The lecture notes are based on the following literature:

- A.Y. Dorogovtsev, *Mathematical analysis*, Kyiv, Fact, 2004, (Russian).
- K.A. Ross, *Elementary analysis: The theory of calculus*, Undergraduate Texts in Mathematics, Springer New York, 2013.
- I. Lankham, B. Nachtergaele, and A. Schilling, *Linear algebra as an introduction to abstract mathematics*, WSPC, 2016.
- B. P. Demidovic, *Problems and exercises in mathematical analysis*, Moscow, 1997, (Russian).

1 Lecture 1 – Elements of Set Theory and Mathematical Induction

1.1 Elements of Set Theory

The notion of a **set** is one of the most important initial and nondefinable notions of the modern mathematics. By a "set" we will understand any collection into a whole M of definite and separate objects m of our intuition or our thought (Georg Cantor). These objects are called the "elements" of M. Shortly we will use the notation $m \in M$ or $M \ni m$. The fact that m does not belong to M is denoted by $m \notin M$.

A set M can be defined by listing of its elements. For instance,

- $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ — the set of **natural numbers**;
- $\mathbb{Z} = \{\dots, -n, \dots, -1, 0, 1, 2, 3, \dots, n, \dots\}$ — the set of **integer numbers**.

A set also can be defined by specifying of properties of its elements. In any mathematical problem usually consider elements of some quite defined set X. The needed set can be chosen by some property P satisfying the following property: for each x from X either x satisfies P (in this case one writes $P(x)$) or x does satisfy it. This set is denoted by $\{x \in X : \ P(x)\}$ or $\{x : \ P(x)\}$. The set which does not contain any elements is called **empty** and is denoted $\emptyset$.

Example 1.1. 1. $\mathbb{N} = \{n \in \mathbb{Z} : \ n > 0\}$. Here P means "to be positive", which is satisfied by any integer number.

2. Let P denote "to be even". Then $\{2, 4, \dots, 2k, \dots\} = \{n \in \mathbb{N} : \ P(n)\}$.

3. $\mathbb{Q} = \left\{\frac{m}{n} : \ n \in \mathbb{N}, \ m \in \mathbb{Z}\right\}$. This is the set of **rational numbers**.

Exercise 1.1. List elements of the following sets:

a) $\{n \in \mathbb{N} : \ (n-3)^2 < 7^2\}$;

b) $\left\{n \in \mathbb{N} : \ \frac{n^2+3n-12}{n} \in \mathbb{N}\right\}$;

c) $\left\{n \in \mathbb{N} : \ \frac{n+9}{n+1} \in \mathbb{N}\right\}$;

d) $\left\{n \in \mathbb{Z} : \ n^3 > 10n^2\right\}$.

1.1.1 Operations on Sets

Let A and B be sets.

Definition 1.1. A set A is a **a subset of a set** B, if each element x of A is an element of B (or shortly, $\forall x \in A \;\Rightarrow\; x \in B$). Notation: $A \subset B$.

Definition 1.2 (Operations on sets).

- $A \cup B = \{x : \; x \in A \text{ or } x \in B\}$ — the **union of** A **and** B;
- $A \cap B = \{x : \; x \in A \text{ and } x \in B\}$ — the **intersection of** A **and** B;
- $A \setminus B = \{x : \; x \in A \text{ and } x \notin B\}$ — the **difference of** A **and** B;
- $A \bigtriangleup B = \{x : \; x \in A \cup B \text{ and } x \notin A \cap B\}$ — the **symmetric difference of** A **and** B;
- $A^c = \{x \in X : \; x \notin A\}$ — the **complement of** A, where X is some given set containing A.

Exercise 1.2. Show that

a) $A \cup \emptyset = A$, $A \cup A = A$, $A \cup B = B \cup A$, $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;

b) $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup B = B \cap A$, $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;

c) $A \bigtriangleup B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$, $A \setminus B = A \cap B^c$;

d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

e) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Let T be a set of indexes and for each $t \in T$ a set A_t is given.

Definition 1.3.

- $\bigcup\limits_{t \in T} A_t = \{x : \; \exists t_0 \in T \;\; A_{t_0} \ni x\}$ — the **union of the family** A_t**,** $t \in T$;
- $\bigcap\limits_{t \in T} A_t = \{x : \; \forall t \in T \;\; A_t \ni x\}$ — the **intersection of the family** A_t**,** $t \in T$;

Example 1.2. Let $A_n = \{1, \ldots, n\}$ for each $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \mathbb{N}, \quad \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \{1\}.$$

1.2 Numbers

1.2.1 Mathematical induction

For more details see [1, Section 1.1].

Let M be a subset of natural numbers which satisfies the following properties

1) $1 \in M$;

2) if $n \in M$, then $n + 1 \in M$.

Then $M = \mathbb{N}$! This is one of the axioms of natural numbers and it is the basis of **mathematical induction**. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I1) P_1 is true;

(I2) P_{n+1} is true whenever P_n is true.

We will refer to **(I1)** as the basis for induction and we will refer to **(I2)** as the induction step.

Example 1.3. Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n.
Solution. Our n-th proposition is

$$P_n : \quad 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Base case: Show that the statement P_n holds for $n = 1$. So,

$$1 = \frac{1}{2} \cdot 1 \cdot (1+1).$$

Induction step: We assume that P_n holds, i.e.

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

is true, and must prove P_{n+1}. So,

$$1 + 2 + \cdots + n + (n+1) = \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2) = \frac{1}{2}(n+1)((n+1)+1).$$

By the principle of mathematical induction, we conclude that P_n is true for all n.

Exercise 1.3. a) Prove that all numbers of the form $5^n - 4n - 1$, $n \in \mathbb{N}$ are divisible by 16.

b) Show that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for each $n \in \mathbb{N}$.

c) Prove the inequality $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

2 Lecture 2 – Completeness of the Set of Real Numbers and some Inequalities

2.1 Real Numbers

2.1.1 Definition of Real Numbers

Very often the set of rational numbers needs an extension. For example, the length of a diagonal of a square with side 1 can not be given as a rational number.

Exercise 2.1. Prove that there does not exist a rational number x solving the equation $x^2 = 2$.

Definition 2.1. A **real number** is an infinite sequence of numerical digits with the comma between them, that is,

$$a = \alpha_0, \alpha_1\alpha_2 \dots \alpha_n \dots,$$

where $\alpha_0 \in \mathbb{Z}$ and $\alpha_n \in \{0, 1, \dots, 9\}$ for all $n \in \mathbb{N}$.

The **set of all real numbers** is denoted by $\mathbb{R}$.

Definition 2.2. Numbers from $\mathbb{R} \setminus \mathbb{Q}$ is called **irrational**.

Remark 2.1. We will identify of two real numbers of the form

$$a = \alpha_0, \alpha_1 \dots \alpha_n 99999 \dots$$

and

$$a = \alpha_0, \alpha_1 \dots (\alpha_n + 1)00000 \dots,$$

where $\alpha_n < 9$. Further, we will avoid numbers, where 9 is in the period.

The order relations "$<, \leq, >, \geq$" between real numbers can be introduced by the natural way as well as the notions of positive and negative real numbers.

Definition 2.3. The **absolute value** of a real number a is defined as follows

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

2.1.2 Supremum and Infimum of Subsets of Real Numbers

Let A be a non-empty subset of $\mathbb{R}$.

Definition 2.4.
- If A contains a larger element a_0, then we call a_0 the **maximum** of A and write $a_0 = \max A$.
- If A contains a smallest element, then we call the smallest element the **minimum** of A and write it as $\min S$.

Example 2.1.
a) $\max\{1, 2, 3, 4, 5\} = 5$, $\min\{1, 2, 3, 4, 5\} = 1$;

b) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\max A = 1$ but $\min A$ does not exist.

Definition 2.5.
- If a real number M satisfies $a \le M$ for all $a \in A$, then M is called an **upper bound** of A and the set A is said to be **bounded above**.
- If a real number m satisfies $m \le a$ for all $a \in A$, then m is called a **lower bound** of A and the set A is said to be **bounded below**.
- The set A is said to be **bounded** if it is bounded above and bounded below.

Example 2.2.
1. The set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded.
2. The set $\mathbb{N}$ is bounded below but not above.
3. The set $\mathbb{R}$ is neither bounded below nor above.

Exercise 2.2. Prove that the following sets are bounded:

a) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$;

b) $\left\{\frac{(-1)^n n+1}{n-(-1)^n} : n \in \mathbb{N}\right\}$.

Definition 2.6.
- If A is bounded above and A has a least upper bound, then we will call it the **supremum** of A and denote it by $\sup A$.
- If A is bounded below and A has a greatest lower bound, then we will call it the **infimum** of A and denote it by $\inf A$.

Exercise 2.3. If $\min A$ exists, then $\min A = \inf A$. Similarly, if $\max A$ exists, then $\max A = \sup A$. Check these statements.

Theorem 2.1. *(i) The number a^* is the supremum of a subset A of $\mathbb{R}$ iff*
- *a^* is an upper bound of A;*
- $\forall a < a^* \ \ \exists x \in A \ \ x > a$.

(ii) The number a_* is the supremum of a subset A of $\mathbb{R}$ iff
- *a_* is an lower bound of A;*
- $\forall a > a^* \ \ \exists x \in A \ \ x < a$.

Exercise 2.4. For each $a < b$ prove that $\inf[a, b] = \inf(a, b] = a$ and $\sup[a, b] = \sup[a, b) = b$.

Theorem 2.2. *(i) For every non-empty subset A of $\mathbb{R}$ that is bounded above* $\sup A$ *exists and is a real number.*

(ii) For every non-empty subset A of $\mathbb{R}$ that is bounded below $\inf A$ *exists and is a real number.*

The latter theorem states the completeness of the set of real numbers, which is not true e.g. for rational numbers. Indeed, the set $A = \{r \in \mathbb{Q} : 0 \le r \text{ and } r^2 \le 2\}$ is a set of rational numbers and it is bounded above by some rational numbers but A has no least upper bound that is a rational number.

Theorem 2.3. *For each positive real number $a = \alpha_0, \alpha_1\alpha_2 \ldots \alpha_n \ldots$, we have*

$$a = \sup\{a_n : \ n \in \mathbb{N}\},$$

where $a_n = \alpha_0, \alpha_1\alpha_2 \ldots \alpha_n$.

Exercise 2.5. Prove Theorem 2.3.

Now, we are ready to introduce operation or real numbers. Let a, b be a positive real numbers and a_n and b_n, $n \in \mathbb{N}$, be defined as in Theorem 2.3.

Definition 2.7. We set by the definition $a + b := \sup\{a_n + b_n : \ n \in \mathbb{N}\}$; $a \cdot b := \sup\{a_n \cdot b_n : \ n \in \mathbb{N}\}$; $\frac{a}{b} := \sup\left\{\frac{a_n}{b_n} : \ n \in \mathbb{N}\right\}$; for $a > b$, $a - b := \sup\{a_n - b_n : \ n \in \mathbb{N}\}$.

We note that all numbers a_n and b_n, $n \in \mathbb{N}$, in the definition are rational and for them all arithmetic operations are defined. All known properties of arithmetic operations on integer numbers are also valid for real numbers but now they have to be proved.

Exercise 2.6. Show that

a) $a \cdot b = b \cdot a$ and $a + b = b + a$;

b) $a + (b + c) = (a + b) + c =: a + b + c$

c) $a \cdot (b \cdot c) = (a \cdot b) \cdot c =: a \cdot b \cdot c$.

2.1.3 n-th Root of a Positive Real Number

Theorem 2.4. *Let a be a positive real number and $n \in \mathbb{N}$. Then there exist a unique positive real number x satisfying $x^n = a$, where $x^n := \underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$.*

Remark 2.2. The number x can be constructed as the supremum of the set $\{y > 0 : \ y^n < a\}$, which is a non-empty bounded above set.

Definition 2.8. Let $a > 0$ and $n \in \mathbb{N}$. The unique positive solution of the equation $x^n = a$, which exists according to Theorem 2.4, is called the **n-th root** of the positive real number a. We use the notation for x: $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Definition 2.9. Let $a > 0$ and $r \in \mathbb{Q}$, $r > 0$. We define

$$a^r := (a^m)^{\frac{1}{n}},$$

where $r = \frac{m}{n}$, $m, n \in \mathbb{N}$.

