Retake Solutions

Each exercise is graded between 0 and 5 points.

1. Is the following matrix A invertible? If yes, compute the inverse matrix.

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 4 & -1 \\ 2 & -1 & -3 & 2 \\ 3 & -4 & 3 & 1 \end{pmatrix}.$$

Solution. We first compute the rank of the matrix A. So,

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 4 & -1 \\ 2 & -1 & -3 & 2 \\ 3 & -4 & 3 & 1 \end{pmatrix} \begin{matrix} \\ II+I \\ III-2I \\ IV-3I \end{matrix} \sim \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 5 & -1 \\ 0 & 3 & -5 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ III+3II \\ IV+2II \end{matrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 10 & -1 \\ 0 & 0 & 10 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence, $\operatorname{rank} A = 3$. It implies that A is not invertible.

2. Let the matrix of the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ in the canonical basis be $M_T = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. Is T diagonalizable? If yes, find a basis of $\mathbb{R}^2$ in which the matrix of T is diagonal.

Solution. Let us compute the eigenvalues and eigenvectors of the matrix M_T. For this, we have to find roots of characteristic polynomial:

$$\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0.$$

So, $\lambda_1 = 2$ and $\lambda_2 = -1$ are two distinct eigenvalues of the map T. Thus, the map T is diagonalizable in the basis consisting of eigenvectors of T. Next, we find the corresponding eigenvectors from the system of linear equations

$$\begin{pmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

So, for $\lambda = \lambda_1 = 2$ one has

$$\begin{cases} -2x + y = 0, \\ 2x - y = 0. \end{cases}$$

Hence, $(1, 2)$ is en eigenvector corresponding to $\lambda_1 = 2$. Similarly, the second eigenvector corresponding to $\lambda = \lambda_2 = -1$ equals $(1, -1)$. Hence, the operator T has a diagonal matrix $M_T' = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ in the basis $(1, 2)$, $(1, -1)$.

3. Prove that

$$\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 6x_2y_2, \quad (x_1,x_2),(y_1,y_2) \in \mathbb{R}^2,$$

is an inner product on $\mathbb{R}^2$. Find an orthonormal basis (with respect to this inner product) using Gram-Schmidt orthogonalisation procedure.

Solution. We remark that the map $\langle\cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is linear in the first slot and symmetric, i.e.

$$\langle x+z, y\rangle = \langle x,y\rangle + \langle z,y\rangle \quad \text{and} \quad \langle x,y\rangle = \langle y,x\rangle$$

for all $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1,z_2)$ from $\mathbb{R}^2$. So, we have to check only positive definiteness:

$$\langle x,x\rangle = x_1^2 + 2x_1x_2 + 2x_2x_1 + 6x_2^2 = (x_1+2x_2)^2 + 2x_2^2 \geq 0.$$

Moreover, $\langle x,x\rangle = 0$ if and only if $x=(0,0)$. Thus, $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^2$.

In order to find an orthonormal basis, we apply the Gram-Schmidt orthogonalisation procedure to linearly independent vectors $v_1=(1,0)$ and $v_2=(0,1)$. We take $e_1 := \frac{v_1}{\|v_1\|} = (1,0)$. Next, we define

$$u_2 = v_2 - \langle v_2, e_1\rangle e_1 = (0,1) - 2(1,0) = (-2,1)$$

and $e_2 = \frac{u_2}{\|u_2\|} = \frac{(-2,1)}{\sqrt{6}} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$. So, vectors e_1 and e_2 form a basis in $\mathbb{R}^2$ with respect to the inner product $\langle\cdot,\cdot\rangle$.

4. If T is a normal operator, prove that eigenvectors for T which are associated with distinct eigenvalues are orthogonal.

Solution. We first remark that if T is normal and λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*. Indeed, let x is a corresponding eigenvector. We consider

$$\begin{aligned} 0 = \|Tx-\lambda x\|^2 &= \langle Tx-\lambda x, Tx-\lambda x\rangle = \langle (T-\lambda I)x, (T-\lambda I)x\rangle \\ &= \langle (T-\lambda I)^*(T-\lambda I)x, x\rangle \overset{(TT^*=T^*T)}{=} \langle (T-\lambda I)(T-\lambda I)^*x, x\rangle \\ &= \langle (T-\lambda I)^*x, (T-\lambda I)^*x\rangle = \|T^*x - \bar{\lambda}x\|^2, \end{aligned}$$

where I denotes the identity operator. Next, let λ and μ be two distinct eigenvalues of T which correspond to eigenvectors x and y, respectively. Then

$$(\lambda-\mu)\langle x,y\rangle = \langle \lambda x, y\rangle - \langle x, \bar{\mu}y\rangle = \langle Tx,y\rangle - \langle x, T^*y\rangle = \langle Tx,y\rangle - \langle Tx,y\rangle = 0.$$

Hence, $\langle x,y\rangle = 0$ which implies the orthogonality of x and y.

