Problem sheet for the exam[1]

Theoretical questions are not included here.

1. Show that

   a) $A \cup \emptyset = A$, $A \cup A = A$, $A \cup B = B \cup A$, $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;

   b) $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup B = B \cap A$, $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;

   c) $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$, $A \setminus B = A \cap B^c$;

   d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   e) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

2. Let $A_n = \{1, \ldots, n\}$ for each $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \mathbb{N}, \quad \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \{1\}.$$

3. Prove that
   a) $\sqrt{6} \notin \mathbb{Q}$; b)$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$; c) for each $n \in \mathbb{N}$ either $\sqrt{n} \in \mathbb{N}$ or $\sqrt{n} \notin \mathbb{Q}$.

4. Using mathematical induction prove that:
   a) $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n;
   b) $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for each $n \in \mathbb{N}$;
   c) $11^n - 4^n$ is divisible by 7 for each $n \in \mathbb{N}$;
   d) $5^n - 4n - 1$, $n \in \mathbb{N}$, are divisible by 16;
   e) $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

5. Prove that there does not exist a rational number x solving the equation $x^2 = 2$.

6. Prove that the following sets are bounded:

   a) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$;

   b) $\left\{\frac{(-1)^n n+1}{n-(-1)^n} : n \in \mathbb{N}\right\}$.

7. For each $a < b$ prove that $\inf[a, b] = \inf(a, b] = a$ and $\sup[a, b] = \sup[a, b) = b$.

8. Show that
   a) $2^n \geq n + 1$, $n \in \mathbb{N}$; b) $3^n \geq 2n + 1$, $n \in \mathbb{N}$; c) $2^n > (\sqrt{2} - 1)^2 n^2$, $n \in \mathbb{N}$.

9. Let $x_1, \ldots, x_n$ be a positive real numbers. Prove that

$$(1 + x_1) \cdot \ldots \cdot (1 + x_n) \geq 1 + x_1 + \ldots + x_n.$$

[1] The listed exercises or similar ones will be chosen for the exam

10. Prove the boundedness of the following sequences:

    a) $\left(\frac{2^n}{n!}\right)_{n\geq 1}$; b) $\left(a_n = \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n \text{ square roots}}\right)_{n\geq 1}$.

11. Prove the following statements:

    a) $a_n \to a,\ n\to\infty \ \Leftrightarrow\ a_n - a \to 0,\ n\to\infty \ \Leftrightarrow\ |a_n - a| \to 0,\ n\to\infty$;

    b) $a_n \to 0,\ n\to\infty \ \Leftrightarrow\ |a_n| \to 0,\ n\to\infty$;

    c) $a_n \to a,\ n\to\infty \ \Leftrightarrow\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}:\ \{a_N, a_{N+1}, \ldots\} \subset (x-\varepsilon, x+\varepsilon)$;

    d) $a_n \to 0,\ n\to\infty \ \Leftrightarrow\ \sup\{|a_k|:\ k\geq n\} \to 0,\ n\to\infty$;

    e) $a_n \to a,\ n\to\infty \ \Rightarrow\ |a_n| \to |a|,\ n\to\infty$.

12. Prove that for a sequence $(a_n)_{n\geq 1}$ with $a_n \neq 0$ the equality $\lim\limits_{n\to\infty} |a_n| = +\infty$ is equivalent to $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

13. Compute the following limits:
    a) $\lim\limits_{n\to\infty} \frac{n^3-2n^2\cos n+n}{\sqrt{n}-3n^3+1}$; b) $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n})$; c) $\lim\limits_{n\to\infty} \sqrt[n]{n^2 2^n + 3^n}$; d) $\lim\limits_{n\to\infty} \frac{\sin^2 n}{\sqrt{n}}$; e) $\lim\limits_{n\to\infty} \frac{n^2+\sin n}{n^2+n\cos n}$; f) $\lim\limits_{n\to\infty} \frac{2^n+n^3}{3^n+1}$; g) $\sqrt[n+1]{n}$.