Definition 2.10. Let $a > 1$ and $b > 0$. We define

$$a^b := \sup\{a^{b_n} : \ n \in \mathbb{N}\},$$

where $b := \beta_0, \beta_1\beta_2 \ldots \beta_n \ldots$ and $b_n := \beta_0, \beta_1\beta_2 \ldots \beta_n$.

Exercise 2.7. Give a definition of a^b in the case $0 < a < 1$ and $b > 0$.

2.2 Some important inequalities

We recall that the absolute value of a real number a is given by

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

We note that $-|a| \leq a \leq |a|$ and also $|a| < c \Leftrightarrow -c < a < c$. Moreover, $|a| = |-a|$.

Theorem 2.5. *For all $a, b \in \mathbb{R}$ the inequalities*

$$1)\ |a+b| \leq |a| + |b| \quad 2)\ ||a| - |b|| \leq |a-b|$$

holds. For every $a_1, \ldots a_n \in \mathbb{R}$ one has

$$|a_1 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|.$$

Proof. Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. This implies inequality 1). Now, applying 1), we obtain $|a| = |a - b + b| \leq |a - b| + |b|$. Hence, $|a| - |b| \leq |a - b|$. Since $|a - b| = |b - a| \geq |b| - |a|$, we obtain 2). The latter inequality trivially follows from 1). □

Inequality 1) from Theorem 2.5 is called the **triangular inequality**.

Theorem 2.6 (Bernoulli's inequality)**.** *For each real number $x > -1$ and $n \in \mathbb{N}$ the inequality*

$$(1+x)^n \geq 1 + nx$$

holds. Moreover, $(1+x)^n = 1 + nx$ iff $x = 0$ or $n = 1$.

Proof. If $n = 1$ or $x = 0$, then the equality holds. We assume that $x \neq 0$ and use mathematical induction to prove $(1+x)^n > 1 + nx$ for all $n \geq 2$. So, for $n = 2$ one has

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

Next, we assume that the strict inequality holds for some $n \geq 2$. Then

$$(1+x)^{n+1} = (1+x)(1+x)^n > (1+x)(1+nx) = 1 + (n+1)x + nx^2 > 1 + (n+1)x.$$

□

Exercise 2.8. Show that
a) $2^n \geq n+1$, $n \in \mathbb{N}$; b) $3^n \geq 2n+1$, $n \in \mathbb{N}$; c) $2^n > (\sqrt{2}-1)^2 n^2$, $n \in \mathbb{N}$.

Exercise 2.9. Let $x_1, \ldots, x_n$ be a positive real numbers. Prove that

$$(1+x_1) \cdot \ldots \cdot (1+x_n) \geq 1 + x_1 + \ldots + x_n.$$

3 Lecture 3 – Convergence of Sequences

3.1 Limits of Sequences

For more details see [1, Section 2.7].

In this section, we will study some properties of sequences of real numbers which do not depend on finite numbers of their elements. So, we will call a **sequence** any enumerated collection of objects (in our case, real numbers) in which repetitions are allowed. It is often convenient to write the sequence as $(a_m, a_{m+1}, a_{m+2}, \ldots)$, $(a_n)_{n \geq m}$ or $(a_n)_{n=m}^{\infty}$, where m is some integer number. Usually, m equals 1.

Definition 3.1. A sequence $(a_n)_{n\geq 1} = (a_1, a_2, \ldots, a_n, \ldots)$ is called **bounded** if there exists $C > 0$ such that $|a_n| \leq C$ for all $n \geq 1$. In another words, if all elements of the sequence belong to some interval $[-C, C]$.

Example 3.1.
1. The sequence $((-1)^n)_{n\geq 1} = (-1, 1, -1, 1, \ldots)$ is bounded and its elements belong to $[-1, 1]$;
2. The sequence $(\sin n)_{n\geq 1}$ is bounded and its elements also belong to $[-1, 1]$;
3. The sequence $(n)_{n\geq 1} = (1, 2, 3, \ldots, n, \ldots)$ is unbounded, since for each $C > 0$ one can find a number $n \in \mathbb{N}$ larger than C.

Exercise 3.1. Prove the boundedness of the following sequences:

a) $\left(\frac{2^n}{n!}\right)_{n\geq 1}$; b) $\left(a_n = \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \sqrt{2}}}}}_{n \text{ square roots}}\right)_{n\geq 1}$;

c) $\left(a_n = 1 + \frac{2}{2} + \frac{3}{2^2} + \ldots + \frac{n}{2^{n-1}}\right)_{n\geq 1}$ (*Hint:* Use the equality $\frac{1}{2}a_n = a_n - \frac{1}{2}a_n$)

Exercise 3.2. Prove that a sequence $(a_n)_{n\geq 1}$ is bounded iff $(a_n^3 - a_n)_{n\geq 1}$ is bounded.

Definition 3.2. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. A **neighbourhood** or **ε-neighbourhood** of the point x is the interval $(x - \varepsilon, x + \varepsilon) = \{y \in \mathbb{R} : \ |y - x| < \varepsilon\}$.

Exercise 3.3. Check that: a) intersection of a finite number of neighbourhoods of x is again a neighbourhood of x; b) intersection of two neighbourhoods is either $\emptyset$ or a neighbourhood.

Definition 3.3. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to **converge** to a real number a provided that

for each $\varepsilon > 0$ there exists a number N such that $n \geq N$ implies $|a_n - a| < \varepsilon$,

or, shortly,

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} \ \forall n \geq N : \quad |a_n - a| < \varepsilon.$$

If $(a_n)_{n\geq 1}$ converges to a, we will write $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$, $n \to \infty$. The number a is called the **limit** of the sequence $(a_n)_{n\geq 1}$. A sequence that does not converge to some real number is said to **diverge**.

Remark 3.1. We note that $a_n \to a$, $n \to \infty$, provided that any ε-neighbourhood of point a contains elements a_n for all $n \geq N$, where N is some number depending on ε.

Exercise 3.4. For which sequences $(a_n)_{n\geq 1}$ the number N from Definition 3.3 could be taken independent of ε.

Answer: If $\exists m \in \mathbb{N}\ \forall n \geq m: \ a_n = a$.

Exercise 3.5. Prove the following statements:

a) $a_n \to a,\ n \to \infty \ \Leftrightarrow \ a_n - a \to 0,\ n \to \infty \ \Leftrightarrow \ |a_n - a| \to 0,\ n \to \infty$;

b) $a_n \to 0,\ n \to \infty \ \Leftrightarrow \ |a_n| \to 0,\ n \to \infty$;

c) $a_n \to a,\ n \to \infty \ \Leftrightarrow \ \forall \varepsilon > 0\ \exists N \in \mathbb{N}: \ \{a_N, a_{N+1}, \dots\} \subset (x - \varepsilon, x + \varepsilon)$;

d) $a_n \to 0,\ n \to \infty \ \Leftrightarrow \ \sup\{|a_k|: \ k \geq n\} \to 0,\ n \to \infty$;

e) $a_n \to a,\ n \to \infty \ \Rightarrow \ |a_n| \to |a|,\ n \to \infty$.

Theorem 3.1. *A sequence can have only a unique limit.*

Proof. Let $a_n \to a$, $n \to \infty$, and $a_n \to b$, $n \to \infty$. Then by the definition, $\forall \varepsilon > 0\ \exists N_1 \in \mathbb{R}\ \forall n \geq N_1:$ $|a_n - a| < \varepsilon$ and $\forall \varepsilon > 0\ \exists N_2 \in \mathbb{R}\ \forall n \geq N_2: \quad |a_n - b| < \varepsilon$. Thus, using the triangular inequality (see Theorem 2.5 1)), we obtain $\forall \varepsilon > 0\ \forall n \geq \max\{N_1, N_2\}: \quad |a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon$. So, $|a - b| < 2\varepsilon$ for all $\varepsilon > 0$. If $a \neq b$, we set $\varepsilon = \frac{|a-b|}{3} > 0$. Then $|a - b| < \frac{2}{3}|a - b| \Rightarrow \frac{1}{3}|a - b| < 0$, that is impossible. $\square$

3.2 Some Examples

For more examples see [1, Section 2.8].

Theorem 3.2. *The equality* $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ *holds.*

Proof. We note that for each $\varepsilon > 0$ we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ iff $n > \frac{1}{\varepsilon}$. Thus, $\forall \varepsilon > 0\ \exists N := \left(\frac{1}{\varepsilon} + 1\right) \in \mathbb{R}\ \forall n \geq N: \ \left|\frac{1}{n} - 0\right| < \varepsilon$. $\square$

Corollary 3.1. *The equality* $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0$ *holds for each* $\alpha > 0$.

Theorem 3.3. *Let* $a \in \mathbb{R}$, $|a| > 1$, $b \in \mathbb{R}$. *Then* $\lim\limits_{n\to\infty} \frac{n^b}{a^n} = 0$.

Proof. We choose $k \in \mathbb{N}$ such that $k \geq b + 1$. By Bernoulli's inequality (see Theorem 2.6), $|a|^n = \left(|a|^{\frac{n}{k}}\right)^k = \left(\left(1 + \left(|a|^{\frac{1}{k}} - 1\right)\right)^n\right)^k > n^k \left(|a|^{\frac{1}{k}} - 1\right)^k$. Hence, $\left|\frac{n^b}{a^n} - 0\right| = \frac{n^b}{|a|^n} \leq \frac{n^{k-1}}{|a|^n} < \frac{1}{n\left(|a|^{\frac{1}{k}} - 1\right)^k} < \varepsilon$. So, $n > \frac{1}{\varepsilon\left(|a|^{\frac{1}{k}} - 1\right)^k}$. Consequently, one can claim

$$\forall \varepsilon > 0\ \exists N := \frac{1}{\varepsilon\left(|a|^{\frac{1}{k}} - 1\right)^k} + 1\ \forall n \geq N: \ \left|\frac{n^b}{a^n} - 0\right| < \varepsilon.$$

$\square$

Theorem 3.4. *The equality* $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ *holds.*

Proof. By Exercise 3.5 a), it is enough to show that $a_n := \sqrt[n]{n} - 1 \to 0$, $n \to \infty$. Since $(1 + a_n)^n = (\sqrt[n]{n})^n = n$, one has

$$n = (1 + a_n)^n \geq 1 + na_n + \frac{1}{2}n(n-1)a_n^2 > \frac{1}{2}n(n-1)a_n^2,$$

by the binomial formula. Thus, $a_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. Next using the standard argument, one has $a_n \to 0$. □

Exercise 3.6. Check the following equalities:
a) $\lim\limits_{n\to\infty} a^n = 0$ for all $0 < a < 1$; b) $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ for all $a > 0$; c) $\lim\limits_{n\to\infty} \frac{\lg n}{n^\alpha} = 0$ for all $\alpha > 0$, where $\lg := \log_{10}$.

Definition 3.4.

1. $\lim\limits_{n\to\infty} a_n = +\infty \;\Leftrightarrow\; \forall C \in \mathbb{R}\; \exists N \in \mathbb{R}\; \forall n \geq N: \; a_n \geq C.$
2. $\lim\limits_{n\to\infty} a_n = -\infty \;\Leftrightarrow\; \forall C \in \mathbb{R}\; \exists N \in \mathbb{R}\; \forall n \geq N: \; a_n \leq C.$

Exercise 3.7. Prove that for a sequence $(a_n)_{n\geq 1}$ with $a_n \neq 0$ the equality $\lim\limits_{n\to\infty} |a_n| = +\infty$ is equivalent to $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

Exercise 3.8. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1, a_2, \ldots, a_n\}}{n} \to 0$, $n \to \infty$.

Exercise 3.9. Assume that $a_n \to a$, $n \to \infty$, and $b_n \to b$, $n \to \infty$. Show that $\max\{a_n, b_n\} \to \max\{a, b\}$, $n \to \infty$.

3.3 Limit Theorems for Sequences

See also [1, Section 2.9].

In this section, we will prove some properties of convergent sequences and their limits. We recall that a sequence $(a_n)_{n\geq 1}$ of real numbers is said to be bounded if there exists a constant C such that $|a_n| \leq C$ for all n.

