Exam Solutions

Each of the exercise is 4 points.

1. Show that for every $n \in \mathbb{N}$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

Solution. To prove equality (1), we use the mathematical induction. For $n = 1$ we have $1^2 = \frac{1\cdot 2\cdot 3}{6}$. We assume that (1) is true for $n \in \mathbb{N}$ and check it for $n+1$. So,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$
$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6}.$$

In order to finish the proof, we have to check that $n(2n+1)+6(n+1) = (n+2)(2(n+1)+1)$. For this, we compute $n(2n+1)+6(n+1) = 2n^2+7n+6$ and $(n+2)(2(n+1)+1) = (n+2)(2n+3) = 2n^2 + 7n + 6$.

2. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n \to \infty$.

Solution. Let $\varepsilon > 0$ be fixed. Since $\frac{a_n}{n} \to 0$, $n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ $\left|\frac{a_n}{n} - 0\right| = \frac{|a_n|}{n} < \varepsilon$. We next choose $N_2 \in \mathbb{N}$ such that $\frac{|a_k|}{N_2} < \varepsilon$ for all $k = 1, \ldots, N_1$. Thus, taking $N := \max\{N_1, N_2\}$, we can estimate for every $n \geq N$ and $k = 1, \ldots, n$

$$\frac{|a_k|}{n} \leq \frac{|a_k|}{N_2} < \varepsilon, \quad \text{if} \quad k \leq N_1,$$

and

$$\frac{|a_k|}{n} \leq \frac{|a_k|}{k} < \varepsilon, \quad \text{if} \quad N_1 < k \leq n.$$

Hence, we have

$$\left|\frac{\max\{a_1, a_2, \ldots, a_n\}}{n}\right| \leq \frac{\max\{|a_1|, |a_2|, \ldots, |a_n|\}}{n} < \varepsilon.$$

This implies that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n \to \infty$.

3. Is the function

$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad x \in (-\pi, \pi),$$

continuous on $(-\pi, \pi)$? Is f differentiable on $(-\pi, \pi)$? Compute the derivative of f at each point where it exists.

Solution. Since sin and cos are continuous functions and $\sin x \neq 0$ for all $x \in (-\pi, \pi) \setminus \{0\}$, the function f is continuous at each point of $(-\pi, \pi) \setminus \{0\}$. To check the continuity of f at 0, we compute

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{x^2(1-\cos x)}{x^2 \sin x}$$
$$= \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} = 0 \cdot 1 \cdot \frac{1}{2} = 0 = f(0).$$

Hence, the function f is continuous at 0 and, consequently, it is continuous on $(-\pi, \pi)$.

Similarly, f is differentiable at each point of $(-\pi, \pi) \setminus \{0\}$ because sin and cos are differentiable and $\sin x \neq 0$ for all $x \in (-\pi, \pi) \setminus \{0\}$. Moreover, for every $x \in (-\pi, \pi) \setminus \{0\}$

$$f'(x) = \left(\frac{1-\cos x}{\sin x}\right)' = \frac{(1-\cos x)' \sin x - (1-\cos x)(\sin x)'}{\sin^2 x}$$
$$= \frac{\sin^2 x - (1-\cos x)\cos x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x}.$$

We next compute

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{1-\cos x}{x \sin x} = \lim_{x\to 0} \frac{x(1-\cos x)}{x^2 \sin x}$$
$$= \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

Hence the function f is differentiable on $(-\pi, \pi)$ and

$$f'(x) = \begin{cases} \frac{1-\cos x}{\sin^2 x}, & x \neq 0, \\ \frac{1}{2}, & x = 0, \end{cases} \quad x \in (-\pi, \pi),$$

4. Compute the limit $\lim_{x\to 0} \left(1 + \arcsin^2 x\right)^{\frac{1}{\tan^2 x}}$.

   Solution.

$$\lim_{x\to 0} \left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}} = \lim_{x\to 0} e^{\ln\left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}}} = \lim_{x\to 0} e^{\frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x}}$$
$$= e^{\left(\lim_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x}\right)},$$

by the continuity of the exponential function. So, we compute

$$\lim_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x} = \lim_{x\to 0} \frac{\arcsin^2 x \cdot \ln\left(1+\arcsin^2 x\right)}{\arcsin^2 x \cdot \tan^2 x} = \lim_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\arcsin^2 x} \cdot \lim_{x\to 0} \frac{\arcsin^2 x}{\tan^2 x}$$
$$= 1 \cdot \lim_{x\to 0} \frac{x^2 \cdot \arcsin^2 x}{x^2 \cdot \tan^2 x} = \lim_{x\to 0} \frac{\arcsin^2 x}{x^2} \cdot \lim_{x\to 0} \frac{x^2}{\tan^2 x}$$
$$= \left(\lim_{x\to 0} \frac{\arcsin x}{x}\right)^2 \cdot \left(\lim_{x\to 0} \frac{x}{\tan x}\right)^2 = 1.$$

Hence, $\lim_{x\to 0} \left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}} = e^1 = e$.

5. Find points of local maximum and minimum of the function $f(x) = x^2(x-5)^3$, $x \in \mathbb{R}$.

   Solution. We first compute critical points of f:

$$f'(x) = \left(x^2(x-5)^3\right)' = \left(x^2\right)'(x-5)^3 + x^2\left((x-5)^3\right)' = 2x(x-5)^3 + 3x^2(x-5)^2$$
$$= x(x-5)^2\left(2(x-5)+3x\right) = x(x-5)^2(5x-10) = 0.$$

   Hence, the points $x = 0$, $x = 2$, $x = 5$ are critical.

