Exam Solutions

Each exercise is graded between 0 and 5 points.

1. Determine all $\lambda \in \mathbb{R}$ for which the vectors $v_1 = (1, 0, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (1, 1, \lambda, 1)$, $v_4 = (1, 2, 3, \lambda)$ form a basis in $\mathbb{R}^4$.

   Solution. Recall that four vectors v_1, v_2, v_3, v_4 form a basis in $\mathbb{R}^4$ if and only if they are linearly independent. So, we need to find all λ for which the following determinant is non-zero.

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & \lambda & 1 \\ 1 & 2 & 3 & \lambda \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & \lambda & 1 \\ 2 & 3 & \lambda \end{vmatrix} + (-1)^{4+1} \cdot (-1) \cdot \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & \lambda \\ 1 & 2 & 3 \end{vmatrix} =$$

$$= \lambda^2 + 2 - 3 - \lambda + 6 + 2 + \lambda - 1 - 3 - 4\lambda = \lambda^2 - 4\lambda + 3 = 0$$

   provided $\lambda = 1$ or $\lambda = 3$. Thus the vectors v_1, v_2, v_3, v_4 form a basis in $\mathbb{R}^4$ if and only if $\lambda \in \mathbb{R} \setminus \{1, 3\}$.

2. Find the orthogonal projection of the vector $v = (1, -1, 1)$ onto

$$U = \{(x_1, x_2, x_3) : \; x_1 + x_2 + x_3 = 0\}$$

   in $\mathbb{R}^3$ with the standard inner product. Compute the distance $d(v, U)$ between v and U, where $d(v, U) = \inf \{\|v - u\| : \; u \in U\}$.

   Solution. We first find an orthonormal basis in U. We remark that a vector $u = (x_1, x_2, x_3)$ belongs to U if and only if $x_1 + x_2 + x_3 = 0$. So, we need to find a fundamental system of solutions to the (system of) linear equation $x_1 + x_2 + x_3 = 0$, which will form a basis in U.

$$\left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right).$$

   Thus, the vectors $v_1 = (-1, 1, 0)$ and $v_2 = (-1, 0, 1)$ form a basis in U. Using the Gram-Schmidt orthogonalization procedure, we built an orthonormal basis in U. We take $e_1 := \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(-1, 1, 0)$. Then, we set

$$\tilde{v}_2 := v_2 - \langle v_2, e_1 \rangle e_1 = (-1, 0, 1) - \frac{1}{2} \cdot 1 \cdot (-1, 1, 0) = \left(-\frac{1}{2}, -\frac{1}{2}, 1 \right) = \frac{1}{2}(-1, -1, 2).$$

   Take $e_2 := \frac{\tilde{v}_2}{\|\tilde{v}_2\|} = \frac{1}{\sqrt{6}}(-1, -1, 2)$. Then we can compute

$$P_U(v) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 = \frac{1}{2} \cdot (-2) \cdot (-1, 1, 0) + \frac{1}{6} \cdot 2 \cdot (-1, -1, 2) = \frac{1}{3}(2, -4, 2).$$

   By properties of the orthogonal projection,

$$d(v, U) = \|v - P_U v\| = \left\| (1, -1, 1) - \frac{1}{3}(2, -4, 2) \right\| = \left\| \frac{1}{3}(1, 1, 1) \right\| = \frac{1}{3} \|(1, 1, 1)\| = \frac{1}{\sqrt{3}}.$$

3. Find coordinates of the vector $x = (1, 1, -2)$ in the basis $e'_1 = (1, 0, 1)$, $e'_2 = (1, 1, 0)$, $e'_3 = (0, 1, 1)$.
Solution. We first write the change of basis matrix from the standard basis to the basis e'_1, e'_2, e'_3.

$$Q_{ee'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then the coordinates of the vector x in the basis e'_1, e'_2, e'_3 can be computed by the formula

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = Q_{ee'}^{-1} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

So, we need to compute $Q_{ee'}^{-1}$. We first find $\det Q_{ee'} = 1 + 1 = 2$. Next computing $i - j$ cofactors of $Q_{ee'}$, we obtain

$$Q_{ee'}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Hence, multiplication of $Q_{ee'}^{-1}$ and the vector column x gives $x = (-1, 2, -1)_{e'}$.