Theorem 3.5. *Any convergent sequence is bounded.*

Proof. Let $a_n \to a$, $n \to \infty$. We have to show that $(a_n)_{n\geq 1}$ is bounded. By the definition of convergence (see Definition 3.3), for each $\epsilon > 0$, in particular for $\varepsilon = 1$, there exists a number N, which can be taken from $\mathbb{N}$, such that $|a_n - a| < \varepsilon = 1$ for all $n \geq N$. Thus, setting $C := \max\{|a_1|, \ldots, |a_{N-1}|, |a| + 1\}$, one trivially obtains for $n \in \{1, 2, \ldots, N-1\}$

$$|a_n| \leq C.$$

Next, using the triangular inequality (inequality 1) of Theorem 2.5), we have

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \leq C,$$

for all $n \geq N$. □

Exercise 3.10. Give an example of a bounded divergent sequence.

Theorem 3.6. *Let $a_n \to a$, $n \to \infty$, $b_n \to b$, $n \to \infty$, and let $a_n \leq b_n$ for all $n \geq 1$. Then $a \leq b$.*

Exercise 3.11. Prove Theorem 3.6.

Remark 3.2. We note that replacing the inequality $a_n \leq b_n$ by the strong one, i.e. $a_n < b_n$, it does not imply $a < b$. Indeed, for $a_n := 0$ and $b_n := \frac{1}{n}$, $n \geq 1$, one has $a_n < b_n$ but $a_n \to 0$, $b_n \to 0$, $n \to \infty$.

Remark 3.3. Theorem 3.6 remains valid, if the inequality $a_n \leq b_n$ holds only for all $n \geq M$, where M is some number N.

Theorem 3.7 (Squeeze theorem)**.** *Let sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ satisfy the following conditions:*

a) $a_n \leq b_n \leq c_n$ *for all* $n \geq 1$;

b) $a_n \to a$, $n \to \infty$, *and* $c_n \to a$, $n \to \infty$.

Then $b_n \to a$, $n \to \infty$.

Proof. We prove the theorem only for the case $a \in \mathbb{R}$. According to Remark 3.1, for each $\varepsilon > 0$ there exists N_1 and N_2 from $\mathbb{R}$ such that a_n belongs to the ε-neighbourhood $(a-\varepsilon, a+\varepsilon)$ of the point a for all $n \geq N_1$ and c_n belongs to $(a-\varepsilon, a+\varepsilon)$ for all $n \geq N_2$. Thus, for all $n \geq \max\{N_1, N_2\}$ elements b_n also belong to $(a-\varepsilon, a+\varepsilon)$ due to property a). □

Example 3.2. Show that $\lim\limits_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} = 1$.

Solution. We take $a_n := \sqrt[n]{1} = 1$ and $c_n := \sqrt[n]{\underbrace{1+1+1+\ldots+1}_{n \text{ times}}} = \sqrt[n]{n}$. Then

$$a_n \leq \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} \leq c_n$$

for all $n \geq 1$. Moreover, $a_n \to 1$, $n \to \infty$, and $c_n \to 1$, $n \to \infty$, by Theorem 3.4. Hence, Theorem 3.7 implies $\lim\limits_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}} = 1$.

Theorem 3.8. *Let $a_n \to a \in \mathbb{R}$, $n \to \infty$, and $b_n \to b \in \mathbb{R}$, $n \to \infty$. Then*

a) $\lim\limits_{n\to\infty} (c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n$ *for all* $c \in \mathbb{R}$;

b) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$;

c) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$;

d) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, *if* $b \neq 0$.

Proof. For proof of the theorem see Section 2.9 [1]. □

Example 3.3. Compute the limit $\lim\limits_{n\to\infty} \frac{2n^2+\lg n}{3n^2+n\cos n+5}$.

Solution. We cannot apply Theorem 3.8 directly, since the numerator and denominator of $\frac{2n^2+\lg n}{3n^2+n\cos n+5}$ tend to infinity. So, first we rewrite them as follows:

$$\frac{2n^2+\lg n}{3n^2+n\cos n+5} = \frac{n^2\cdot\left(2+\frac{\lg n}{n^2}\right)}{n^2\cdot\left(3+\frac{\cos n}{n}+\frac{5}{n^2}\right)} = \frac{2+\frac{\lg n}{n^2}}{3+\frac{\cos n}{n}+\frac{5}{n^2}}.$$

Now, we can use Theorem 3.8 d) to the right hand side of the latter equality. Indeed, we first compute

$$\lim_{n\to\infty}\left(2+\frac{\lg n}{n^2}\right) = 2+\lim_{n\to\infty}\frac{\lg n}{n^2} = 2,$$

by, Theorem 3.8 b) and Exercise 3.6 c). Next, due to the inequality

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}, \quad n\ge 1,$$

theorems 3.7 and 3.2, one has $\lim\limits_{n\to\infty}\frac{\cos n}{n} = 0$. Thus, by Theorem 3.8 a), b)

$$\lim_{n\to\infty}\left(3+\frac{\cos n}{n}+\frac{5}{n^2}\right) = 3+\lim_{n\to\infty}\frac{\cos n}{n}+5\lim_{n\to\infty}\frac{1}{n^2} = 3\ne 0.$$

So, we can apply Theorem 3.7 d) and obtain

$$\lim_{n\to\infty}\frac{2n^2+\lg n}{3n^2+n\cos n+5} = \lim_{n\to\infty}\frac{2+\frac{\lg n}{n^2}}{3+\frac{\cos n}{n}+\frac{5}{n^2}} = \frac{2}{3}.$$

Exercise 3.12. Compute the following limits:
a) $\lim\limits_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}}$; b) $\lim\limits_{n\to\infty}\frac{n^2+\sin n}{n^2+n\cos n}$; c) $\lim\limits_{n\to\infty}\sqrt[n]{n^2 2^n+3^n}$; d) $\lim\limits_{n\to\infty}\frac{2^n+n^3}{3^n+1}$; e) $\sqrt[n+1]{n}$.

Exercise 3.13. Let $(a_n)_{n\ge1}$ be a bounded sequence and $b_n\to 0$, $n\ge\infty$. Prove that $a_nb_n\to 0$, $n\to\infty$.

Exercise 3.14. Let $(a_n)_{n\ge1}$ be a bounded sequence and $b_n\to+\infty$, $n\ge\infty$. Prove that $a_n+b_n\to+\infty$, $n\to\infty$.

Exercise 3.15. Let $a_n\ge 0$ for all $n\ge 1$ and $a_n\to a$, $n\to\infty$. Show that for all $k\in\mathbb{N}$ one has $\sqrt[k]{a_n}\to\sqrt[k]{a}$, $n\to\infty$.

Exercise 3.16. Let $a_n\to a\in\mathbb{R}$, $n\to\infty$. Prove that $\frac{a_1+\ldots+a_n}{n}\to a$, $n\to\infty$.

4 Lecture 4 – Subsequences and Monotone Sequences

4.1 Monotone Sequences

The main goal of this section is to prove that any bounded monotone sequence must converge. So, we start from the definition.

Definition 4.1. A sequence $(a_n)_{n\geq 1}$ of real numbers is called an **increasing sequence** if $a_n \leq a_{n+1}$ for all $n \geq 1$, and $(a_n)_{n\geq 1}$ is called a **decreasing sequence** if $a_n \geq a_{n+1}$ for all $n \geq 1$. A sequence that is increasing or decreasing is said to be a **monotone sequence**.

Example 4.1. The sequence $(1, 1, 2, 2, 3, 3, 4, 4, \dots)$ is increasing, but $(-1, 1, -1, 1, \dots)$ is not monotone.

Exercise 4.1. a) Show that any bounded above increasing sequence is bounded. b) Show that any bounded below decreasing sequence is bounded.

Exercise 4.2. a) Prove that $(n2^{-n})_{n\geq 2}$ is a decreasing sequence.

b) Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive numbers and define $\sigma_n = \frac{a_1+\ldots+a_n}{n}$. Prove that $(\sigma_n)_{n\geq 1}$ is also an increasing sequence.

Theorem 4.1. *Every bounded monotone sequence converges.*

Proof. We will prove the theorem for increasing sequences. The case of decreasing sequences is left to Exercise 4.3. So, let a sequence $(a_n)_{n\geq 1}$ increase. By the assumption of the theorem, $(a_n)_{n\geq 1}$ is bounded, that is, there exists $C \in \mathbb{R}$ such that $|a_n| \leq C$ for all $n \geq 1$. This implies that the set $A := \{a_n : n \geq 1\}$ is also bounded. Thus, by Theorem 2.2 (i) there exists $\sup A =: \sup\limits_{n\geq 1} a_n$ denoted by a. Let us prove that $a_n \to a$, $n \to \infty$. We first note that $a_n \leq a$ for all $n \geq 1$, since the supremum of A is also its upper bound (see Definition 2.6). Next, we take an arbitrary $\varepsilon > 0$ and use Theorem 2.1 (i). So, there exists a number m such that $a_m > a - \varepsilon$. By the monotonicity, $a - \varepsilon < a_m \leq a_n$ for all $n \geq m$. Thus, setting $N := m$, one has $a - \varepsilon < a_n \leq a$ for all $n \geq N$ which implies $|a_n - a| < \varepsilon$. $\square$

Exercise 4.3. Prove Theorem 4.1 for decreasing sequences.

Remark 4.1. Theorem 4.1 remains valid if one requires the monotonicity of $(a_n)_{n\geq 1}$ starting from some number m, that is, the monotonicity of $(a_n)_{n\geq m} = (a_m, a_{m+1}, \dots)$.

Example 4.2. Prove that $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$, where $n! := 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

Solution. First we note that $\frac{10^{n+1}}{(n+1)!} < \frac{10^n}{n!} \Leftrightarrow 10 < n + 1 \Leftrightarrow n > 9$. Hence, the sequence $\left(\frac{10^n}{n!}\right)_{n\geq 10}$ is decreasing. Moreover, it is bounded below by zero. Thus, $\left(\frac{10^n}{n!}\right)_{n\geq 10}$ is bounded, by Exercise 4.1 b). Using Theorem 4.1, one gets that there exists $a \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} \frac{10^n}{n!} = a$. But we can write $\frac{10^{n+1}}{(n+1)!} = \frac{10^n}{n!} \cdot \frac{10}{n+1}$. So,

$$a = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{10^n}{n!} \cdot \lim_{n\to\infty} \frac{10}{n+1} = a \cdot 0.$$

This implies $a = 0$.

Exercise 4.4. Show that a) $\lim\limits_{n\to\infty} \frac{n!}{2^{n^2}} = 0$; b) $\lim\limits_{n\to\infty} \frac{n}{2^{\sqrt{n}}} = 0$.

Exercise 4.5. Find a limit of the sequence $\left(\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots\right)$.

Exercise 4.6. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{3}(a_n + 1)$ for all $n \geq 1$.

a) Find a_2, a_3, a_4.

b) Use induction to show that $a_n > \frac{1}{2}$ for all $n \geq 1$.

c) Show that $(a_n)_{n\geq 1}$ is a decreasing sequence.

d) Show that $\lim\limits_{n\to\infty} a_n$ exists and find it.

Exercise 4.7. Let $c > 0$, $a_1 > 0$ and let $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for all $n \geq 1$.

a) Show that $a_n \geq \sqrt{c}$ for all $n \geq 2$.

b) Show that $(a_n)_{n\geq 2}$ is a decreasing sequence.

c) Show that $\lim\limits_{n\to\infty} a_n$ exists and find it.

Theorem 4.2. *(i) If $(a_n)_{n\geq 1}$ is an unbounded increasing sequence, then $\lim\limits_{n\to\infty} a_n = +\infty$.*

(ii) If $(a_n)_{n\geq 1}$ is an unbounded decreasing sequence, then $\lim\limits_{n\to\infty} a_n = -\infty$.

Proof. We will prove only Part (i) of the theorem. The proof of Part (ii) is similar. If $(a_n)_{n\geq 1}$ is an unbounded increasing sequence, then it must be unbounded above, since it is bounded below by a_1. Taking any C and using the unboundedness of $(a_n)_{n\geq 1}$, one can find a number $m \in \mathbb{N}$ such that $a_m \geq C$. Next, by the monotonicity of $(a_n)_{n\geq 1}$, the inequality $a_n \geq a_m \geq C$ trivially holds for all $n \geq N := m$. This proves $\lim\limits_{n\to\infty} a_n = +\infty$ (see Definition 3.4). $\square$

Corollary 4.1. *If $(a_n)_{n\geq 1}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim\limits_{n\to\infty} a_n$ is always meaningful for monotone sequences.*

Proof. The proof immediately follows from theorems 4.1 and 4.2. $\square$

Exercise 4.8. Let A be a bounded nonempty subset of $\mathbb{R}$ such that $\sup A$ is not in A. Prove that there is an increasing sequence $(a_n)_{n\geq 1}$ of points from A such that $\lim\limits_{n\to\infty} a_n = \sup A$.