14. Let $(a_n)_{n\geq 1}$ be a bounded sequence and $b_n \to 0$, $n \geq \infty$. Prove that $a_n b_n \to 0$, $n\to\infty$.

15. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n\to\infty$. Prove that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n\to\infty$.

16. Let $(a_n)_{n\geq 1}$ be a bounded sequence and $b_n \to +\infty$, $n \geq \infty$. Prove that $a_n + b_n \to +\infty$, $n\to\infty$.

17. Using the monotonicity compute the following limits:

    a) $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$; b) $\lim\limits_{n\to\infty} \frac{n!}{2^{n^2}}$; c) $\lim\limits_{n\to\infty} \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n \text{ square roots}}$; d) $\lim\limits_{n\to\infty} \frac{n}{2^{\sqrt{n}}} = 0$.

18. Identify the set of subsequential limits of the following sequences:
    a) $\left(\sin\frac{2\pi n}{3}\right)_{n\geq 1}$; b) $(\sin 3\pi n)_{n\geq 1}$; c) $(a_n)_{n\geq 1}$,
    where $a_n = \begin{cases} (-1)^{\frac{n+1}{2}} + n, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} + \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$

19. Prove that $a_n \to a$, $n\to\infty \Leftrightarrow \varliminf\limits_{n\to\infty} a_n = \varlimsup\limits_{n\to\infty} a_n = a$.

20. For a sequence $(a_n)_{n\geq 1}$ compute $\varliminf\limits_{n\to\infty} a_n$ and $\varlimsup\limits_{n\to\infty} a_n$, if for all $n\geq 1$

    a) $a_n = 1 - \frac{1}{n}$; b) $a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$; c) $a_n = \frac{n-1}{n+1}\cos\frac{2n\pi}{3}$; d) $a_n = 1 + n\sin\frac{n\pi}{2}$;
    e) $a_n = \left(1+\frac{1}{n}\right)^n \cdot (-1)^n + \sin\frac{n\pi}{4}$; f) $a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$.

21. Check whether the following sequences are Cauchy sequences.
    a) $\left(\frac{1}{2^n}\right)_{n\geq 1}$; b) $((-1)^n)_{n\geq 1}$; c) $\left(a_n = \frac{\sin 1}{2^1} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}\right)_{n\geq 1}$.

22. Show that $(a_n)_{n\geq 1}$ is a Cauchy sequence iff $\sup\limits_{m\geq N, n\geq N} |a_m - a_n| \to 0$, $N \to \infty$.

23. Find the domain and the range of the following functions:
a) $f(x) = \frac{1}{(x+1)^2}$; b) $f(x) = \sqrt{1-x^2}$; c) $f(x) = \ln(1+x)$.

24. Let $f : X \to Y$ and $A_1 \subset X$, $A_2 \subset X$. Check that
a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; b) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$; c) $(f(A_1) \setminus f(A_2)) \subset f(A_1 \setminus A_2)$;
d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$; e) $A_1 \subset f^{-1}(f(A_1))$; f) $(f(X) \setminus f(A_1)) \subset f(X \setminus A_1)$.

25. Let $f : X \to Y$ and $B_1 \subset Y$, $B_2 \subset Y$. Show that
a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$; b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
c) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$; d) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$;
e) $f(f^{-1}(B_1)) = B_1 \cap f(X)$; f) $f^{-1}(B_1^c) = (f^{-1}(B_1))^c$.

26. Prove that the set of all limit points of $\mathbb{Q}$ equals $\mathbb{R} \cup \{-\infty, +\infty\}$.

27. Prove that $\frac{1}{f(x)} \to 0$, $x \to a$, if $f(x) \to +\infty$, $x \to a$.

28. Prove that the limit of the function $f(x) = \cos\frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exist at the point $a = 0$.