4. Let V be a finite dimensional vector space over $\mathbb{C}$ and T be a linear operator on V. Show that T is normal if and only if $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint operators which commute, i.e. $T_1T_2 = T_2T_1$.

Solution. We first assume that $T = T_1 + iT_2$, where T_1 and T_2 are self-adjoint and commutate, and show that T is normal, that is, $TT^* = T^*T$. We note that $T^* = (T_1 + iT_2)^* = T_1^* - iT_2^* = T_1 - iT_2$ and compute

$$TT^* = (T_1 + iT_2)(T_1 - iT_2) = T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 = T_1^2 + T_2^2.$$

Similarly, $T^*T = T_1^2 + T_2^2$. This shows that T is normal.

Next, we check that any normal operator T can be written as $T_1 + iT_2$, where T_1 and T_2 are self-adjoint operators which commute. For this we take $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$. Then $T_1^* = \frac{T^*+T^{**}}{2} = \frac{T^*+T}{2} = T_1$ and, similarly, $T_2^* = -\frac{T^*-T^{**}}{2i} = \frac{T-T^*}{2i} = T_2$. Moreover, $T_1T_2 = \frac{T+T^*}{2}\frac{T-T^*}{2i} = \frac{T-T^*}{2i}\frac{T+T^*}{2} = T_2T_1$, since T and T^* commute.

Another solution could be the following. Since T is normal, there exists a basis $v_1, \ldots, v_n$ in which the matrix of T has a diagonal form, that is,

$$M_T = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of T. We take operators T_1 and T_2 whose matrices in the basis $v_1, \ldots, v_n$ have the form

$$M_{T_1} = \begin{pmatrix} \operatorname{Re}\lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \operatorname{Re}\lambda_n \end{pmatrix} \quad \text{and} \quad M_{T_2} = \begin{pmatrix} \operatorname{Im}\lambda_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \operatorname{Im}\lambda_n \end{pmatrix}.$$

Then, trivially, $M_{T_1+iT_2} = M_{T_1} + iM_{T_2} = M_T$. This yields $T = T_1 + iT_2$. Moreover, $M_{T_1^*} = M_{T_1}^* = M_{T_1}$ and, similarly, $M_{T_2^*} = M_{T_2}$ because M_{T_1} and M_{T_2} are diagonal with real entries. Consequently, $T_1^* = T_1$ and $T_2^* = T_2$ which implies that T_1 and T_2 are self-adjoint. Next, due to the diagonal form of the matrices M_{T_1} and M_{T_2}, we have,

$$M_{T_1T_2} = M_{T_1}M_{T_2} = M_{T_2}M_{T_1} = M_{T_2T_1}.$$

So, $T_1T_2 = T_2T_1$. It finishes the proof.

5. Show that the function $f(x,y) = \left(x\cos\frac{y}{x}, x\sin\frac{y}{x}\right)$, $x \neq 0$, is invertible in a neighbourhood of every point $(x,y) \in \mathbb{R}^2$, $x \neq 0$.

   Solution. In order to prove that $f = (f_1, f_2) = \left(x\cos\frac{y}{x}, x\sin\frac{y}{x}\right)$ is invertible in a neighbourhood of every point (x,y), $x \neq 0$, we use the theorem about existence of in inverse function (see Theorem 20.1). For this we show that the Jacobian of f is non-zero at each point (x,y), $x \neq 0$.

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\frac{y}{x} + \frac{y}{x}\sin\frac{y}{x} & -\sin\frac{y}{x} \\ \sin\frac{y}{x} - \frac{y}{x}\cos\frac{y}{x} & \cos\frac{y}{x} \end{vmatrix}$$
$$= \cos^2\frac{y}{x} + \frac{y}{x}\cdot\sin\frac{y}{x}\cdot\cos\frac{y}{x} + \sin^2\frac{y}{x} - \frac{y}{x}\cdot\cos\frac{y}{x}\cdot\sin\frac{y}{x} = 1 \neq 0,$$

   for all $(x,y) \in \mathbb{R}^2$, $x \neq 0$.

6. Consider the function

$$f(x,y) = \sqrt[3]{x^3+y^3}, \quad (x,y) \in \mathbb{R}^2.$$

   Check if the function f is differentiable at $(0,0)$.

