Problem sheet 7

Tutorials by Ikhwan Khalid <ikhwankhalid92@gmail.com> and Mahsa Sayyary <mahsa.sayyary@mis.mpg.de>.
Solutions will be collected during the lecture on Friday May 31.

1. **[3 points]** Consider the vector space $C([-\pi, \pi])$ of continuous function from $[-\pi, \pi]$ to $\mathbb{R}$ with the inner product
$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx, \quad f, g \in C([-\pi, \pi]).$$
Check that for any positive $n \in \mathbb{N}$ the vectors
$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\sin x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}$$
are orthonormal.

2. **[4 points]** Let $\mathbb{R}_2[x]$ be the space of polynomials over $\mathbb{R}$ of degree at most two with the inner product
$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx, \quad p, q \in \mathbb{R}_2[x].$$
Apply the Gram-Schmidt procedure to the standard basis $1, x, x^2$ in $\mathbb{R}_2[x]$ in order to produce an orthonormal basis in $\mathbb{R}_2[x]$. Find the projection of x^2 onto the subspace $\mathbb{R}_1[x]$ of polynomials of degree at most one.

3. **[3 points]** Prove that
$$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$$
is an inner product in $\mathbb{R}^2$. Notice that $(1, 0)$ and $(0, 1)$ are not orthogonal with respect to this inner product. Find an orthonormal basis using Gram-Schmidt orthogonalisation procedure.

4. **[3 points]** Let $U = \{v = (x_1, x_2, x_3, x_4) : \ x_1 + 2x_2 + x_3 + x_4 = 0, \ x_2 - x_4 = 0\}$ be a subspace of $\mathbb{R}^4$. Find $U^\perp$.

5. **[2 points]** Let P_U be a projection in a vector space V onto a subspace U of V. Find $\ker P_U$ and range P_U.

6. **[4 points]** Let V be a finite-dimensional inner product space over $\mathbb{F}$. Show that $P \in \mathcal{L}(V)$ is an orthonormal projection if and only if it satisfies the following properties
a) for every $v \in V$, $P(Pv) = Pv$, i.e. $P^2 = P$;
b) for all $u \in \ker P$ and $v \in \operatorname{range} P$, $\langle u, v \rangle = 0$.

7. **[4 points]** Let $U = \{u = (x, y, z) : \ x + y + z = 0\}$ be a plane in $\mathbb{R}^3$. Find an orthonormal basis e_1, e_2 in U and compute the projection of any vector $v = (x, y, z)$ onto U as
$$P_U(v) = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2.$$
Compute the distance of the point (x, y, z) to the plane U.