Problem sheet 8

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Solutions will be collected during the lecture on Wednesday June 5.

1. **[2+3 points]** Let $\mathbb{R}_2[x]$ denote the space of polynomials over $\mathbb{R}$ of degree at most two with the inner product
$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$
Show that the map $f : \mathbb{R}_2[x] \to \mathbb{R}$ is a linear functional on $\mathbb{R}_2[x]$ and find a polynomial $q \in \mathbb{R}_2[x]$ such that $f(p) = \langle p, q \rangle$, if a) $f(p) = p(0)$, b) $f(p) = \int_0^2 p(x)dx$

   (*Hint:* Use the proof of the Riesz representation theorem and the orthonormal basis from Ex. 2, Problem sheet 7)

2. **[2 points]** Let V be the space $\mathbb{C}^2$ with the standard inner product. Let T be the linear operator defined by $Te_1 = (1, -2)$, $Te_2 = (i, -1)$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. If $v = (z_1, z_2)$, find T^*v.

3. **[3 points]** Let V be an inner product space and u, w be fixed vectors in V. Show that $Tv = \langle v, u \rangle w$ defines a linear operator in V. Show that T has an adjoint, and describe T^* explicitly.

4. **[3 points]** Let V be a finite-dimensional inner product space over $\mathbb{F}$. Prove that $T \in \mathcal{L}(V)$ is an orthogonal projection if and only if $T^2 = T$ and T is self-adjoint.

5. **[4 points]** Show that the matrix
$$A = \begin{pmatrix} 4 & 1-i \\ 1+i & 5 \end{pmatrix}$$
is self-adjoint and find a unitary matrix U such that $U^{-1}AU$ is diagonal. Compute e^A.

6. **[4 points]** Let T be a normal operator on an inner product space V over $\mathbb{F}$.
a) Prove the identity $\langle Tv - av, Tv - av \rangle = \langle T^*v - \bar{a}v, T^*v - \bar{a}v \rangle$ for all $v \in V$ and $a \in \mathbb{F}$.
b) Infer that if λ is an eigenvalue of T with corresponding eigenvector u, then $\bar{\lambda}$ is an eigenvalue of T^* with the same corresponding eigenvector u.

7. **[3 points]** Prove that a real symmetric matrix A has a real symmetric cube root, i.e. there exists a real symmetric matrix B such that $B^3 = A$.