29. Using $\varepsilon - \delta$ definition, show that
a) $\lim\limits_{x\to 4} \sqrt{x} = 2$; b) $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$.

30. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\tan x}{x}$; b) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$; c) $\lim\limits_{x\to 1} \frac{x^2-x}{x^2-3x+2}$; d) $\lim\limits_{x\to+\infty} \frac{x^3-x\sin x+x}{1-3x^3+\ln x}$; e) $\lim\limits_{x\to+\infty} \frac{x^2+\cos x+1}{\sqrt{x^4+1}+x+3}$;
f) $\lim\limits_{x\to 0} \left(\frac{2}{\sin^2 x} - \frac{1}{1-\cos x}\right)$; g) $\lim\limits_{x\to 0} \frac{x^2+x}{\sqrt[3]{1+\sin x}-1}$; h) $\lim\limits_{x\to+\infty} \left(x(\sqrt{x^2+2x+2} - x - 1)\right)$;
i) $\lim\limits_{x\to+\infty} (\sqrt{ax+1} - \sqrt{x})$, for some $a > 0$.

31. Compute the following limits:
a) $\lim\limits_{x\to 0-} \frac{x}{\sqrt{1-\cos^2 x}}$; b) $\lim\limits_{x\to 0+} \frac{x}{\sqrt{1-\cos^2 x}}$; c) $\lim\limits_{x\to\frac{\pi}{2}-} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$; d) $\lim\limits_{x\to\frac{\pi}{2}+} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$;
e) $\lim\limits_{x\to 0+} e^{-\frac{1}{x}}$; f) $\lim\limits_{x\to 0+} \frac{e^{-\frac{1}{x}}}{x}$.

32. Let f be an increasing function on an interval $[a, b]$.
a) For each $c \in (a, b)$ show that the one-sided limits $f(a+)$, $f(c-)$, $f(c+)$, $f(b-)$ exist.
b) Check the inequalities

$$f(a) \leq f(a+) \leq f(c-) \leq f(c) \leq f(c+) \leq f(b-) \leq f(b),$$

for all $c \in (a, b)$.
c) Prove that $\lim\limits_{x\to c+} f(x-) = f(c+)$ and $\lim\limits_{x\to c-} f(x+) = f(c-)$ for all $c \in (a, b)$.

33. Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. For which a, b the function f is continuous on $\mathbb{R}$?

34. Compute the following limits:
a) $\lim\limits_{x\to 0}(\tan x - e^x)$; b) $\lim\limits_{x\to 2} \frac{x^2-3^x+1}{x-\sin \pi x}$; c) $\lim\limits_{x\to 3} \frac{x\cos x+1}{x^3+1}$.

35. Prove that the function $f(x) = \sin\frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

36. Show that the Dirichlet function $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at any point of $\mathbb{R}$.

37. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\ln(1+x)+\arcsin x^2}{\arccos x+\cos x}$; b) $\lim\limits_{x\to 1} \frac{\arctan x}{1+\arctan x^2}$; c) $\lim\limits_{x\to 0} \frac{\arcsin x}{x}$; d) $\lim\limits_{x\to 0} \frac{x}{\sin x+\arcsin x}$; e) $\lim\limits_{x\to 0} \frac{\arctan x}{x}$;
f) $\lim\limits_{x\to 0} \frac{\arccos x-\frac{\pi}{2}}{x}$; g) $\lim\limits_{x\to 0} \frac{\sin(\arctan x)}{\tan x}$.