4.2 The number e

In this section, we will consider two sequences of positive numbers

$$\left(a_n := \left(1 + \frac{1}{n}\right)^n\right)_{n\geq 1} \quad \text{and} \quad \left(b_n := \left(1 + \frac{1}{n}\right)^{n+1}\right)_{n\geq 1} \tag{1}$$

and study their properties.

Theorem 4.3. *The sequences defined in (1) satisfy the following properties:*

1) $a_n < b_n$ *for all* $n \geq 1$*;*

2) *the sequence* $(a_n)_{n\geq 1}$ *increases;*

3) *the sequence* $(b_n)_{n\geq 1}$ *decreases.*

Proof. Since $b_n = a_n\left(1+\frac{1}{n}\right) = a_n + \frac{a_n}{n} > a_n$ for all $n \geq 1$, Property 1) is proved.

To prove 2), we are going to use Bernoulli's inequality (see Theorem 2.6). So, one has

$$\frac{a_n}{a_{n-1}} = \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^{n-1} = \frac{n}{n-1}\left(1-\frac{1}{n^2}\right)^n > \frac{n}{n-1}\left(1-\frac{n}{n^2}\right) = 1,$$

for all $n \geq 2$. Thus, $a_n > a_{n-1}$ for all $n \geq 2$.

For the prove of 3) we use the same argument. We consider

$$\begin{aligned}\frac{b_{n-1}}{b_n} &= \left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^{n+1} = \frac{n-1}{n}\left(\frac{n^2}{n^2-1}\right)^{n+1} \\ &= \frac{n-1}{n}\left(1+\frac{1}{n^2-1}\right)^{n+1} > \frac{n-1}{n}\left(1+\frac{n+1}{n^2-1}\right) = 1,\end{aligned}$$

for all $n \geq 2$. Hence, $b_{n-1} > b_n$ for all $n \geq 2$. □

Theorem 4.3 yields the following inequalities

$$a_1 < a_2 < \ldots < a_n < \ldots < b_n < \ldots < b_2 < b_1. \tag{2}$$

Consequently, the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are monotone and bounded. By Theorem 4.1, they converge. We set

$$e := \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2,718281828459045...$$

It is known that e is an irrational number. The number e is one of the most important constants in mathematics.

Since $b_n = a_n\left(1+\frac{1}{n}\right)$ for all $n \geq 1$, one has $b_n \to e$, $n \to \infty$. We also note that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \tag{3}$$

by inequalities (2).

Definition 4.2. The logarithm to base e is called the **natural logarithm** and is denoted by $\ln := \log_e$, that is, for each $a > 0$ $\ln a$ is a (unique!) real number such that $e^{\ln a} = a$.

The inequality (3) immediately implies

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

for all $n \geq 1$.

Exercise 4.9. Show that $\lim\limits_{n\to\infty}\left(n\ln\left(1+\frac{1}{n}\right)\right) = 1$.

Exercise 4.10. Prove that for each $x > 0$ the sequence $\left(\left(1+\frac{x}{n}\right)^n\right)_{n\geq 1}$ is increasing and bounded.

4.3 Subsequences

4.3.1 Subsequences and Subsequential Limits

Let $(a_n)_{n\geq 1}$ be a sequence. We consider any subsequence $(n_k)_{k\geq 1}$ of natural numbers such that $1 \leq n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$. We note that $n_k \geq k$ and $n_k \to +\infty$, $k \to \infty$.

Example 4.3. 1) $n_k = k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (1, 2, 3, \ldots, k, \ldots)$;

2) $n_k = 2k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (2, 4, 6, \ldots, 2k, \ldots)$;

3) $n_k = k^2$, $k \geq 1$; then $(n_k)_{k\geq 1} = (1, 2, 9, \ldots, k^2, \ldots)$;

4) $n_k = 2^k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (2, 4, 8, \ldots, 2^k, \ldots)$.

Definition 4.3. A sequence $(a_{n_k})_{k\geq 1} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_k}, \ldots)$ is said to be a **subsequence** of $(a_n)_{n\geq 1}$.

Thus, $(a_{n_k})_{k\geq 1}$ is just a selection of some (possibly all) of the a_n's taken in order.

Remark 4.2. The following properties follows from the definition of subsequence.

1. If a sequence is bounded, then every its subsequence is bounded.
2. If a sequence converges to a (that could be $+\infty$ or $-\infty$), then every its subsequence also converges to a.

Exercise 4.11. Prove that a monotone sequences which contains a bounded subsequence is bounded.

Exercise 4.12. Prove that a sequence $(a_n)_{n\geq 1}$ converges iff $(a_{2k})_{k\geq 1}$, $(a_{2k-1})_{k\geq 1}$ and $(a_{3k})_{k\geq 1}$ converge.

Definition 4.4. A **subsequential limit** of a sequence $(a_n)_{n\geq 1}$ is any real number or the symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of $(a_n)_{n\geq 1}$. Let A denotes the **set of all subsequential limit** of $(a_n)_{n\geq 1}$.

Example 4.4. a) For the sequence $(1, 2, 3, \ldots, n, \ldots)$ the set of all subsequential limit $A = \{+\infty\}$.

b) For the sequence $(-1, 1, -1, \ldots, (-1)^n, \ldots)$ the set of all subsequential limit $A = \{-1, 1\}$.

c) If $a_n \to a$, then $A = \{a\}$, by Remark 4.2.

Exercise 4.13. Prove the following statements.
a) $-\infty \in A \quad \Leftrightarrow \quad (a_n)_{n\geq 1}$ is unbounded below. b) $+\infty \in A \quad \Leftrightarrow \quad (a_n)_{n\geq 1}$ is unbounded above.

Exercise 4.14. Find the set A of all subsequential limits of the following sequences.
a) $(\sin 3\pi n)_{n\geq 1}$; b) $(\sin \alpha\pi n)_{n\geq 1}$ for $\alpha \in \mathbb{Q}$; c) $(a_n)_{n\geq 1}$, where $a_n = \begin{cases} (-1)^{\frac{n+1}{2}} + n, & \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}} + \frac{1}{n}, & \text{if } n \text{ is even}. \end{cases}$

4.3.2 Existence of Monotone Subsequence

Theorem 4.4. *A number $a \in \mathbb{R}$ is a subsequential limit of a sequence $(a_n)_{n\geq 1}$ iff*

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists \tilde{n} \in \mathbb{N} : \ \tilde{n} \geq N, \ |a_{\tilde{n}} - a| < \varepsilon. \tag{4}$$

Proof. We first prove the necessity. Let $a \in A$. Then there exists a subsequence $(a_{n_k})_{k\geq 1}$ such that $a_{n_k} \to a$, $k \to \infty$. We fix an arbitrary $\varepsilon > 0$ and $N \in \mathbb{N}$. By the definition of the limit, $\exists K_1 \in \mathbb{N} \ \forall k \geq K_1 : |a_{n_k} - a| < \varepsilon$. Similarly, $\exists K_2 \in \mathbb{N} \ \forall k \geq K_2 : n_k \geq N$. Thus, taking $\tilde{k} := \max\{K_1, K_2\}$, $\tilde{n} := n_{\tilde{k}}$, one has $\tilde{n} \geq N$ and $|a_{\tilde{n}} - a| < \varepsilon$.

To prove the sufficiency, we are going to construct a subsequence of $(a_n)_{n\geq 1}$ converging to a. Let (4) holds. Then, by (4), for $\varepsilon = 1$ and $N = 1$ there exists $n_1 \geq 1$ such that $|a_{n_1} - a| < 1$. Similarly, for $\varepsilon = \frac{1}{2}$ and $N = n_1 + 1$ there exists $n_2 \geq n_1 + 1$ such that $|a_{n_2} - a| < \frac{1}{2}$ and so on. Consequently, we obtain a subsequence $(a_{n_k})_{k\geq 1}$ satisfying $|a_{n_k} - a| < \frac{1}{k}$ for all $k \geq 1$. Using Theorem 3.7 and Exercise 3.5 a), one can see that $a_{n_k} \to a$, $k \to \infty$. □

Exercise 4.15. Show that $+\infty \in A$ ($-\infty \in A$) provided $\forall C \in \mathbb{R} \ \forall N \in \mathbb{N} \ \exists \tilde{n} \in \mathbb{N} : \ \tilde{n} \geq N$ and $a_{\tilde{n}} \geq C$ ($a_{\tilde{n}} \leq C$).

Theorem 4.5. *Every sequence of real numbers contains a monotone subsequence.*

Proof. We consider the set $M := \{n \in \mathbb{N} : \ \forall m > n \ \ a_m > a_n\}$. If M is infinite, then M can be written as $M = \{n_1, n_2, \ldots, n_k, \ldots\}$, where $n_1 < n_2 < \ldots < n_k < \ldots$. By the definition of M, we have $a_{n_1} < a_{n_2} < \ldots < a_{n_k} < \ldots$. So, the subsequence $(a_{n_k})_{k\geq 1}$ increases.

If M is finite, then let n_1 be the smallest natural number such that $\forall m \geq n_1 : \ m \notin M$. Since $n_1 \notin M$, one can find $n_2 > n_1$ such that $a_{n_1} \geq a_{n_2}$. Similarly, since $n_2 \notin M$, one can find $n_3 > n_2$ such that $a_{n_2} \geq a_{n_3}$ and so on. Thus, the constructed subsequence $(a_{n_k})_{k\geq 1}$ decreases. □

Corollary 4.2. *For every sequence the set of its subsequential limits is not empty.*

Proof. The corollary immediately follows from Theorem 4.5 and Corollary 4.1. □

Theorem 4.6 (Bolzano-Weierstrass theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. The theorem is a direct consequence of theorems 4.5 and 4.1. □

5 Lecture 5 – Cauchy Sequences. Base Notion of Functions

5.1 Subsequences (continuation)

5.1.1 Upper and Lower Limits

Definition 5.1. • Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and A be the set of its subsequential limits. The value

$$\varliminf_{n\to\infty} a_n = \begin{cases} -\infty, & \text{if } A \text{ is unbounded below;} \\ \inf A, & \text{if } A \text{ is bounded below and } A \neq \{+\infty\}; \\ +\infty, & \text{if } A = \{+\infty\} \end{cases}$$

is called the **lower limit** of $(a_n)_{n\geq 1}$.

• The value

$$\varlimsup_{n\to\infty} a_n = \begin{cases} +\infty, & \text{if } A \text{ is unbounded above;} \\ \sup A, & \text{if } A \text{ is bounded above and } A \neq \{-\infty\}; \\ -\infty, & \text{if } A = \{-\infty\} \end{cases}$$

is called the **upper limit** of $(a_n)_{n\geq 1}$.

Remark 5.1. If $(a_n)_{n\geq 1}$ is a bounded sequence, then $\varliminf_{n\to\infty} a_n = \inf A$ and $\varlimsup_{n\to\infty} a_n = \sup A$.

Example 5.1. If $a_n \to a$, $n \to \infty$, then $\varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = a$, since $A = \{a\}$ in this case.

Exercise 5.1. Prove that $a_n \to a$, $n \to \infty \Leftrightarrow \varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = a$.

Theorem 5.1. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and A be the set of its subsequential limits. Then $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$ belong to A.*

Remark 5.2. If a sequence $(a_n)_{n\geq 1}$ is bounded, then $\inf A = \min A$ and $\sup A = \max A$, by Theorem 5.1, Remark 5.1 and Exercise 2.3. It means that $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$ are the minimal and the maximal subsequential limits of the bounded sequence $(a_n)_{n\geq 1}$, respectively.

Theorem 5.2. *The following equalities hold: a) $\varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k :\ k \geq n\} =: \lim_{n\to\infty} \inf_{k\geq n} a_k$; b) $\varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k :\ k \geq n\} =: \lim_{n\to\infty} \sup_{k\geq n} a_k$.*

Exercise 5.2. Prove Theorem 5.2.

Exercise 5.3. For a sequence $(a_n)_{n\geq 1}$ compute $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$, if for all $n \geq 1$

a) $a_n = 1 - \frac{1}{n}$; b) $a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$; c) $a_n = \frac{n-1}{n+1} \cos\frac{2n\pi}{3}$; d) $a_n = 1 + n\sin\frac{n\pi}{2}$;
e) $a_n = \left(1 + \frac{1}{n}\right)^n \cdot (-1)^n + \sin\frac{n\pi}{4}$.