   Solution. If f is differentiable at zero, then

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - L(x,y)}{\sqrt{x^2+y^2}} = 0,$$

   where $L(x,y) = \frac{\partial f}{\partial x}(0,0)\cdot x + \frac{\partial f}{\partial y}(0,0)\cdot y$ and partial derivatives of f at $(0,0)$ exist, by the definition of differentiable function (see Definition 18.4) and Theorem 19.2. We first compute

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{\sqrt[3]{t^3}}{t} = 1.$$

   Similarly, $\frac{\partial f}{\partial y}(0,0) = 1$. Thus, we need to compute

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt[3]{x^3+y^3} - x - y}{\sqrt{x^2+y^2}}.$$

   Taking $x = y$, we have

$$\frac{\sqrt[3]{x^3+y^3} - x - y}{\sqrt{x^2+y^2}} = \frac{\sqrt[3]{x^3+x^3} - 2x}{\sqrt{x^2+x^2}} = \frac{(\sqrt[3]{2}-2)x}{\sqrt{2}|x|} \to \frac{\sqrt[3]{2}-2}{\sqrt{2}}, \quad \text{as } x \to 0+.$$

   So, the function f is not differentiable at $(0,0)$.

7. Find all local extrema of the function $f(x,y,z) = x^2 + y^2 + 2z^2 - 2xz - 2x$.

Solution. We first find critical points of f from the equality $\nabla f(x,y,z) = 0$.

$$\frac{\partial f}{\partial x} = 2x - 2z - 2 = 0, \quad \frac{\partial f}{\partial y} = 2y = 0, \quad \frac{\partial f}{\partial z} = 4z - 2x = 0.$$

Hence, $x = 2$, $y = 0$, $z = 1$. So, the point $(2,0,1)$ is a critical point of f. In order to check if this point is a point of local extrema, we check whether $\mathrm{Hess}_{(2,0,1)}(f)$ is positive or negative defined. We compute

$$\mathrm{Hess}_{(2,0,1)}(f) = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

By Sylvester's criterion (see Theorem 14.6), $\mathrm{Hess}_{(2,0,1)}(f)$ is positive definite, since $M_1 = 2 > 0$, $M_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 > 0$, and $M_3 = \det(\mathrm{Hess}_{(2,0,1)}(f)) = 16 - 8 = 8 > 0$. So, $(2,0,1)$ is a point of local minimum.

8. Find general solution to the differential equation $x^2 y' + 2y = 2e^{\frac{1}{x}} y^{\frac{1}{2}}$.

Solution. We devide the equation by x^2 which is not zero. So, we obtain the differential equation

$$y' + \frac{2y}{x^2} = \frac{2}{x^2} e^{\frac{1}{x}} y^{\frac{1}{2}}. \tag{1}$$

which is a Bernoulli equation. So, we first need to solve the corresponding homogeneous equation

$$y' + \frac{2y}{x^2} = 0.$$

The general solution to this equation is given by

$$y = Ce^{-\int \frac{2}{x^2} dx} = Ce^{\frac{2}{x}}.$$

One needs to find solutions to Bernoulli equation (1) in the form $y = u(x)e^{\frac{2}{x}}$, where u is a new unknown function. Substituting y into equation (1), we have

$$u' e^{\frac{2}{x}} - u\frac{2}{x^2} e^{\frac{2}{x}} + u\frac{2}{x^2} e^{\frac{2}{x}} = \frac{2}{x^2} e^{\frac{1}{x}} u^{\frac{1}{2}} e^{\frac{1}{x}}.$$

So, we obtain a separable equation

$$u' = \frac{2}{x^2}\sqrt{u}. \tag{2}$$

Integration it, we have

$$\int \frac{du}{2\sqrt{u}} = \int \frac{dx}{x^2},$$
$$\sqrt{u} = -\frac{1}{x} + C.$$

Hence, $u = \left(-\frac{1}{x} + C\right)^2$ for all x such that $x \neq 0$ and $-\frac{1}{x} + C > 0$. Consequently, $y = \left(-\frac{1}{x} + C\right)^2 e^{\frac{2}{x}}$ for all $x > \frac{1}{C}$ and $x \neq 0$, where C is any non-zero constant. Solving equation (2), we have lost a solution $u = 0$. So, $y = 0$ is a particular solution to Bernoulli equation (1).