Problem sheet 2

Tutorials by Ikhwan Khalid <ikhwankhalid92@gmail.com> and Mahsa Sayyary <mahsa.sayyary@mis.mpg.de>.
Solutions will be collected during the lecture on Wednesday April 24.

1. [**1 point**] Which vectors in $\mathbb{C}^3$ are linear combinations of $(0,1,1)$ and $(1,1,1)$?

2. [**3 points**] Show that the vectors
$$v_1 = (1,0,-1), \quad v_2 = (1,2,1), \quad v_3 = (0,-3,2)$$
form a basis for $\mathbb{R}^3$. Express each of the standard basis vector e_i, $i = 1,2,3$, as linear combinations of v_1, v_2, v_3.

3. [**2 points**] Find three vectors in $\mathbb{R}^3$ which are linearly dependent and are such that any two of them are linearly independent.

4. [**2 points**] Is the vector $(3,-1,0,-1)$ in $\mathbb{R}^4$ spanned by the vectors $(2,-1,3,2)$, $(-1,1,1,-3)$ and $(1,1,9,-5)$?

5. [**3 points**] Let $t \in \mathbb{R}$ be fixed. We consider a basis
$$q_0(z) = 1, \quad q_1(z) = z - t, \quad q_2(z) = (z-t)^2, \ldots, \quad q_n(z) = (z-t)^n$$
in $\mathbb{R}_n[z]$. What are the coordinates of $p \in \mathbb{R}_n[z]$ in the basis $q_0, q_1, \ldots, q_n$. In particular, compute the coordinates of the polynomial $p(z) = z^n$.

   (*Hint:* Use the Taylor's formula for polynomials)

6. [**3 points**] Let V and W be a vector spaces over $\mathbb{F}$ and $T \in \mathcal{L}(V,W)$ be invertible. Prove that T^{-1} is also linear map from W to V.

7. [**2+2 points**] a) Prove that the set of matrices
$$\mathcal{M} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : \ x, y \in \mathbb{R} \right\}$$
is a vector subspace of $\mathbb{R}^{2\times 2}$ which is additionally closed under the matrix multiplication, that is, if $A, B \in \mathcal{M}$ then $A \cdot B \in \mathcal{M}$.
b) Prove that $\mathcal{M}$ is isomorphic to the vector space of complex numbers over $\mathbb{R}$ and the isomorphism $T : \mathcal{M} \to \mathbb{C}$ additionally satisfies
$$T(A \cdot B) = T(A) \cdot T(B)$$
for any $A, B \in \mathcal{M}$, with matrix multiplication on the left-hand side and the usual complex multiplication on the right-hand side.

8. [**2+2 points**] For each of the two matrices
$$\begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$
use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

9. **[2 points]** Let T be a linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find the rule for T^{-1} like the one which defines T.

10. **[2 points]** Let V and W be vector spaces over $\mathbb{F}$ and let S be an isomorphism of V onto W (i.e. $S \in \mathcal{L}(V, W)$ and S is invertible). Prove that the map

$$T \mapsto STS^{-1}$$

is an isomorphism of $\mathcal{L}(V, V)$ onto $\mathcal{L}(W, W)$.