UNIVERSITÄT LEIPZIG

Problem sheet 3

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Solutions will be collected during the lecture on Monday April 29.

1. **[2 points]** Compute the rank of the following matrix:

$$\begin{pmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{pmatrix}.$$

2. **[2 points]** Are the vectors

$$\alpha_1 = (1,1,2,4), \quad \alpha_2 = (2,-1,-5,2),$$
$$\alpha_3 = (1,-1,-4,0), \quad \alpha_4 = (2,1,1,6)$$

linearly independent in $\mathbb{R}^4$?

3. **[3 points]** Let

$$\alpha_1 = (1,1,-2,1), \quad \alpha_2 = (3,0,4,-1), \quad \alpha_3 = (-1,2,5,2)$$

and let also

$$\alpha = (4,-5,9,-7), \quad \beta = (3,1,-4,4), \quad \gamma = (-1,1,0,1).$$

Which of the vectors α, β, γ belong to the subspace of $\mathbb{R}^4$ spanned by the vectors α_i, $i = 1,2,3$.

4. **[2+2+2 points]** Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}.$$

a) Find a basis for V.
b) Which vectors $(x_1, x_2, x_3, x_4, x_5)$ are elements of V.
c) If $(x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part a)?

5. **[1+2+1 points]** Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

a) If e_1, e_2, e_3 is the standard basis for $\mathbb{R}^3$ and e'_1, e'_2 is the standard basis for $\mathbb{R}^2$, what is the matrix of T relative to the bases e_1, e_2, e_3 and e'_1, e'_2?
b) If $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,1,1)$, $\alpha_3 = (1,0,0)$, $\beta_1 = (1,1)$, $\beta_2 = (1,0)$, what is the matrix of T relative to the bases $\alpha_1, \alpha_2, \alpha_3$ and β_1, β_2?
c) Compute the rank of T.

6. **[2 points]** Let T be the linear operator on $\mathbb{R}^3$, the matrix of which in the standard basis is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$

Find a basis for $\operatorname{range} T$ and a basis for $\ker T$.

7. [**2 points**] Find the fundamental system of solutions to the system

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 2x_4 + x_5 & = 0 \\ x_1 + 4x_2 - x_4 + 3x_5 & = 0 \\ 2x_1 + 6x_2 - x_3 + 5x_4 & = 0 \\ 5x_1 + 9x_2 + 2x_3 + 6x_4 + 4x_5 & = 0. \end{cases}$$

8. [**3 points**] Find solutions to the system of linear equations

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 + 2x_5 & = 1 \\ -x_2 + x_3 - 3x_5 + x_6 & = 2 \\ 3x_1 + x_2 + 2x_3 + x_4 - x_5 + 2x_6 & = -1 \\ 2x_1 + 3x_2 - x_3 + 2x_4 + x_6 & = -4. \end{cases}$$