Problem sheet 10

Tutorials by Ikhwan Khalid <ikhwankhalid92@gmail.com> and Mahsa Sayyary <mahsa.sayyary@mis.mpg.de>.
Solutions will be collected during the lecture on Wednesday June 19.

1. **[3 points]** Let K_1, K_2 be compact sets in $\mathbb{R}$. Show that the cartesian product $K = K_1 \times K_2 = \{(x, y) : \ x \in K_1, \ y \in K_2\}$ is compact in $\mathbb{R}^2$.

2. **[2 points]** Using the definition of limit, show that $\lim_{(x,y)\to(0,0)} x \sin \frac{1}{y} = 0$.

3. **[3x3 points]** Do the following limits exist? Justify your answer.

   (a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$;

   (b) $\lim_{(x,y)\to(0,0)} \frac{e^x-y}{xy}$;

   (c) $\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(x^2+y^2+z^2)}{x^2+y^2+z^2}$.

4. **[3 points]** Show that a function $f : \mathbb{R}^d \to \mathbb{R}^m$ is continuous on $\mathbb{R}^d$ if and only if $f^{-1}(F)$ is close for each close set F in $\mathbb{R}^m$.

5. **[3+2 points]** Let $D \subset \mathbb{R}^d$.

   (a) Show that a subset A of D is open in D if and only if there exists an open (in $\mathbb{R}^d$) subset $\tilde{A}$ such that $A = \tilde{A} \cap D$.

   (b) Let additionally D be open. Conclude from (a) that A is open in D if and only if A is open (in $\mathbb{R}^d$).

6. **[2 points]** Construct a continuous function $f : D \to \mathbb{R}$ which is bounded and does not attain its maximum, if $D = B_1(0) \subset \mathbb{R}^2$. Can one construct such a function in the case of the close ball $D = \bar{B}_1(0)$?