday January 7.

1. **[1+2 points]** Let $f(z) = z^2$, $z \in \mathbb{C}$.
   (a) Determine the angle of rotation of the complex plane by f at the point $z = 1 + i$.
   (b) Which part of the complex plane is stretched and which is contacted by f?

2. **[3 points]** Find the image of the interior of the circle $\gamma: \ |z - 2| = 2$ under the linear fractional transformation $w = f(z) = \frac{z}{2z-8}$. Sketch the image and pre-image of γ under $w = f(z)$.

3. **[3 points]** Show that the linear fractional transformation $f(z) = \frac{a(z-z_0)}{\bar{z}_0 z - 1}$ maps the disc $B = \{z \in \mathbb{C}: \ |z| < 1\}$ onto itself, where $|z_0| < 1$ and $|a| = 1$ are some complex numbers.

4. **[3 points]** Let γ be a continuously differentiable positively oriented boundary of a set $S \subset \mathbb{C}$ with area A. Compute the integral $\int_\gamma \operatorname{Re} z \, dz$.

5. **[1+2+3 points]** Evaluate the complex line integral $\int_\gamma f(z)dz$ in the following cases.
   (a) $f(z) = z^3$, γ is a part of the parabola $x = y^2$, that connects the points 0 and $1 + i$ in the complex plane.
   (b) $f(z) = |z|$, γ is the half circular $|z| = 1$, $0 \le \arg z \le \pi$ ($z = 1$ is the initial point);
   (c) $f(z) = |z|\bar{z}$, γ is the union of the half circular $|z| = 1$, $y \ge 0$, and the segment $-1 \le x \le 1$, $y = 0$,

6. **[5 points]** Prove that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

   (*Hint:* Integrate the function $f(z) = e^{iz^2}$ along the boundary of the domain $0 \le |z| \le R$, $0 \le \arg z \le \frac{\pi}{4}$, and then pass to the limit as $R \to \infty$.)

Problem sheet 10

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday January 13.

1. **[1+2+1 points]** Let γ be a positively oriented closed path in $\mathbb{C}$. Use Cauchy's integral formula to compute $\int_\gamma \frac{dz}{z^2+9}$ if

   (a) γ surrounds the point $3i$, but does not surround the point $-3i$;

   (b) γ surrounds the points $3i$ and $-3i$;

   (c) γ surrounds neither the point $3i$ not $-3i$.

2. **[3 points]** Use Cauchy's integral formula to compute the integral $\int_\gamma \frac{zdz}{z^4-1}$, where γ is a positively oriented circle $|z| = a$ and $a > 1$ is a real number.

3. **[4 points]** Let $f_n : U \to \mathbb{C}$ be continuous function on an open subset U of $\mathbb{C}$ for all $n \geq 0$. Let the series $\sum_{n=0}^{\infty} f_n$ converges uniformly on U. Show that for every $z_0 \in U$

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} f_n(z_0).$$

4. **[4 points]** Show that the series

$$\sum_{n=0}^{\infty} \frac{nz^n}{1-z^n}$$

   converges uniformly on each closed disc $|z| \leq R$ for every $R \in (0,1)$.

5. **[2+3 points]** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function (holomorphic on $\mathbb{C}$). Show that

   (a) if $|f(z)| \geq 1$ for all $z \in \mathbb{C}$, then f is constant in $\mathbb{C}$;
   (*Hint:* Apply the Liouville theorem to the function $\frac{1}{f(z)}$)

   (b) if

$$\lim_{z\to\infty} \frac{f(z)}{1+|z|^{\frac{7}{2}}} = 0,$$

   then f is a polynomial of degree less or equal than 3.
   (*Hint:* Use the Cauchy inequality)

Problem sheet 11

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday January 20.

1. **[1+2 points]** Using Uniqueness theorem prove the following formulas:
   (a) $\sin^2 z = \frac{1-\cos 2z}{2}$, $z \in \mathbb{C}$;
   (b) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$, $z_1, z_2 \in \mathbb{C}$.