38. Compute the following limits:
a) $\lim\limits_{x\to 0}(\cos x)^x$; b) $\lim\limits_{x\to +\infty} x(\ln(1+x)-\ln x)$; c) $\lim\limits_{x\to 0}\left(\frac{1+\sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$; d) $\lim\limits_{x\to 0} \frac{1-\cos x}{1-\cos 2x}$; e) $\lim\limits_{x\to 0} \frac{\ln(1+x)+e^x-\cos x}{e^{x^2}-1+\sin x}$;
f) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$; g) $\lim\limits_{x\to 0} \frac{\arcsin(x-1)}{x^m-1}$ for $m \in \mathbb{N}$; h) $\lim\limits_{x\to 0} \frac{1-(\cos mx)^m}{x^2}$ for $m \in \mathbb{N}$;
i) $\lim\limits_{x\to 0} \frac{1-(\cos mx)^{\frac{1}{m}}}{x^2}$ for $m \in \mathbb{N}$; k) $\lim\limits_{x\to 0} \frac{\sqrt[7]{\cos x}-1}{\sqrt[3]{1+x^2}-1}$; l) $\lim\limits_{x\to 0} \frac{e^{\sin 2x}-e^{\tan x}}{x}$.

39. Prove that the function $P(x) = x^3 + 7x^2 - 1$, $x \in \mathbb{R}$, has at least one root, that is, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = 0$.

40. Let $g : [0,1] \to [0,1]$ be a continuous function on $[0,1]$. Show that there exists $x_0 \in [0,1]$ such that $g(x_0) = x_0$.

41. Let $f, g : [0,1] \to [0,1]$ be continuous and f be a surjection. Prove that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

42. Using the definition of derivative, check that $(x|x|)' = 2|x|$, $x \in \mathbb{R}$.

43. Show that the following functions are not differentiable at 0.
a) $f(x) = |x|$, $x \in \mathbb{R}$; b) $f(x) = \sqrt[3]{x}$, $x \in \mathbb{R}$; c) $f(x) = x\sin\frac{1}{x}$, $x \in \mathbb{R}\setminus\{0\}$, and $f(0) = 0$.

44. For the function $f(x) = |x^2 - x|$, $x \in \mathbb{R}$, compute $f'(x)$ for each $x \in \mathbb{R}\setminus\{0,1\}$. Compute left and right derivatives at points 0 and 1.

45. Let
$$f(x) = \begin{cases} x^2, & x \leq 1, \\ ax+b, & x > 1. \end{cases}$$
For which $a, b \in \mathbb{R}$ the function f:
a) is continuous on $\mathbb{R}$; b) is differentiable on $\mathbb{R}$? Compute also f'.

46. Check whether the following functions are differentiable at 0. Justify your answer.
a) $f(x) = \begin{cases} \frac{\cos x-1}{x}, & x \neq 0, \\ 0, & x = 0; \end{cases}$ b) $f(x) = \sqrt[5]{x^2}$, $x \in \mathbb{R}$; c) $f(x) = |\sin x|$, $x \in \mathbb{R}$.

47. Prove that f is continuous at a point a if $f'_-(a)$ and $f'_+(a)$ exist.

48. Compute derivatives of the following functions:
a) $f(x) = x^2\sin x$; b) $f(x) = e^{-\frac{x^2}{2}}\cos x$; c) $f(x) = \frac{x}{1+x^2}$; d) $f(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}$;
e) $f(x) = 2^{\tan(x^2-1)}$; f) $f(x) = \sin(\cos^2(\tan^3 x))$; g) $f(x) = \sqrt[3]{\frac{1+x^3}{1-x^3}}$;

h) $y = e^{-x^2 \sin x}$; i) $y = \frac{\sin^2 x}{\sin x^2}$; j) $y = e^x \left(1 + \cot \frac{x}{2}\right)$; k) $f(x) = e^{ax} \cdot \frac{a \sin bx - b \cos bx}{\sqrt{a^2+b^2}}$, where a, b are some constants.

49. Let $f(x) = \frac{1}{x^3} e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f'(0) = 0$.

50. Compute derivatives of the following functions:
a) $f(x) = \frac{1}{4} \ln \frac{x^2-1}{x^2+1}$; b) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$; c) $f(x) = \ln \tan \frac{x}{2}$;
d) $f(x) = \arcsin \frac{1-x}{\sqrt{2}}$; e) $f(x) = \arctan \frac{1+x}{1-x}$; f) $f(x) = x^x$; g) $f(x) = \sqrt[x]{x}$.