Exercise 5.4. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $\sigma_n := \frac{a_1+a_2+\ldots+a_n}{n}$, $n \geq 1$. Prove that

$$\varliminf_{n\to\infty} a_n \leq \varliminf_{n\to\infty} \sigma_n \leq \varlimsup_{n\to\infty} \sigma_n \leq \varlimsup_{n\to\infty} a_n.$$

Compare with the statement from Exercise 3.16.

Exercise 5.5. Check that

$$\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n \le \varliminf_{n\to\infty} (a_n + b_n) \le \varlimsup_{n\to\infty} (a_n + b_n) \le \varlimsup_{n\to\infty} a_n + \varlimsup_{n\to\infty} b_n.$$

5.2 Cauchy Sequences

Definition 5.2. A sequence $(a_n)_{n\ge1}$ of real numbers is called a **Cauchy sequence** if

$$\forall\varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall m \ge N : \ |a_n - a_m| < \varepsilon.$$

Example 5.2.

1. The sequence $\left(\frac{1}{2^n}\right)_{n\ge1}$ is a Cauchy sequence. Indeed, since $\frac{1}{2^n} \to 0$, $n \to \infty$, (see Theorem 3.3), one has that for every given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \ge N$ $\frac{1}{2^n} < \varepsilon$. Consequently, for every $n \ge N$ and $m \ge N$ we can estimate $\left|\frac{1}{2^m} - \frac{1}{2^n}\right| \le \frac{1}{2^k} < \varepsilon$, where $k := \min\{n, m\} \ge N$.

2. The sequence $(a_n = (-1)^n)_{n\ge1}$ is not a Cauchy sequence. To check this, we take $\varepsilon := 1$. Then $\forall N \in \mathbb{N}\ \exists n := N$ and $\exists m := N + 1$ such that $|a_n - a_m| = 2 > \varepsilon$.

Exercise 5.6. Prove that a monotone sequence which contains a Cauchy subsequence is also a Cauchy sequence.

Exercise 5.7. Show that $(a_n)_{n\ge1}$ is a Cauchy sequence iff $\sup\limits_{m\ge N, n\ge N} |a_m - a_n| \to 0$, $N \to \infty$.

Lemma 5.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Let $a_n \to a$, $n \to \infty$, and let $\varepsilon > 0$ be given. By the definition of convergence (see Definition 3.3), for the number $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ such that $\forall n \ge N_1$ $|a_n - a| < \frac{\varepsilon}{2}$. Thus we have that $\forall n \ge N := N_1$ and $\forall m \ge N$

$$|a_n - a_m| = |a_n - a + a - a_m| \le |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

by the triangular inequality. □

Lemma 5.2. *Every Cauchy sequence is bounded.*

Proof. The proof is similar to the proof of Theorem 3.5. □

Exercise 5.8. Prove Lemma 5.2.

Theorem 5.3. *A sequence converges iff it is a Cauchy sequence.*

Proof. The necessity was stated in Lemma 5.1. We will prove the sufficiency. Let $(a_n)_{n\ge1}$ be a Cauchy sequence. By Lemma 5.2, it is bounded. Thus, using the Bolzano-Weierstrass theorem (see Theorem 4.6), there exists a subsequence $(a_{n_k})_{k\ge1}$ which converges to some $a \in \mathbb{R}$.

Next, we are going to show that $a_n \to a$, $n \to \infty$. Let $\varepsilon > 0$ be given. Since $(a_n)_{n\ge1}$ is a Cauchy sequence, for the number $\frac{\varepsilon}{2} > 0$ $\exists N_1 \in \mathbb{N}$ $\forall m \ge N$ $\forall n \ge N$ such that $|a_m - a_n| < \frac{\varepsilon}{2}$. By the definition of convergence, we have that $\exists K \in \mathbb{N}$ $\forall k \ge K$ such that $|a - a_{n_k}| < \frac{\varepsilon}{2}$. Thus, $\forall n \ge N := N_1$

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \le |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where k is any number satisfying $k \ge K$ and $n_k \ge N$. □

Exercise 5.9. Show that the sequence $\left(a_n = \frac{\sin 1}{2^1} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}\right)_{n\ge1}$ is a Cauchy sequence.

Exercise 5.10. Let $(a_n)_{n\ge1}$ be a sequence which satisfies the following property: there exists $\lambda \in [0, 1)$ such that $|a_{n+2} - a_{n+1}| \le \lambda|a_{n+1} - a_n|$ for all $n \ge 1$. Prove that $(a_n)_{n\ge1}$ converges.

5.3 Base Notion of Functions

Let X and Y be two sets.

Definition 5.3.
- A **function** f is a process or a relation that associates each element x of X to a single element y of Y. The set X is called the **domain** of the function f and is denoted by $D(f)$. The set Y is said to be the **codomain** of f. We will use the notation $f : X \to Y$.
- The element $y \in Y$ which is associated to $x \in Y$ by a function f is called the **value of f applied to the argument x** or the **image of x under f** and is denoted by $f(x)$. We will also write $x \mapsto f(x)$.
- The set
$$R(f) := \{y \in Y : \ \exists x \in X \ y = f(x)\}$$
is called the **range** or the **image** of the function f.
- If $Y \subset \mathbb{R}$, then f is called a **real valued function**.

In further sections, we will usually consider real valued functions with $D(f) \subset \mathbb{R}$.

Exercise 5.11. Determine domains $X \subset \mathbb{R}$ for which the following functions $f : X \to \mathbb{R}$ are well-defined:
a) $f(x) = \frac{x^2}{x+1}$; b) $f(x) = \sqrt{3x - x^3}$; c) $f(x) = \ln(x^2 - 4)$; d) $\sqrt{\cos(x^2)}$; e) $f(x) = \frac{\sqrt{x}}{\sin \pi x}$.

Exercise 5.12. Compute $f(-1)$, $f(-0,001)$ and $f(100)$, if $f(x) = \lg(x^2)$.

Exercise 5.13. Compute $f(-2)$, $f(-1)$, $f(0)$, $f(1)$ and $f(2)$, if

$$f(x) = \begin{cases} 1 + x, & \text{if } x \leq 0, \\ 2^x, & \text{if } x > 0. \end{cases}$$

Exercise 5.14. Define the range $R(f)$ of the following functions:

a) $X = \mathbb{Z}$, $Y = \mathbb{Z}$ and $f(x) = |x| - 1$, $x \in \mathbb{Z}$;

b) $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2 + x$, $x \in \mathbb{R}$;

c) $X = (0, \infty)$, $Y = \mathbb{R}$ and $f(x) = (x - 1) \ln x$, $x > 0$.

Exercise 5.15. Let $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, where a, b, c are some numbers. Show that

$$f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x) = 0.$$

Exercise 5.16. Find a function of the form $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, which satisfies the following properties: $f(-2) = 0$, $f(0) = 1$, $f(1) = 5$.

Definition 5.4. We will say that a function $f_1 : X_1 \to Y_1$ equals a function $f_2 : X_2 \to Y_2$, if $X_1 = X_2$ and $f_1(x) = f_2(x)$ for all $x \in X_1$. We will use the notation $f_1 = f_2$.

Definition 5.5. Let $f : X \to Y$ be a function and A be a subset of X. The function $f|_A : A \to Y$ defined by $f|_A(x) = f(x)$ for all $x \in A$ is called the **restriction of f to A**.

Definition 5.6. For sets A and B, we will denote the new set $A \times B$ that consists of all ordered pairs (a, b), where $a \in A$ and $b \in B$, that is,

$$A \times B := \{(a, b) : \ a \in A, \ b \in B\}.$$

The set $A \times B$ is called the **Cartesian product** of A and B.

Definition 5.7. The set $G(f) = \{(x, f(x)) : \ x \in X\}$ is said to be the **graph** of a function $f : X \to Y$.

6 Lecture 6 – Limits of Functions

6.1 Base Notion of Functions (continuation)

Definition 6.1. Let $f : X \to Y$ and $g : Y \to Z$ be functions. The function $h : X \to Z$ defined by $h(x) = f(g(x))$ for all $x \in X$ is called the **composition of** f **and** g and it is denoted by $h = f \circ g$.

Definition 6.2. Let $f : X \to Y$, $A \subset X$ and $B \subset Y$. The set

$$f(A) := \{f(x) : \ x \in X\}$$

is said to be the **image** of A by f. The set

$$f^{-1}(B) := \{x : \ f(x) \in B\}$$

is called the **preimage** of B by f.

Be note that $f(A)$ is a subset of Y and $f^{-1}(B)$ is a subset of X.

Example 6.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2$, $x \in \mathbb{R}$. Then $f([0,1)) = f((-1,1)) = [0,1)$; $f^{-1}([-4,4]) = f^{-1}([0,4]) = [-2,2]$; $f^{-1}((1,9]) = [-3,-1) \cup (1,3]$; $f((-\infty,0)) = \emptyset$.

Exercise 6.1. Let $f : X \to Y$ and $A_1 \subset X$, $A_2 \subset X$. Check that
a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; b) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$; c) $(f(A_1) \setminus f(A_2)) \subset f(A_1 \setminus A_2)$;
d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$; e) $A_1 \subset f^{-1}(f(A_1))$; f) $(f(X) \setminus f(A_1)) \subset f(X \setminus A_1)$.

Exercise 6.2. Let $f : X \to Y$ and $B_1 \subset Y$, $B_2 \subset Y$. Show that
a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$; b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
c) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$; d) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$; e) $f(f^{-1}(B_1)) = B_1 \cap f(X)$;
f) $f^{-1}(B_1^c) = (f^{-1}(B_1))^c$.

Definition 6.3.

- A function $f : X \to Y$ is **surjective** or a **surjection**, if $f(X) = Y$, i.e. for every element y in Y there is at least one element x in X such that $f(x) = y$.
- A function $f : X \to Y$ is **injective** or an **injection**, if for each $x_1, x_2 \in X$ $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- A function $f : X \to Y$ is **bijective** or a **bijection** or an **one-to-one function**, if it is surjective and injective, that is, for each $y \in Y$ there exists a unique element $x \in X$ such that $f(x) = y$. We set $f^{-1}(y) := x$. The function $f^{-1} : Y \to X$ is called the **inverse function to** f.

Exercise 6.3. Prove that the composition of two bijective functions is a bijection.

Exercise 6.4. Check the following statements:
a) $f : X \to Y$ is a surjection iff for all $y \in Y$ $f^{-1}(\{y\}) \neq \emptyset$.
b) $f : X \to Y$ is an injection iff for all $y \in Y$ the set $f^{-1}(\{y\})$ is either empty or contains only one element.
c) $f : X \to Y$ is a bijection iff for all $y \in Y$ the set $f^{-1}(\{y\})$ contains only one element.

Exercise 6.5. a) Let functions $f : X \to Y$ and $g : Y \to X$ satisfy the following property $g(f(x)) = x$ for all $x \in X$. Prove that f is an injection and g is a surjection.
b) Let additionally $f(g(y)) = y$ for all $y \in Y$. Show that f, g are bijections and $g = f^{-1}$.

Remark 6.1. Every sequence $(a_n)_{n \geq 1}$ of real numbers can be considered as a function $f : \mathbb{N} \to \mathbb{R}$, namely, $f(n) := a_n$ for all $n \in \mathbb{N}$.

6.2 Limit Points of a Set

Definition 6.4. Let a be a real number or the symbol $+\infty$ or $-\infty$. Then a is called a **limit point** of a subset A of $\mathbb{R}$, if there exists a sequence $(a_n)_{n\geq 1}$ satisfying the following properties: 1) $a_n \in A$ and $a_n \neq a$ for all $n \geq 1$; 2) $a_n \to a$, $n \to \infty$.

Example 6.2.

- For the set $A = [0,1]$, the set of its limit points is A.
- For the set $A = (0,1] \cup \{2\}$, the set of its limit points is $[0,1]$.
- The set $A = \left\{\frac{1}{n} :\ n \in \mathbb{N}\right\}$ has only one limit point 0.
- The limit points of $A = \mathbb{Z}$ are $+\infty$ and $-\infty$.
- The set $A = \{1, 2, 3, \ldots, 10\}$ has no limit points.

