Problem sheet 1

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Solutions will be collected during the lecture on Wednesday April 17.

1. **[2 points]** Check whether the following two systems are equivalent. If so, express each equation in each system as a linear combination of the equations in the other system.

$$\begin{cases} x_1 - x_2 = 0, \\ 2x_1 + x_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 3x_1 + x_2 = 0, \\ x_1 + x_2 = 0. \end{cases}$$

2. **[2 points]** Suppose that A, B, C, D and E are matrices over $\mathbb{F}$ having the following sizes:

$$A \text{ is } 4 \times 5, \quad B \text{ is } 4 \times 5, \quad C \text{ is } 5 \times 2, \quad D \text{ is } 4 \times 2, \quad E \text{ is } 5 \times 4.$$

   Determine whether the following matrix expressions are defined and for those that are defined, determine the size of the resulting matrix:

$$a)\ BA; \quad b)\ AC + D; \quad c)\ AE + B; \quad d)\ AB + B; \quad e)\ E(A + B); \quad f)\ E(AC).$$

3. **[2 points]** Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

   Compute ABC and CAB.

4. **[1 point]** Find a 2×2 matrix A such that $A^2 = 0$ and $A \neq 0$.

5. **[2 points]** Find a row-reduced echelon matrix which is row-equivalent to

$$A = \begin{pmatrix} 1 & -i \\ 2 & 2 \\ i & 1 + i \end{pmatrix}.$$

   What are the solutions of $Ax = 0$?

6. **[1 point]** Describe explicitly all 2×2 row-reduced echalon matrices.

7. **[3+3+3 points]** Find all solutions of the following systems of linear equations:

$$\begin{cases} x_1 - x_2 + 2x_3 = 1, \\ 2x_1 + x_3 = 1, \\ x_1 - 3x_2 + 4x_3 = 2, \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = 1, \\ x_1 + x_2 - x_3 + x_4 = 2, \\ x_1 + 7x_2 - 5x_3 - x_4 = 3, \end{cases}$$

$$\begin{cases} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 &= -2, \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 &= -2, \\ 2x_1 - 4x_3 + 2x_4 + x_5 &= 3, \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 &= -7. \end{cases}$$

8. **[3 points]** Let

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}.$$

For which triples (b_1, b_2, b_3) does the system $Ax = b$ have a solution?