51. Let a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

52. Prove that
a) $x - \frac{x^3}{3!} \leq \sin x \leq x$ for all $x \geq 0$;
b) $1 - \frac{x^2}{2} \leq \cos x \leq 1$ for all $x \geq 0$;
c) $\frac{x}{1+x} \leq \ln(1+x) \leq x$ for all $x > -1$.

53. For each $\alpha > 1$, prove that $(1+x)^\alpha \geq 1 + \alpha x$ for all $x > -1$.

54. Identify the intervals on which the following functions are monotone.
a) $f(x) = 3x - x^3$, $x \in \mathbb{R}$ b)$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{x^2}{2^x}$, $x \in \mathbb{R}$;
d) $f(x) = x + \sqrt{|1-x^2|}$, $x \in \mathbb{R}$; e) $f(x) = \frac{1}{x^3} - \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$.

55. Identify $a \in \mathbb{R}$ for which the function $f(x) = x + a \sin x$, $x \in \mathbb{R}$, is increasing on $\mathbb{R}$.

56. Using L'Hospital's Rule, show that
a) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = 1$; b) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{\sin x} = 1$; c) $\lim\limits_{x\to e} \frac{(\ln x)^\alpha - \left(\frac{x}{e}\right)^\beta}{x-e} = \frac{\alpha-\beta}{e}$,
where α, β are some real numbers; d) $\lim\limits_{x\to 1} \frac{\left(\frac{4}{\pi}\arctan x\right)^\alpha - 1}{\ln x} = \frac{2\alpha}{\pi}$, $\alpha \in \mathbb{R}$;
e) $\lim\limits_{x\to 0+} \left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = e^{-\frac{1}{2}}$; f) $\lim\limits_{x\to+\infty} \frac{x}{2^x} = 0$; g) $\lim\limits_{x\to+\infty} \frac{\ln x}{x^\varepsilon} = 0$ for all $\varepsilon > 0$;
h) $\lim\limits_{x\to+0} x^\varepsilon \ln x = 0$ for all $\varepsilon > 0$; i) $\lim\limits_{x\to+0} (\ln(1+x))^x = 1$.

57. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2}$; b) $\lim\limits_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$; c) $\lim\limits_{x\to+\infty} \left(x\left(\frac{\pi}{2} - \arctan x\right)\right)$; d) $\lim\limits_{x\to+\infty} \frac{\ln(x+1)-\ln(x-1)}{\sqrt{x^2+1}-\sqrt{x^2-1}}$;
e) $\lim\limits_{x\to+\infty} \left(x \sin\frac{1}{x} + \frac{1}{x}\right)^x$; f) $\lim\limits_{x\to+\infty} \left(x \sin\frac{1}{x} + \frac{1}{x^2}\right)^x$; g) $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$; h) $\lim\limits_{x\to+\infty} \frac{x^{\ln x}}{(\ln x)^x}$.

58. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0.$$

59. Write Taylor's expansion of the function e^{2x-x^2}, $x \in \mathbb{R}$ at the point $x_0 = 0$ up to the term with x^5.

60. Use Taylor's formula to compute the limits
a) $\lim\limits_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$; b) $\lim\limits_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3}$; c) $\lim\limits_{x\to 0}\frac{e^x-1-x}{x^2}$; d) $\lim\limits_{x\to 0}\frac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}$;
e) $\lim\limits_{x\to 0}\frac{\ln(1+x+x^2)+\ln(1-x-x^2)}{x\sin x}$; f) $\lim\limits_{x\to 0}\frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}$.