For convenience, we will denote the ε-neighbourhood of a point a by

$$B(a,\varepsilon) := (a-\varepsilon, a+\varepsilon) = \{y \in \mathbb{R} : |a-y| < \varepsilon\}.$$

Theorem 6.1. *(i) A real number $a \in \mathbb{R}$ is a limit point of a subset A of $\mathbb{R}$ iff*

$$\forall \varepsilon > 0\ \exists y \in A,\ \ y \neq a :\ \ |y-a| < \varepsilon, \tag{5}$$

that is, each ε-neighbourhood $B(a,\varepsilon)$ of the point a contains at least one point different from a.

(ii) The symbol $a = +\infty$ ($a = -\infty$) is a limit point of a subset A of $\mathbb{R}$ iff

$$\forall C \in \mathbb{R}\ \exists y \in A :\ \ y > C\ \ (y < C).$$

Proof. We will prove only Part (i). If a is a limit point of A, then (5) immediately follows from the definition of the limit of a sequence and the definition of a limit point (see definitions 3.3 and 6.4).

Next, let (5) hold. Then for each $\varepsilon := \frac{1}{n}$ there exists $a_n \in A$ and $a_n \neq a$ such that $|a_n - a| < \varepsilon = \frac{1}{n}$. By theorems 3.7 and 3.2 and Exercise 3.5 a), $a_n \to a$, $n \to \infty$. So, a is a limit point of A. □

Exercise 6.6. Prove that the set of all limit points of $\mathbb{Q}$ equals $\mathbb{R} \cup \{-\infty, +\infty\}$.

Exercise 6.7. Let a be a limit point of A. Show that every neighbourhood of the point a contains infinitely many points from A.

Definition 6.5. A point $a \in A$ is an **isolated point** of a set A, if it is not a limit point of A.

Remark 6.2. A point $a \in A$ is an isolated point of A iff $\exists \varepsilon > 0$ such that $B(a,\varepsilon) \cap A = \{a\}$.

Example 6.3.

- The set $A = [0,1]$ has no isolated points.
- The set $A = (0,1] \cup \{2\}$ has only one isolated point 2.
- For the set $A = \left\{\frac{1}{n} :\ n \in \mathbb{N}\right\}$, the set of its isolated points is A.

6.3 Limits of Functions

In this section, we will assume that A is any subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$.

Definition 6.6. Let a be a limit point of A. The value p (maybe $p = -\infty$ or $p = +\infty$) is called a **limit of the function f at the point a**, if for every sequence $(x_n)_{n\geq 1}$ satisfying the properties: 1) $x_n \in A$, $x_n \neq a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, implies $f(x_n) \to p$, $n \to \infty$. In this case, we will write $\lim\limits_{x\to a} f(x) = p$ or $f(x) \to p$, $x \to a$.

Example 6.4. Let $A = \mathbb{R}$, $f(x) = x^2$, $x \in \mathbb{R}$. Then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} x^2 = a^2$ for each $a \in \mathbb{R}$. Indeed, let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers such that $x_n \neq a$ for all $n \geq 1$ and $x_n \to a$, $n \to \infty$. Then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n^2 = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} x_n = a \cdot a = a^2$, by Theorem 3.8 c).

Example 6.5. Let $A = \mathbb{R} \setminus \{0\}$, $a = 0$, and $f(x) = \frac{\sin x}{x}$, $x \in A$. Then $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. To show this, we will compare areas of triangles and a sector of a circle with radius 1. So, we obtain for each $x \in \left(0, \frac{\pi}{2}\right)$

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x.$$

This yields

$$\cos x < \frac{\sin x}{x} < 1, \tag{6}$$

for all x satisfying $0 < x < \frac{\pi}{2}$, and, consequently, for all $0 < |x| < \frac{\pi}{2}$ because each function in the latter inequalities is even. Thus, if $\{x_n\}_{n\geq 0}$ is any sequence such that $x_n \neq 0$ for all $n \geq 1$ and $x_n \to 0$, then inequality (6) and the Squeeze theorem (see Theorem 3.7) implies that $\lim\limits_{n\to\infty} \frac{\sin x_n}{x_n} = 1$.

Remark 6.3. Inequality (6) implies that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$. Moreover, $|\sin x| = |x|$ iff $x = 0$.

Exercise 6.8. Prove that $\frac{1}{f(x)} \to 0$, $x \to a$, if $f(x) \to +\infty$, $x \to a$.

Example 6.6. Show that for every $a \in \mathbb{R}$ $\lim\limits_{x\to a} \sin x = \sin a$ and $\lim\limits_{x\to a} \cos x = \cos a$.

Solution. We prove only the first equality. The proof of the second one is similar. So, using properties of sin and cos and Remark 6.3, we can estimate

$$|\sin x - \sin a| = 2\left|\cos\frac{x+a}{2}\right| \cdot \left|\sin\frac{x-a}{2}\right| \leq 2 \cdot 1 \cdot \frac{|x-a|}{2} = |x-a|,$$

for all $x \in \mathbb{R}$. Thus, if $(x_n)_{n\geq 1}$ is any sequence which convergences to a, one has $\sin x_n \to \sin a$, by the Squeeze theorem (see Theorem 3.7).

Exercise 6.9. Prove that the limit of the function $f(x) = \sin\frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exists at the point $a = 0$.

7 Lecture 7 – Limits of Functions. Left- and Right-Sided Limits

7.1 Limit of Functions via $\varepsilon - \delta$ Approach

Let A be a subset of $\mathbb{R}$. We recall that $B(a,\varepsilon) = (a-\varepsilon, a+\varepsilon)$ denotes the ε-neighbourhood of a.

Theorem 7.1. *(i) Let p be a real number and $a \in \mathbb{R}$ be a limit point of A. Then $\lim\limits_{x\to a} f(x) = p$ is equivalent to*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A \cap B(a,\delta),\ x \neq a:\ |f(x) - p| < \varepsilon.$$

ii) If $p = +\infty$ and $a \in \mathbb{R}$, then $\lim\limits_{x\to a} f(x) = +\infty$ is equivalent to

$$\forall C \in \mathbb{R}\ \exists \delta > 0\ \forall x \in A \cap B(a,\delta),\ x \neq a:\ f(x) > C.$$

iii) If $p \in \mathbb{R}$ and $a = +\infty$, then $\lim\limits_{x\to+\infty} f(x) = p$ is equivalent to

$$\forall \varepsilon > 0\ \exists D \in \mathbb{R}\ \forall x > D:\ |f(x) - p| < \varepsilon.$$

iv) If $p = +\infty$ and $a = +\infty$, then $\lim\limits_{x\to+\infty} f(x) = +\infty$ is equivalent to

$$\forall C \in \mathbb{R}\ \exists D \in \mathbb{R}\ \forall x > D:\ f(x) > D.$$

Example 7.1. $A = \mathbb{R} \setminus \{1\}$, $a = 1$ and $f(x) = \frac{x^2-1}{x-1}$, $x \in A$. Then $\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = 2$. Indeed, let us fix an arbitrary $\varepsilon > 0$. Then we can take $\delta := \varepsilon$ because for all $x \in A \cap B(1,\delta)$ we have $\left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1| < \delta = \varepsilon$.

Example 7.2. We show that $\lim\limits_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$.

By the definition of the number e (see Section 4.2), we have

$$\left(1 + \frac{1}{n+1}\right)^n = \left(1 + \frac{1}{n+1}\right)^{n+1} \frac{n+1}{n+2} \to e \quad \text{and} \quad \left(1 + \frac{1}{n}\right)^{n+1} \to e, \quad n \to \infty.$$

Hence, using the definition of the limit (see Definition 3.3), we obtain that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$

$$e - \varepsilon < \left(1 + \frac{1}{n+1}\right)^n, \quad \left(1 + \frac{1}{n}\right)^{n+1} < e + \varepsilon.$$

So, taking $D := N$, we can estimate for each $x > D$

$$e - \varepsilon < \left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1} < e + \varepsilon,$$

where $\lfloor x \rfloor$ is the greatest integer number less than or equal to x, e.g. $\lfloor 1,7 \rfloor = 1$, $\lfloor -\frac{1}{2} \rfloor = -1$, $\lfloor \pi \rfloor = 3$. Consequently, $\left|\left(1 + \frac{1}{x}\right)^x - e\right| < \varepsilon$ for all $x > D$. This implies $\lim\limits_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$, by Theorem 7.1 (iii).

Exercise 7.1. Compute the following limits
a) $\lim\limits_{x\to 0} \left(x \sin \frac{1}{x}\right)$; b) $\lim\limits_{x\to 0} \left(x \left\lfloor \frac{1}{x} \right\rfloor\right)$.

Example 7.3. Let $b > 1$, $A = \mathbb{R}$, $m \in \mathbb{N}$ and $f(x) = x^m b^{-x}$, $x \in \mathbb{R}$.
We show that $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x^m}{b^x} = 0$.

Solution. Let $\varepsilon > 0$ be given. According to Theorem 3.3, we have $\frac{(n+1)^m}{b^n} = \frac{(n+1)^m}{b^{n+1}} b \to 0$, $n \to \infty$. By the definition of the limit (see Definition 3.3), there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $\frac{(n+1)^m}{b^n} < \varepsilon$. Thus, taking $D := N$, we obtain that for each $x > D$ $\left|\frac{x^m}{b^x} - 0\right| = \frac{x^m}{b^x} < \frac{(\lfloor x\rfloor+1)^m}{b^{\lfloor x\rfloor}} < \varepsilon$. This implies $\lim\limits_{x\to+\infty} \frac{x^m}{b^x} = 0$, by Theorem 7.1 (iii).

Exercise 7.2. Prove that $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$.

7.2 Properties of Limits

Let a be a limit point of a set A.

Theorem 7.2. *If $\lim\limits_{x\to a} f(x) = p_1$ and $\lim\limits_{x\to a} f(x) = p_2$, then $p_1 = p_2$.*

Proof. The theorem immediately follows from the uniqueness of limit for sequences (see Theorem 3.1). Indeed, let $\{x_n\}_{n\geq 1}$ be an arbitrary sequence from A such that $x_n \neq a$, for all $n \geq 1$ and $x_n \to a$, then by the definition of the limit (see Definition 6.6), $f(x_n) \to p_1$, $n \to \infty$, and $f(x_n) \to p_2$, $n \to \infty$. By the uniqueness of limit for sequences (see Theorem 3.1), one has $p_1 = p_2$. $\square$

Theorem 7.3. *Let functions $f, g : A \to \mathbb{R}$ satisfy the following properties: a) $f(x) \leq g(x)$ for all $x \in A$; 2) $\lim\limits_{x\to a} f(x) = p$ and $\lim\limits_{x\to a} g(x) = q$. Then $p \leq q$, that is, $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$.*

Proof. The theorem immediately follows from Theorem 3.6. $\square$

Exercise 7.3. Prove Theorem 7.3.

Theorem 7.4 (Squeeze theorem for functions)**.** *Let $f, g, h : A \to \mathbb{R}$ satisfy the following conditions:*

a) $f(x) \leq h(x) \leq g(x)$ for all $x \in A$;

b) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = p$.

Then $\lim\limits_{x\to a} h(x) = p$.

Proof. The theorem follows from the Squeeze theorem for sequences (see Theorem 3.7). $\square$

Exercise 7.4. Prove Theorem 7.4.

Theorem 7.5. *We assume that for functions $f, g : A \to \mathbb{R}$ there exists limits $\lim\limits_{x\to a} f(x) = p \in \mathbb{R}$ and $\lim\limits_{x\to a} g(x) = q \in \mathbb{R}$. Then*

a) $\lim\limits_{x\to a}(c \cdot f(x)) = c \cdot \lim\limits_{x\to a} f(x)$ for all $c \in \mathbb{R}$;

b) $\lim\limits_{x\to a}(f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$;

c) $\lim\limits_{x\to a}(f(x) \cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$;

d) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, *if* $q \neq 0$.

Proof. The theorem follows from Theorem 3.8. □

Exercise 7.5. Prove Theorem 7.5.

Exercise 7.6. Let $a \notin \{\pi n : \ n \in \mathbb{Z}\}$. Prove that $\lim\limits_{x \to a} \cot x = \cot a$. (*Hint:* Use Example 6.6)

Example 7.4. Let $\alpha \in \mathbb{R}$, and $b > 1$. Show that $\lim\limits_{x \to +\infty} \frac{x^\alpha}{b^x} = 0$.

