Problem sheet 11

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Solutions will be collected during the lecture on Wednesday June 26.

1. [**3 points**] Let $f(x_1, x_2, x_3) = \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{\alpha}{2}}$, $(x_1, x_2, x_3) \in \mathbb{R}^3$, $\alpha \geq 0$. For which α is the function f differentiable at 0?

   (*Hint:* Use the definition of differentiable function at a point)

2. [**2 points**] Find the gradient and the differential of the function $f(x, y, z) = \ln(x + y^2) + ze^x$.

3. [**2 points**] Find the differential of the function $u = f(x+y+z, x^2+y^2+z^2)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function.

4. [**3+2 points**] Consider the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & \text{if } x^2+y^2 > 0, \\ 0 & \text{if } x = y = 0. \end{cases}$$

   a) Compute the partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$ both if $(x_0, y_0) \neq (0, 0)$ and if $(x_0, y_0) = (0, 0)$. Show that the partial derivatives are not continuous at $(0, 0)$.
   b) Prove that f is differentiable at 0.

5. [**3 points**] Find a tangent plane to the graph of the function $f(x, y) = x + y^2$, $(x, y) \in \mathbb{R}^2$ through the point $(1, -1, 2)$.

6. [**3 points**] Let $f(x_1, x_2) = \sqrt{|x_1^2 - x_2^2|}$, $(x_1, x_2) \in \mathbb{R}^2$. Determine all directions $l \in \mathbb{R}^2$ along which $\frac{\partial f}{\partial l}(0, 0)$ exists.

7. [**2 points**] Compute the directional derivative of the function $f(x, y, z) = \sin(x + y) - e^{z-x}$ at the point $(1, -1, 1)$ along the direction $(1, 0, 3)$.

8. [**4 points**] Compute the Jacobian of the function

$$f(r, \theta, \varphi) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta), \quad r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi.$$