61. Find points of local extrema of the following functions:
a) $f(x) = x^4(1-x)^3$, $x \in \mathbb{R}$; b) $f(x) = x^2e^x$, $x \in \mathbb{R}$; c) $f(x) = x + \frac{1}{x}$, $x > 0$;
d) $f(x) = \frac{x^2}{2} - \frac{1}{4} + \frac{9}{4(2x^2+1)}$, $x \in \mathbb{R}$; e) $f(x) = |x|e^{-x^2}$, $x \in \mathbb{R}$; f) $f(x) = x^x$, $x > 0$;
g) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$, $x \in \mathbb{R}$.

62. Identify intervals on which the following functions are convex or concave:
a) $f(x) = e^x$, $x \in \mathbb{R}$; b) $f(x) = \ln x$, $x > 0$; c) $f(x) = \sin x$, $x \in \mathbb{R}$; d) $f(x) = \arctan x$, $x \in \mathbb{R}$;
e) $f(x) = x^\alpha$, $x > 0$, $\alpha \in \mathbb{R}$.

63. Compute the following indefinite integrals:
a) $\int \cos 6x dx$, $x \in \mathbb{R}$; b) $\int x \sin x dx$, $x \in \mathbb{R}$; c) $\int \sin^2 x dx$, $x \in \mathbb{R}$;
d) $\int \sin 2x \sin 3x dx$, $x \in \mathbb{R}$; e) $\int \sin^3 x dx$, $x \in \mathbb{R}$; f) $\int \frac{dx}{\sin x \cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$;
g) $\int x \cos x^2 dx$, $x \in \mathbb{R}$; h) $\int \frac{dx}{1-x}$ on $(-\infty, 1)$ and $(1, +\infty)$; i) $\int \frac{dx}{x \ln x}$, $x > 0$;
j) $\int \frac{(2x+1)dx}{\sqrt[3]{1+x+x^2}}$, $x \in \mathbb{R}$; k) $\int \frac{\sqrt{\tan x}}{\cos^2 x}dx$, $x \in \left(0, \frac{\pi}{2}\right)$; l) $\int \frac{dx}{\cos x}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;
m) $\int \cos^2 x \sin^3 x dx$, $x \in \mathbb{R}$; n) $\int \frac{dx}{x^2+x+1}$, $x \in \mathbb{R}$; o) $\int x^2 \sin x dx$, $x \in \mathbb{R}$;
p) $\int (\ln x)^2 dx$, $x > 0$; q) $\int e^{2x} \cos x dx$, $x \in \mathbb{R}$. r) $\int \frac{1}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$; s) $\int \frac{e^{\frac{1}{x}} dx}{x^2}$, $x > 0$;
t) $\int \sqrt{1-3x}dx$, $x < \frac{1}{3}$; u) $\int \frac{dx}{1-x^2}$ on $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$;
v) $\int \ln(x^2+x+1)dx$, $x \in \mathbb{R}$.

64. Let $f : [0,1] \to \mathbb{R}$ be integrable on $[0,1]$. Prove the equality

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f(x)dx = \int_0^1 f(x)dx.$$

65. Compute the following integrals:
a) $\int_{-1}^8 \sqrt[3]{x}dx$; b) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$; c) $\int_0^{\frac{\pi}{2}} \sin 2x dx$; d) $\int_0^1 e^{2x-1}dx$; e) $\int_0^2 |1-x|dx$; f) $\int_0^{\ln 2} xe^{-x}dx$;
g) $\int_0^{\sqrt{\pi}} x \sin x^2 dx$; h) $\int_0^{2\pi} x^2 \cos x dx$; i) $\int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$; j) $\int_0^{\ln 2} \sqrt{e^x - 1}dx$;
k) $\int_{-1}^1 \frac{dx}{x^2 - 2x\cos\alpha + 1}$ for $\alpha \in (0, \pi)$; l) $\int_{\frac{1}{e}}^e |\ln x|dx$; m) $\int_0^1 \arccos x dx$.