Exercise 7.7. Show that for every $a \geq 0$ $\lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$.

Exercise 7.8. Compute the following limits:
a) $\lim\limits_{x \to 0} \frac{\tan x}{x}$ b) $\lim\limits_{x \to +\infty} \frac{x^2 + \cos x + 1}{\sqrt{x^4+1} + x + 3}$; c) $\lim\limits_{x \to +\infty} \left(x(\sqrt{x^2 + 2x + 2} - x - 1) \right)$; d) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$;
e) $\lim\limits_{x \to 0} \left(\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right)$; f) $\lim\limits_{x \to 0} \frac{x^2 + x}{\sqrt[3]{1 + \sin x} - 1}$; g) $\lim\limits_{x \to +\infty} (\sqrt{ax + 1} - \sqrt{x})$, for some $a > 0$.

7.3 Left- and Right-Sided Limits

Let A be a subset of $\mathbb{R}$ and a is a limit point of A satisfying the following property

$$\begin{aligned} &\text{there exists a sequence } \ (x_n)_{n \geq 1} \ \text{ such that} \\ &x_n \in A, \quad x_n < a \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad x_n \to a, \quad n \to \infty. \end{aligned} \tag{7}$$

Definition 7.1. A number $p \in \mathbb{R}$ is the **left-sided limit** of a function $f : A \to \mathbb{R}$ at the point a if for each sequence $(x_n)_{n \geq 1}$ such that 1) $x_n \in A$, $x_n < a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, it follows that $f(x_n) \to p$, $n \to \infty$. We will use the notation $p = f(a-)$ or $p = \lim\limits_{x \to a-} f(x)$.

Theorem 7.6. *We assume that* $a \in \mathbb{R}$ *and* $(a - \gamma, a) \subset A$ *for some* $\gamma > 0$. *Then* $p = \lim\limits_{x \to a-} f(x)$ *iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in (a - \delta, a) : \ |f(x) - p| < \varepsilon.$$

Next, if a is a limit point of A satisfying the following property

$$\begin{aligned} &\text{there exists a sequence } \ (x_n)_{n \geq 1} \ \text{ such that} \\ &x_n \in A, \quad x_n > a \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad x_n \to a, \quad n \to \infty, \end{aligned} \tag{8}$$

then we can introduce the right-sided limit of a function.

Definition 7.2. A number $p \in \mathbb{R}$ is the **right-sided limit** of a function $f : A \to \mathbb{R}$ at the point a if for each sequence $(x_n)_{n \geq 1}$ such that 1) $x_n \in A$, $x_n > a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, it follows that $f(x_n) \to p$, $n \to \infty$. We will use the notation $p = f(a+)$ or $p = \lim\limits_{x \to a+} f(x)$.

Theorem 7.7. *We assume that* $a \in \mathbb{R}$ *and* $(a, a + \gamma) \subset A$ *for some* $\gamma > 0$. *Then* $p = \lim\limits_{x \to a+} f(x)$ *iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in (a, a + \delta) : \ |f(x) - p| < \varepsilon.$$

Example 7.5. For the function

$$\operatorname{sgn}(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0, \end{cases}$$

one has $\operatorname{sgn}(0-) = -1$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(0+) = 1$.

Theorem 7.8. *Let $f : A \to \mathbb{R}$ and a be a limit point of A which satisfies properties (7) and (8). Then the limit $\lim\limits_{x\to a} f(x)$ exists iff $f(a-)$ and $f(a+)$ exist and are equal to each other. In this case, $\lim\limits_{x\to a} f(x) = f(a-) = f(a+)$.*

Proof. The necessity of the theorem immediately follows from the definition of the limit of f at a. Next we prove the sufficiency. Setting $p := f(a-) = f(a+)$, we are going to show that $\lim\limits_{x\to a} f(x) = p$. Let $(x_n)_{n\geq 1}$ be as in Definition 6.6, i.e. it satisfies the properties: 1) $x_n \in A$, $x_n \neq a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$. If all elements of the sequence are from one hand side of a starting from some number N, that is, $x_n < a$ for all $n \geq N$ or $x_n > a$ for all $n \geq N$, then $f(x_n) \to f(a-) = p$, $n \to \infty$, or $f(x_n) \to f(a+) = p$, $n \to \infty$, respectively. Next, we assume that infinitely many elements of $(x_n)_{n\geq 1}$ are from both hand sides of a. We construct two subsequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ of $(x_n)_{n\geq 1}$, where $(y_n)_{n\geq 1}$ consists of all elements of $(x_n)_{n\geq 1}$ which are less than a and $(z_n)_{n\geq 1}$ consists of all elements of $(x_n)_{n\geq 1}$ which are grater than a. Then $f(y_n) \to f(a-) = p$, $n \to \infty$, and $f(z_n) \to f(a-) = p$, $n \to \infty$. This implies $f(x_n) \to p$, $n \to \infty$. $\square$

Exercise 7.9. Compute the following limits:
a) $\lim\limits_{x\to\frac{\pi}{2}-} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$; b) $\lim\limits_{x\to\frac{\pi}{2}+} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$; c) $\lim\limits_{x\to 0+} e^{-\frac{1}{x}}$; d) $\lim\limits_{x\to 0+} \frac{e^{-\frac{1}{x}}}{x}$.

7.4 Existence of Limit of Function

Let A be a subset of $\mathbb{R}$.

Definition 7.3. A function $f : A \to \mathbb{R}$ is said to be **increasing** (**decreasing**) **on** A if for all $x_1, x_2 \in A$ the inequality $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$).

Example 7.6. The function $f(x) = x^2$, $x \in \mathbb{R}$, decreases on $(-\infty, 0]$ and increases on $[0, +\infty)$.

Definition 7.4. A function $f : A \to \mathbb{R}$ is called a **monotone function on** A if it is either increasing or decreasing on A.

Definition 7.5. A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if the set $f(A)$ is bounded, that is, there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in A$.

Theorem 7.9. *(i) If $f : A \to \mathbb{R}$ be a monotone and bounded function, then for each limit point a of A which satisfies (7) the left-sided limit $\lim\limits_{x\to a-} f(x)$ exists and belongs to $\mathbb{R}$.*

(ii) If $f : A \to \mathbb{R}$ be a monotone and bounded function, then for each limit point a of A which satisfies (8) the right-sided limit $\lim\limits_{x\to a+} f(x)$ exists and belongs to $\mathbb{R}$.

Proof. We will prove only Part (i). Let $f : A \to \mathbb{R}$ increase and be bounded. We consider the set $B := \{x \in A : x < a\}$. By (7), it is non-empty. Consequently, the set $f(B)$ is also non-empty. Moreover, it is bounded, by the boundedness of the function f. We set

$$p := \sup f(B) = \sup_{x<a} f(x),$$

which exists according to Theorem 2.2.

We are going to show that $f(a-) = p$. Let $(x_n)_{n\geq 1}$ be an arbitrary sequence such that 1) $x_n \in A$, $x_n < a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$. Since for each $n \geq 1$ $x_n < a$, we have $f(x_n) \leq p$ for each $n \geq 1$, by the definition of supremum (see Definition 2.6).

Next, we fix $\varepsilon > 0$ and show that there exists $N \in \mathbb{N}$ such that $|p - f(x_n)| = p - f(x_n) < \varepsilon$ for all $n \geq N$. By Theorem 2.1 (i), there exists $b < a$ such that $p - \varepsilon < f(b)$. Since $x_n \to a$, $n \to \infty$, for $\varepsilon_1 := a - b > 0$ there exists N such that for all $n \geq N$ $|a - x_n| = a - x_n < \varepsilon_1 = a - b$. Hence, $x_n > b$ for all $n \geq N$. Consequently, using the increasing of f, we obtain $|p - f(x_n)| = p - f(x_n) \leq p - f(b) < \varepsilon$. This proves that $f(x_n) \to p$, $n \to \infty$, and, thus, $f(a-) = p$.

If the function f decreases and is bounded, then $f(a-) := \inf_{x<a} f(x)$. The proof is similar. □

Exercise 7.10. Prove Part (ii) of Theorem 7.9.

Exercise 7.11. Let f be an increasing function on an interval $[a, b]$.
a) For each $c \in (a, b)$ show that the one-sided limits $f(a+)$, $f(c-)$, $f(c+)$, $f(b-)$ exist.
b) Check the inequalities

$$f(a) \leq f(a+) \leq f(c-) \leq f(c) \leq f(c+) \leq f(b-) \leq f(b),$$

for all $c \in (a, b)$.
c) Prove that $\lim_{x\to c+} f(x-) = f(c+)$ and $\lim_{x\to c-} f(x+) = f(c-)$ for all $c \in (a, b)$.

Theorem 7.10 (Cauchy Criterion)**.** *Let $a \in \mathbb{R}$ be a limit point of A and $f : A \to \mathbb{R}$. A (finite) limit of f at the point a exists iff*

$$\forall\varepsilon > 0 \ \exists\delta > 0 \ \forall x, y \in A \cap B(a, \delta), \ x \neq a, \ y \neq a : \ |f(x) - f(y)| < \varepsilon.$$

8 Lecture 8 – Continuous Functions

8.1 Definitions and Examples

Let $A \subset \mathbb{R}$, $a \in A$ be a limit point of A and $f : A \to \mathbb{R}$.

Definition 8.1. A function f is said to be **continuous at** a, if $\lim\limits_{x\to a} f(x) = f(a)$, i.e. $\lim\limits_{x\to a} f(x) = f\left(\lim\limits_{x\to a} x\right)$.

By the definition of limit of function (see Definition 6.6) and Theorem 7.1, the following two definitions are equivalent to Definition 8.1.

Definition 8.2. A function f is said to be **continuous at** a, if for each sequence $(x_n)_{n\geq 1}$ such that 1) $x_n \in A$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, it follows that $f(x_n) \to f(a)$, $n \to \infty$.

Definition 8.3. A function f is said to be **continuous at** a, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A \cap B(a, \delta) : \ |f(x) - f(a)| < \varepsilon.$$

Now we want to introduce the left and right continuity. For this we assume that $a \in A$ satisfies (7) (resp., (8)).

Definition 8.4. A function f is said to be **left continuous** (resp. **right continuous**), if $f(a-) = f(a)$ (resp. $f(a+) = f(a)$).

Remark 8.1.

1. If $(a - \gamma, a] \subset A$ for some $\gamma > 0$, then f is left continuous iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in (a - \delta, a] : \ |f(x) - f(a)| < \varepsilon.$$

   This immediately follows from Theorem 7.6.

2. If $[a, a + \gamma) \subset A$ for some $\gamma > 0$, then f is right continuous iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in [a, a + \delta) : \ |f(x) - f(a)| < \varepsilon.$$

   This follows from Theorem 7.7.

Remark 8.2. Let a satisfy properties (7) and (8). Then, by Theorem 7.8, a function f is continuous at the point a iff f is left and right continuous at a.

For convenience we will suppose that every function is continuous at each isolated point, points from A which are not its limit points.

Definition 8.5. A function $f : A \to \mathbb{R}$ is called **continuous** on the set A, if it is continuous at each point of A. We will often use the notation $f \in \mathrm{C}(A)$.

Theorem 8.1. *Let functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at $a \in A$. Then*

a) for each real number c the function $c \cdot f$ is continuous at the point a;

b) the function $f + g$ is continuous at the point a;

c) the function $f \cdot g$ is continuous at the point a;

d) the function $\frac{f}{g}$ is continuous at the point a, if additionally $g(a) \neq 0$.

In the theorem, the functions $c \cdot f$, $f+g$, $f \cdot g$, $\frac{f}{g}$ are defined in the usual way. For instance, $f \cdot g : A \to \mathbb{R}$ is defined as $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in A$.

Example 8.1. For an arbitrary real number c we define the function $f(x) = c$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$.

Example 8.2. Let $f(x) = x$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$. Indeed, to show this, let us use e.g. Definition 8.3. We fix any $a \in \mathbb{R}$. Then we obtain that for every $\varepsilon > 0$ there exists $\delta := \varepsilon > 0$ such that for each $x \in B(a, \delta)$ $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$. So, f is continuous at a. Since a was an arbitrary point of $\mathbb{R}$, f is continuous on $\mathbb{R}$.