   The point 0 is a point of strict local maximum because the derivative changes its sign from "+" to "–", passing through 0.

   The point 2 is a point of strict local minimum because the derivative changes its sign from "–" to "+", passing through 2.

   The point 5 is not a point of local extrema because the derivative stays positive, passing through 5.

6. Compute the length of continuous curve defined by the function $y = x^{\frac{3}{2}}$, $x \in [0,4]$.

   Solution. The length of the curve Γ defined by the function $y = x^{\frac{3}{2}}$, $x \in [0,4]$, can be computed by the formula

$$l(\Gamma) = \int_0^4 \sqrt{1 + \left(\left(x^{\frac{3}{2}}\right)'\right)^2}\,dx = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx = \left| \begin{array}{l} y = 1 + \frac{9}{4}x, \\ x = \frac{4}{9}(y-1), \\ dx = \frac{4}{9}dy \end{array} \right|$$
$$= \frac{4}{9}\int_1^{10} y^{\frac{1}{2}}dy = \frac{4}{9} \cdot \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^{10} = \frac{8}{27} \cdot y^{\frac{3}{2}}\Bigg|_1^{10} = \frac{8}{27}(10\sqrt{10} - 1).$$

7. Compute the improper integral $\int_2^{+\infty} \frac{\ln x}{x^2}dx$.

   Solution. First we change the variable and then use the integration by parts formula:

$$\int_2^{+\infty} \frac{\ln x}{x^2}dx = \left| \begin{array}{l} y = \ln x, \\ x = e^y, \\ dx = e^y dy \end{array} \right| = \int_{\ln 2}^{+\infty} \frac{ye^y}{e^{2y}}dy = \int_{\ln 2}^{+\infty} ye^{-y}dy = -\int_{\ln 2}^{+\infty} y\,de^{-y}$$
$$= -ye^{-y}\Big|_{\ln 2}^{+\infty} + \int_{\ln 2}^{+\infty} e^{-y}dy = \ln 2 \cdot e^{-\ln 2} - e^{-y}\Big|_{\ln 2}^{+\infty} = \frac{\ln 2}{2} + e^{-\ln 2} = \frac{1+\ln 2}{2}.$$

   Another way of the computation without change of variable:

$$\int_2^{+\infty} \frac{\ln x}{x^2}dx = -\int_2^{+\infty} \ln x\, d\frac{1}{x} = -\ln x \frac{1}{x}\Big|_2^{+\infty} + \int_2^{+\infty} \frac{1}{x}d\ln x = \frac{\ln 2}{2} + \int_2^{+\infty} \frac{1}{x^2}dx$$
$$= \frac{\ln 2}{2} - \frac{1}{x}\Big|_2^{+\infty} = \frac{1+\ln 2}{2}$$

8. Investigate the absolute and conditional convergence of the series $\sum\limits_{n=1}^{\infty}(-1)^n \ln\left(1 + \frac{1}{\sqrt{n}}\right)$.

   Solution. The series

$$\sum_{n=1}^{\infty}\left|(-1)^n \ln\left(1 + \frac{1}{\sqrt{n}}\right)\right| = \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{\sqrt{n}}\right)$$

diverges because $\ln\left(1+\frac{1}{\sqrt{n}}\right) \sim \frac{1}{\sqrt{n}}$, $n \to \infty$, and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Since the sequence $\ln\left(1+\frac{1}{\sqrt{n}}\right)$, $n \geq 1$, is monotone and converges to 0, the series

$$\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{\sqrt{n}}\right)$$

converges, according to Leibniz's test. This implies the conditional convergence of the series.

9. Compute $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{12}$.

Solution. We first write the numbers $1+\sqrt{3}i$ and $1-i$ in the polar form. We compute the absolute volume r and the argument θ of $1+\sqrt{3}i$. So, $r = \sqrt{1^2+(\sqrt{3})^2} = 2$ and $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$. Thus, $\theta = \frac{\pi}{3}$. So, we obtain

$$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Similarly,

$$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).$$

Hence

$$\left(\frac{1+\sqrt{3}i}{1-i}\right)^{12} = \left(\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)}\right)^{12} = 2^6\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)^{12}$$
$$= 64\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)^{12} = 64\left(\cos\frac{12\cdot 7\pi}{12}+i\sin\frac{12\cdot 7\pi}{12}\right) = -64.$$

10. Show that there does not exist any linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with

$$\ker T = \{(x_1,x_2,x_3,x_4,x_5) : \; x_1 = x_2, \; x_3 = x_4 = -x_5\}. \tag{2}$$

Solution. We assume that there exists a map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with the kernel given by (2). We first note that

$$\ker T = \{(x_1,x_2,x_3,x_4,x_5) : \; x_1 = x_2, \; x_3 = x_4 = -x_5\} = \{(a,a,b,b,-b) : \; a,b \in \mathbb{R}\}.$$

Thus, the vectors $v_1 = (1,1,0,0,0)$ and $v_2 = (0,0,1,1,-1)$ form a basis of $\ker T$, since they are linearly independent and span $\ker T$. Hence, $\dim(\ker T) = 2$. Since $5 = \dim(\mathbb{R}^5) = \dim(\ker T) + \dim(\operatorname{range} T) = 2 + \dim(\operatorname{range} T)$, we have that $\dim(\operatorname{range} T) = 3$. But that is impossible because $\operatorname{range} T \subset \mathbb{R}^2$ and $\dim(\mathbb{R}^2) = 2$.