66. Compute the following derivatives:
a) $\frac{d}{dx}\int_a^b \sin x^2 dx$; b) $\frac{d}{da}\int_a^b \sin x^2 dx$; c) $\frac{d}{dx}\int_0^{x^2} \sqrt{1+t^2}dt$; d) $\frac{d}{dx}\int_{x^2}^{x^3} \frac{dt}{1+t^4}$.

67. Compute the following limits:
a) $\lim\limits_{x\to 0}\frac{\int_0^x \cos t^2 dt}{x}$; b) $\lim\limits_{x\to+\infty}\frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}$.

68. Compute the area of regions bounded by the graphs of the following functions:
a) $2x = y^2$ and $2y = x^2$; b) $y = x^2$ and $x + y = 2$; c) $y = 2^x$, $y = 2$ and $x = 0$;
d) $y = \frac{a^3}{a^2+x^2}$ and $y = 0$, where $a > 0$.

69. Compute the length of the circle $x^2 + y^2 = r^2$, $r > 0$.

70. Compute the length of continuous curves defined by the following functions:
a) $y = x^{\frac{3}{2}}$, $x \in [0, 4]$; b) $y = e^x$, $0 \le x \le b$; c) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.

71. Compute the following improper integrals:
a) $\int_0^{+\infty} xe^{-x}dx$; b) $\int_0^{+\infty} \frac{dx}{x^2+x+1}$; c) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; d) $\int_0^{+\infty} x^2 e^{-x}dx$.

72. Identify all $p \in \mathbb{R}$ for which the improper integral $\int_1^{+\infty} x^p e^{-x}dx$ converges. Justify your answer.

73. Show that the following improper integrals converge:
a) $\int_1^{+\infty} e^{-x^2}dx$; b) $\int_1^{+\infty} \frac{x-2}{x^3+x+1}dx$; c) $\int_0^{+\infty} \frac{\sin x}{1+x^2}dx$; d) $\int_1^{+\infty} e^{-x}\ln x dx$; e) $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx$
f) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}}dx$ $\int_1^{+\infty} \cos x^2 dx$; g) $\int_1^{+\infty} \frac{\sin 2x \cdot \sin x}{x}dx$; h) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; i) $\int_0^1 \ln x dx$.

74. Prove that the convergence of a series $\sum_{n=1}^{\infty} a_n$ implies that $a_n + a_{n+1} + \ldots + a_{2n} \to 0$, $n \to \infty$.

75. Identify all $p > 0$ for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges. Justify your answer.

76. Prove the convergence of the following series:
a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n^2+1}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$; d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)^2$;
f) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$; g) $\sum_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$; h) $\sum_{n=2}^{\infty} \left(\ln \frac{n}{n-1} - \frac{1}{n}\right)$.

77. Investigate the convergence of the following series:
a) $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$; b) $\sum_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$; c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$;
f) $\sum_{n=1}^{\infty} \frac{7^n (n!)^2}{n^{2n}}$; g) $\sum_{n=1}^{\infty} \frac{3^n}{(\ln n)^n}$; h) $\sum_{n=1}^{\infty} \frac{n^{n^2} 2^n}{(n+1)^{n^2}}$.

78. Prove the convergence of the following sequences:
a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$; b) $\sum_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}} \frac{1}{\sqrt{n}}$; c) $\sum_{n=1}^{\infty} \frac{\sin 3n}{\sqrt{n}}$; d) $\sum_{n=1}^{\infty} \frac{\cos n}{n}$.

79. Investigate the absolute and conditional convergence of the following series:
a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$; b) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$; c) $\sum_{n=1}^{\infty} (-1)^n \sin^{\frac{1}{3}} \frac{1}{n}$; d) $\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n}}$.