5. Find all $\lambda \in \mathbb{R}$ for which the following quadratic form on $\mathbb{R}^3$ is positive definite

$$Q(x_1,x_2,x_3) = x_1^2 + x_2^2 + 2x_3^2 + 2\lambda x_1x_2 - 2x_1x_3 - 2x_2x_3.$$

Solution. In order to check for which λ the quadratic form Q is positive definite, we use the Sylvester's criterion. We compute the principal minors of matrix A of Q, where

$$A = \begin{pmatrix} 1 & \lambda & -1 \\ \lambda & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

So,

$$M_1 = 1 > 0, \quad M_2 = \begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix} = 1 - \lambda^2 > 0$$

and

$$M_3 = \det A = 2 - 2\lambda - 1 - 1 - 2\lambda^2 = -2\lambda - 2\lambda^2 > 0.$$

Hence, the quadratic form is positive definite for $\lambda \in (-1, 0)$.

6. Let

$$f(x,y) = \begin{cases} \frac{x^3 - y^2}{x^2 + y^2} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

Determine all directions $l = (l_1, l_2) \in \mathbb{R}^2$ along which $\frac{\partial f}{\partial l}(0,0)$ exists.

Solution. We fix a direction $l = (l_1, l_2) \in \mathbb{R}^2$, $l \neq (0,0)$. By the definition of directional derivative, we have

$$\frac{\partial f}{\partial l}(0,0) = \lim_{t \to 0} \frac{f(tl_1, tl_2) - f(0,0)}{t} = \lim_{t \to 0} \frac{t^3 l_1^3 - t^2 l_2^2}{t(t^2 l_1^2 + t^2 l_2^2)} = \lim_{t \to 0} \frac{l_1^3 - \frac{1}{t} l_2^2}{l_1^2 + l_2^2},$$

where the limit exists if and only if $l_2 = 0$. Hence, the directional derivative $\frac{\partial f}{\partial l}(0,0)$ only exists for all $l = (l_1, 0)$, $l_1 \in \mathbb{R}$.

7. Find a local extrema of the function

$$f(x,y) = (y^2 + x)e^{x-2y}, \quad (x,y) \in \mathbb{R}^2.$$

Solution. We find first all critical points from the system of equations

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = e^{x-2y} + (y^2 + x)e^{x-2y} = 0, \\ \frac{\partial f}{\partial y}(x,y) = 2ye^{x-2y} - 2(y^2 + x)e^{x-2y} = 0. \end{cases}$$

It is equivalent to

$$\begin{cases} 1 + y^2 + x = 0, \\ 2y - 2y^2 - 2x = 0. \end{cases}$$

Hence, $x = -2$, $y = -1$. So, $(-2, -1)$ is a critical point of f. Next, we compute the second derivatives of f

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2e^{x-2y} + (y^2 + x)e^{x-2y} = (2 + x + y^2)e^{x-2y}, \\ \frac{\partial^2 f}{\partial y^2} &= 2e^{x-2y} - 4ye^{x-2y} - 4ye^{x-2y} + 4(y^2 + x)e^{x-2y} = (2 + 4x - 8y + 4y^2)e^{x-2y}, \\ \frac{\partial^2 f}{\partial x \partial y} &= -2e^{x-2y} + 2ye^{x-2y} - 2(y^2 + x)e^{x-2y} = (-2 - 2x + 2y - 2y^2)e^{x-2y}. \end{aligned}$$

Hence $\frac{\partial^2 f}{\partial x^2}(-2,-1) = 1 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-2,-1)\frac{\partial^2 f}{\partial y^2}(-2,-1) - \left(\frac{\partial^2 f}{\partial x \partial y}(-2,-1)\right)^2 = 1 \cdot 6 - (-2)^2 = 2 > 0$. This implies that $(-2,-1)$ is a point of local minimum.

8. Find the general solution to the equation $y^{(4)} - 2y''' + 5y'' = 0$.

   Solution. We first write find roots of the characteristic polynomials:

$$\lambda^4 - 2\lambda^3 + 5\lambda^2 = \lambda^2(\lambda^2 - 2\lambda + 5) = \lambda^2(\lambda - 1 - 2i)(\lambda - 1 + 2i) = 0.$$

   Thus, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 1 + 2i$ and $\lambda_4 = 1 - 2i$. Consequently, the general solution is given by the formula

$$y = C_1 + C_2 x + C_3 e^x \sin 2x + C_4 e^x \cos 2x.$$

9. Solve the initial value problem $x^2 y' + xy + 1 = 0, \quad y(1) = 0$. What is the maximal interval where this solution is defined?

   Solution. We start from solving the homogeneous equation

$$x^2 y' + xy = 0.$$

   So,

$$\begin{aligned} \int \frac{dy}{y} &= -\int \frac{dx}{x}, \\ \ln|y| &= -\ln|x| + \ln|C|, \\ y &= \frac{C}{x}. \end{aligned}$$

   Next, we assume that the constant C depends on x and plug in $y = \frac{C(x)}{x}$ into the initial equation. We obtain

$$\begin{aligned} \frac{x^2 C'(x)}{x} - \frac{x^2 C(x)}{x^2} + \frac{xC(x)}{x} + 1 &= 0, \\ xC'(x) + 1 &= 0. \end{aligned}$$

   Hence, $C(x) = -\ln|x| + C_1$, and consequently, the general solution to the equation $x^2 y' + xy + 1 = 0$ is given by the formula $y = -\frac{\ln|x|}{x} + \frac{C_1}{x}$. From the initial condition $y(1) = 0$, we find $C_1 = 0$. Hence, $y(x) = -\frac{\ln|x|}{x}$, $x > 0$. The interval $(0, \infty)$ is the maximal one, where the solution to the initial value problem is defined.