Problem sheet 14

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Solutions will be collected during the lecture on Wednesday July 10.

Points for solved exercises have to be included as bonus points for the homework

1. **[2 points]** Solve the following systems of linear equations:

$$\begin{cases} 2x_1 + 7x_2 + 3x_3 + x_4 &= 6, \\ 3x_1 + 5x_2 + 2x_3 + 2x_4 &= 4, \\ 9x_1 + 4x_2 + x_3 + 7x_4 &= 2. \end{cases}$$

2. **[2 points]** Find the fundamental system of solutions of the following system of homogeneous linear equations:

$$\begin{cases} 3x_1 + 2x_2 + 5x_3 + 2x_4 + 7x_5 &= 0, \\ 6x_1 + 4x_2 + 7x_3 + 4x_4 + 5x_5 &= 0, \\ 3x_1 + 2x_2 - x_3 + 2x_4 - 11x_5 &= 0, \\ 6x_1 + 4x_2 + x_3 + 4x_4 - 13x_5 &= 0. \end{cases}$$

3. **[2+3 points]** Compute the following determinants:

$$\begin{vmatrix} -3 & 9 & 3 & 6 \\ -5 & 8 & 2 & 7 \\ 4 & -5 & -3 & -2 \\ 7 & -8 & -4 & -5 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & n & n & n & n & n & n \end{vmatrix}.$$

4. **[3 points]** Compute the rank of matrix

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix}$$

depending on λ.

5. **[3 points]** Show that the following systems of vectors form bases in $\mathbb{R}^3$, find the change of basis matrix and find the coordinates of vector $x = (1, 0, 1)$ in both bases. The first system: $e_1 = (1, 1, 0)$, $e_2 = (0, 0, 1)$, $e_3 = (-1, 1, 0)$ and the second system: $e'_1 = (1, 2, 1)$, $e'_2 = (2, 3, 3)$, $e'_3 = (3, 7, 1)$.

6. **[2 points]** Find the dimension and a basis of the linear subspace $\text{span}\{v_1, v_2, v_3, v_4, v_5\}$ of $\mathbb{R}^4$, where $v_1 = (1, 0, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (1, 1, 1, 1)$, $v_4 = (1, 2, 3, 4)$, $v_5 = (0, 1, 2, 3)$.

7. **[2 points]** Determine whether the matrix

$$\begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}$$

can be reduced to a diagonal form by going over to a new basis. Find that basis and the corresponding matrix.

8. **[3 points]** Let a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x, x + y, x + y + z)$. Find its matrix in the standard basis and also in the basis $v_1 = (1, 0, 0)$, $v_2 = (2, 0, 1)$, $v_3 = (0, 1, 1)$. Check if T is invertible and if yes, find the inverse map T^{-1}. Is T self-adjoint?

9. **[3 points]** Find the orthogonal projection of the vector $u = (4, -1, -3, 4)$ onto $\operatorname{span}\{v_1, v_2, v_3\}$ in $\mathbb{R}^4$ with the standard inner product, where $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 2, -1)$, $v_3 = (1, 0, 0, 3)$.

10. **[4 points]** Consider the vector space $\mathbb{R}_n[x]$ of all polynomials with real coefficients of degree at most n with the inner product
$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx.$$
Show the following polynomials
$$p_0(x) = 1, \quad p_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}\left((x^2-1)^k\right), \quad k = 1, \dots, n,$$
form a basis in $\mathbb{R}_n[x]$.

11. **[3 points]** Show that A is self-adjoint and compute e^A, if
$$A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}.$$

12. **[2 points]** Reduce to a canonical form the following quadratic form on $\mathbb{R}^3$
$$Q(x) = x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3.$$

13. **[2 points]** Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$, where $\overline{A}$ denotes the closure of a set A.

14. **[3 points]** Compute the partial derivatives $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$, if $f(x, y) = \sqrt[3]{xy}$. Is the function f differentiable at $(0, 0)$?

15. **[3 points]** Does $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ exist, if
$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } x^2 + y^2 > 0, \\ 0, & \text{if } x = y = 0? \end{cases}$$

16. **[3 points]** Show that the function $z = z(x, y)$ defined by the equation
$$F(x - az, y - bz) = 0,$$
where F is a differentiable function of two variables and a, b are some constants, solves the equation
$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 1.$$

17. **[2+2+2 points]** Find the general solutions to the following equations:
a) $y' \ln|y| + x^2 y = 0$; b) $xy' + (1 + 2x^2)y = x^3 e^{-x^2}$; c) $y' = \frac{y^2+2xy}{x^2}$.

18. **[3+3 points]** Solve the initial value problems:
a) $y' - xy = xy^{\frac{3}{2}}$, $y(1) = 4$; b) $y^{(4)} - 16y = 0$, $y(0) = y'(0) = 2$, $y''(0) = -2$, $y'''(0) = 0$.