80. Show that for each $x \in \mathbb{R}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

81. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:
a) $(-2 + 3i)(1 + i)$; b) $(\sqrt{2} - i)^2$; c) $(2 + 3i)^2(1 + 2i)$;
d) $\frac{2+3i}{2-i}$; e) $\frac{3-i}{2+2i}$; f) $\frac{i}{(1-i)^2}$; g) $\frac{1}{i} - \frac{1}{(1+i)^2}$.

82. Compute the real and imaginary parts of $\frac{1}{z^2}$, where $z = x + iy$, $x, y \in \mathbb{R}$.

83. Solve the following equations:
a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

84. Write the following complex numbers in the polar form:
a) i; b) $1 - i$; c) $-1 + \sqrt{3}i$; d) $-2 - 2i$.

85. Compute $\frac{(1-\sqrt{3}i)(\cos\theta + i\sin\theta)}{2(1-i)(\cos\theta - i\sin\theta)}$.

86. Compute a) $(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{21}$; b) $(\sqrt{3} - 3i)^{15}$; c) $(1+i)^{25}$; d) $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}$; e) $\left(1 - \frac{\sqrt{3}-i}{2}\right)^{24}$.

87. Solve the following equations:
a) $z^2 + z + 3 = 0$; b) $z^3 - i = 0$; c) $z^5 - 2 = 0$; d) $z^4 + i = 0$; e) $z^3 - 4i = 0$.

88. Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$.

89. For a complex number α show that the coefficients of the polynomial

$$p(z) = (z - \alpha)(z - \overline{\alpha})$$

are real numbers.

90. Let $p(z)$ be a polynomial with real coefficients and let α be a complex number. Prove that $p(\alpha) = 0$ if and only if $p(\overline{\alpha}) = 0$.

91. For each of the following sets, either show that the set is a vector space over $\mathbb{F}$ or explain why it is not a vector space.

    a) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{R}$.
    b) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{C}$.
    c) The set $\{f \in \mathrm{C}[0,1] : f(0) = 2\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.
    d) The set $\{f \in \mathrm{C}[0,1] : f(0) = f(1) = 0\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.
    e) The set $\{(x, y, z) \in \mathbb{R}^3 : x - 2y + z = 0\}$ under the usual operations of addition and multiplication on $\mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$.
    f) The set $\{(x, y, z) \in \mathbb{C}^3 : 2x + z + i = 0\}$ under the usual operations of addition and multiplication on $\mathbb{C}^3$, $\mathbb{F} = \mathbb{C}$.

92. Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let

$$U = \{az^2 + bz^5 : \ a, b \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}[z]$ such that $F[z] = U \oplus W$.

93. Consider the complex vector space $V = \mathbb{C}^3$ and the list $\{v_1, v_2, v_3\}$ of vectors in V, where $v_1 = (i, 0, 0)$, $v_2 = (i, 1, 0)$ and $v_3 = (i, i, -1)$.
a) Prove that $\mathrm{span}\{v_1, v_2, v_3\} = V$.
b) Prove or disprove that $\{v_1, v_2, v_3\}$ is a basis of V.

94. Let V be a vector space over $\mathbb{F}$, and suppose that $v_1, v_2, \ldots, v_n \in V$ are linearly independent. Let w be a vector from V such that the vectors $v_1 + w, v_2 + w, \ldots, v_n + w$ are linearly dependent. Prove that $w \in \operatorname{span}\{v_1, v_2, \ldots, v_n\}$.

95. Let $p_0, p_1, \ldots, p_n \in \mathbb{F}_n[z]$ satisfy $p_j(2) = 0$ for all $j = 0, 1, \ldots, n$. Prove that $p_0, p_1, \ldots, p_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

96. Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
a) Show that T is linear; b) show that T is surjective; c) find $\dim(\ker T)$.

97. Show that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ is surjective if

$$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = 5x_2, \ x_3 = 7x_4\}.$$

98. Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V. Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

99. Let U, V and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

$$\dim(\ker(TS)) \le \dim(\ker T) + \dim(\ker S).$$