2. **[1+1 points]** Find the radius of convergence of the following power series:
   (a) $\sum_{n=0}^{\infty} \frac{(z-1)^n}{n^2}$;
   (b) $\sum_{n=0}^{\infty} nz^{2n}$.

3. **[2+3 points]** Expand the function $\frac{z^2}{(z+1)^2}$ in the power series
   (a) $\sum_{n=0}^{\infty} a_n z^n$;
   (b) $\sum_{n=0}^{\infty} a_n (z-1)^n$.

4. **[2 points]** Use Cauchy's integral formula for derivatives in order to compute the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{ze^z}{(z-a)^3} dz,$$

   where γ is a positively oriented simple path surrounding $a \in \mathbb{C}$.

5. **[1+1+2 points]** Does there exist a function f holomorphic at $z = 0$ and such that $f\left(\frac{1}{n}\right)$, $n \geq 1$, equals
   (a) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$;
   (b) $0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, 0, \frac{1}{8}, \ldots$;
   (c) $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \ldots$

   Justify your answers.

6. **[2+4+3 points]** Find the Laurent series for the following functions:
   (a) $\frac{1}{z+3}$ in the annulus $3 < |z| < \infty$;
   (b) $\frac{1}{z(1-z)}$ in the annuli $0 < |z| < 1$ and $0 < |z-1| < 1$;
   (c) $z^2 \sin \frac{1}{z-1}$ in $0 < |z-1| < \infty$.

Problem sheet 12

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Monday January 27.

1. **[3+2+3+3 points]** Evaluate residues of the following functions at all isolated singularities:
   (a) $\frac{1}{z^3 - z^5}$;
   (b) $\frac{\sin 2z}{(z+1)^2}$;
   (c) $z^3 \cos \frac{1}{z-2}$;
   (d) $\sin \frac{z}{z+1}$.

2. **[2+2+3+4+4 points]** Use the residue theorem to evaluate the following complex line integrals:
   (a) $\int\limits_{|z-2|=\frac{1}{2}} \frac{z dz}{(z-1)(z-2)^2}$;
   (b) $\int\limits_{|z|=1} \sin \frac{1}{z} dz$;
   (c) $\frac{1}{2\pi i} \int\limits_{|z|=2} \sin^2 \frac{1}{z} dz$;
   (d) $\frac{1}{2\pi i} \int\limits_{|z|=1} z^n e^{\frac{2}{z}} dz$, where n is an integer number;
   (e) $\int\limits_{|z|=4} \frac{z^{11} dz}{(z^6+2)^2}$. (*Hint:* Compute via residue at infinity)

Problem sheet 13

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Thursday January 30.

Points for solved exercises have to be included as bonus points for the homework

1. **[3 points]** Find a solution to the transport equation

$$2u_t(t,x) + x^3 u_x(t,x) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(0,x) = \sin x, \quad x \in \mathbb{R}.$$

2. **[3+6 points]** Solve the following heat equations:

   (a)

$$u_t(t,x) = \frac{1}{2}u_{xx}(t,x) + x, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(0,x) = 1, \quad x \in \mathbb{R};$$

   (b)

$$u_t(t,x) = u_{xx}(t,x) + t, \quad 0 < x < 1, \quad t > 0,$$
$$u(t,0) = 0, \quad u(t,1) = 0, \quad t \geq 0,$$
$$u(0,x) = 0, \quad t \geq 0;$$

3. **[3+6 points]** Solve the following wave equations:

   (a)

$$u_{tt}(t,x) = u_{xx}(t,x), \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(0,x) = x, \quad u_t(0,x) = x^2, \quad x \in \mathbb{R}.$$

   (b)

$$u_{tt}(t,x) = 4u_{xx}(t,x), \quad 0 < x < 1, \quad t > 0,$$
$$u(t,0) = 0, \quad u(t,1) = 0, \quad t \geq 0,$$
$$u(0,x) = 0, \quad u_t(0,x) = x(1-x), \quad 0 \leq x \leq 1.$$