Example 8.3. Let $P(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m$, $x \in \mathbb{R}$, where $m \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_m$ are some real numbers. The function P is called a **polynomial function**. Theorem 8.1 and examples 8.1, 8.2 imply that $P \in \mathrm{C}(\mathbb{R})$.

Example 8.4. Let P and Q be two polynomial functions. We define the function $R(x) = \frac{P(x)}{Q(x)}$, $x \in \{z \in \mathbb{R} : \ Q(z) \neq 0\}$, which is called a **rational function**. By Theorem 8.1 and Example 8.3, the rational function R is continuous at any point where it is well-defined.

Example 8.5. The functions sin and cos are continuous on $\mathbb{R}$. The functions tan and cot are continuous on the set where they are well-defined. The continuity of functions sin and cos follows from Example 6.6. For the functions tan and cot the continuity follows from Theorem 8.1 and the equalities $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$.

Example 8.6. Let $a > 0$ and $f(x) = a^x$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$.

Exercise 8.1. Prove that the function from Example 8.6 is continuous on $\mathbb{R}$.

Exercise 8.2. Compute the following limits:
a) $\lim\limits_{x \to 0} (\tan x - e^x)$; b) $\lim\limits_{x \to 2} \frac{x^2 - 3^x + 1}{x - \sin \pi x}$; c) $\lim\limits_{x \to 3} \frac{x \cos x + 1}{x^3 + 1}$.

Exercise 8.3. Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. a) For which a, b the function f is monotone on $\mathbb{R}$? b) For which a, b the function f is continuous on $\mathbb{R}$?

Exercise 8.4. Let $f(x) = \lfloor x \rfloor \sin \pi x$, $x \in \mathbb{R}$. Prove that $f \in \mathrm{C}(\mathbb{R})$ and sketch its graph.

(*Hint:* If $x \in [k, k+1)$ for some $k \in \mathbb{Z}$, then $\lfloor x \rfloor = k$ and $f(x) = k \sin \pi x$. Find $f(k-)$ and $f(k+)$ at the points k)

Exercise 8.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ and $f(r) = r^3 + r + 1$ for all $r \in \mathbb{Q}$. Find the function f.

Exercise 8.6. Show that $|f| \in \mathrm{C}(A)$, if $f \in \mathrm{C}(A)$, where $|f|(x) := |f(x)|$, $x \in A$.

Exercise 8.7. For functions $f, g \in \mathrm{C}(A)$ we set $h(x) := \min\{f(x), g(x)\}$, $x \in A$, and $l(x) := \max\{f(x), g(x)\}$, $x \in A$. Prove that $h, l \in \mathrm{C}(A)$.

(*Hint:* Use the equalities $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ and $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.)

Definition 8.6. If a function $f : A \to \mathbb{R}$ is not continuous at a point $a \in A$, then f is said to be **discontinuous at the point** a.

Example 8.7. The function sgn, defined in Example 7.5, is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0.

Exercise 8.8. Prove that the function $f(x) = \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

Exercise 8.9. Show that the **Dirichlet function** $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at any point of $\mathbb{R}$.

8.2 Some Properties of Continuous Functions

Theorem 8.2. *Let a function $f : A \to \mathbb{R}$ be continuous at $a \in A$ and $f(a) < q$. Then*

$$\exists \delta > 0 \ \forall x \in A \cap B(a, \delta) : \ f(x) < q.$$

Proof. Using Definition 8.3, we obtain that for $\varepsilon := q - f(a) > 0$ there exists $\delta > 0$ such that for all $x \in A \cap B(a, \delta)$ $|f(x) - f(a)| < \varepsilon = q - f(a)$. In particular, $f(x) - f(a) < q - f(a)$, which implies $f(x) < q$ for all $x \in A \cap B(a, \delta)$. $\square$

Theorem 8.3 (Limit of composition)**.** *Let a be a limit point of A (which could be $+\infty$ or $-\infty$) and let for a function $f : A \to \mathbb{R}$ there exists a limit $\lim\limits_{x \to a} f(x) = p \in \mathbb{R}$. We also assume that $f(A) \cap \{p\} \subset B$ and a function $g : B \to \mathbb{R}$ is continuous at the point p. Then $\lim\limits_{x \to a} g(f(x)) = g(p)$, that is, $\lim\limits_{x \to a} g(f(x)) = g\left(\lim\limits_{x \to a} f(x)\right)$.*

Proof. For any sequence $(x_n)_{n \geq 1}$ satisfying properties 1) and 2) from the definition of limit (see Definition 6.6), one has $f(x_n) \to p$, $n \to \infty$. Since g is continuous, $g(f(x_n)) \to g(p)$, $n \to \infty$, by Definition 8.2. $\square$

Theorem 8.4 (Continuity of composition)**.** *We assume that $f : A \to \mathbb{R}$ is continuous at $a \in A$, $f(A) \subset B$ and a function $g : B \to \mathbb{R}$ is continuous at the point $f(a)$. Then the function $g \circ f$ is continuous at the point a.*

Proof. The statement immediately follows from Theorem 8.3, setting $p := f(a)$. $\square$

Let $(a, b) \subset \mathbb{R}$, where $-\infty \leq a < b \leq +\infty$. Let $f : (a, b) \to \mathbb{R}$ be an increasing function. By Theorem 7.9 (ii), there exists $\lim\limits_{x \to a+} f(x) =: c \in \mathbb{R}$, if f is bounded below. If f is unbounded below, then it is easy to see that $\lim\limits_{x \to a+} f(x) = -\infty =: c$. Consequently, $\lim\limits_{x \to a+} f(x) = c$ can be well defined for any increasing function. Similarly, $\lim\limits_{x \to b-} f(x) =: d \leq +\infty$ is also well defined.

Theorem 8.5 (Existence of continuous inverse function)**.** *Let a function $f : (a, b) \to \mathbb{R}$ satisfy the following properties:*

1) f strictly increases on (a, b), that is, for any $x_1, x_2 \in (a, b)$ $x_1 < x_2$ implies $f(x_1) < f(x_2)$;

2) $f \in \mathrm{C}((a, b))$.

We set $c := \lim\limits_{x \to a+} f(x)$ and $d := \lim\limits_{x \to b-} f(x)$.
Then there exists a function $g : (c, d) \to (a, b)$ such that

a) g is strictly increasing on (c, d);

b) $g \in \mathrm{C}((c, d))$;

c) $g(f(x)) = x$ for all $x \in (a, b)$, and $f(g(y)) = y$ for all $y \in (c, d)$, that is, $g = f^{-1}$.

Remark 8.3. A similar statement also is true for a strictly decreasing function $f : (a, b) \to \mathbb{R}$, i.e. for a function such that for any $x_1, x_2 \in (a, b)$ $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Remark 8.4. If $a \in \mathbb{R}$, then Theorem 8.5 is also valid for the set $[a, b)$.

8.3 Some Inverse Functions

Example 8.8. n-th root function $g(y) = \sqrt[m]{y}$**,** $y \geq 0$**.**

Let $m \in \mathbb{N}$ be fixed. We set $[a, b) = [0, +\infty)$ and $f(x) = x^m$, $x \in [0, +\infty)$. The function f satisfies conditions of Theorem 8.5, namely, it strictly increases and is continuous on $[0, +\infty)$. Moreover, $c = \lim\limits_{x\to 0+} x^m = 0$ and $d = \lim\limits_{x\to+\infty} x^m = +\infty$. Thus, according to Theorem 8.5, there exists a function $g : [0, +\infty) \to [0, +\infty)$ which increases and is continuous on $[0, +\infty)$ and inverse to f. Usually, the function g is denoted as follows $\sqrt[m]{y} = y^{\frac{1}{m}} := g(y)$, $y \geq 0$. Moreover, $\sqrt[m]{x^m} = x$ for each $x \geq 0$ and $(\sqrt[m]{y})^m = y$ for each $y \geq 0$.

Example 8.9. Logarithmic function $g(y) = \log_p y$**,** $y > 0$**.**

Let $p > 0$, $p \neq 1$ and $f(x) = p^x$, $x \in \mathbb{R}$. We want to prove that the function f has the inverse function, which is called the **logarithm**. We will consider the case $p > 1$, for which the function f is strictly increasing and continuous, by Example 8.6. Moreover, $c = \lim\limits_{x\to-\infty} p^x = 0$ and $d = \lim\limits_{x\to+\infty} p^x$. By Theorem 8.5, there exists a function $g : (0, +\infty) \to \mathbb{R}$, which is continuous on $(0, +\infty)$ and inverse to f. The function g is denoted by $\log_p y := g(y)$, $y > 0$, and it satisfies $\log_p p^x = x$ for all $x \in \mathbb{R}$ and $p^{\log_p y} = y$ for all $y > 0$.

Example 8.10. Trigonometric functions arcsin, arccos, arctan, arccot**.**

Let $[a, b] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. By the definition of sin, it is strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Furthermore, by Example 8.5, sin is continuous on $\mathbb{R}$ and, in particular, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, using Theorem 8.5, there exists the continuous inverse function $g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to f. It is denoted by $\arcsin y := g(y)$, $y \in [-1, 1]$, and satisfies $\arcsin(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(\arcsin y) = y$ for all $y \in [-1, 1]$.

Similarly, one can define the functions $\arccos : [-1, 1] \to [0, \pi]$, $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$, which are inverse to $\cos : [0, \pi] \to [-1, 1]$, $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ and $\cot : (0, \pi) \to \mathbb{R}$, respectively. Moreover, each function is continuous on the set where it is defined.

Exercise 8.10. Sketch the graphs of the functions $\ln = \log_e$, $\log_{\frac{1}{2}}$, arcsin, arccos, arctan and arccot.

Exercise 8.11. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\ln(1+x)+\arcsin x^2}{\arccos x+\cos x}$; b) $\lim\limits_{x\to 1} \frac{\arctan x}{1+\arctan x^2}$; c) $\lim\limits_{x\to 0} \frac{\arcsin x}{x}$; d) $\lim\limits_{x\to 0} \frac{x}{\sin x+\arcsin x}$; e) $\lim\limits_{x\to 0} \frac{\arctan x}{x}$;
f) $\lim\limits_{x\to 0} \frac{\arccos x-\frac{\pi}{2}}{x}$; g) $\lim\limits_{x\to 0} \frac{\sin(\arctan x)}{\tan x}$.

8.4 Some Important Limits

Theorem 8.6. *Let $a > 0$ and $a \neq 1$. Then*

$$\lim_{x\to 0} \frac{\log_a(1 + x)}{x} = \log_a e,$$

in particular, for $a = e$

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1.$$

Proof. We are going to use Theorem 8.3 about limit of composition in order to prove the needed equality. Let $A = (-1, +\infty)$, $f(x) = (1 + x)^{\frac{1}{x}}$, $x > -1$, $p = \lim\limits_{x\to 0}(1 + x)^{\frac{1}{x}} = e > 0$; $B = (0, +\infty)$ and

$g(y) = \log_a y$, $y > 0$. In Example 8.9, we have proved that g is continuous and, consequently, it is continuous at $p = e > 0$. Thus, using Theorem 8.3, we obtain

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \lim_{x\to 0} \log_a (1+x)^{\frac{1}{x}} = \log_a \left(\lim_{x\to 0} (1+x)^{\frac{1}{x}} \right) = \log_a e.$$

□

Theorem 8.7. *Let $a > 0$. Then*

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a,$$

in particular, for $a = e$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Proof. If $a = 1$, then the statement is true. We assume that $a \neq 1$. By the continuity and monotonicity of the function $h(x) = a^x$ (see Example 8.6), one can easily seen that $z := a^x - 1 \to 0$ provided $x \to 0$. Moreover, $x = \log_a(1+z)$. Hence, by Theorem 8.6, we obtain

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{z\to 0} \frac{z}{\log_a(1+z)} = \frac{1}{\log_a e} = \log_e a = \ln a.$$

□

Theorem 8.8. *Let $\alpha \in \mathbb{R}$. Then*

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha.$$

Proof. For $\alpha = 0$ the statement holds. We assume that $\alpha \neq 0$. Using the continuity of ln (see Example 8.9), we have $\ln(1+x) \to \ln 1 = 0$, $x \to 0$. By theorems 8.1, 8.6 and 8.7, we get

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{x\to 0} \frac{\left(e^{\alpha \ln(1+x)} - 1\right) \alpha \ln(1+x)}{x \alpha \ln(1+x)} = \alpha \lim_{x\to 0} \frac{e^{\alpha \ln(1+x)} - 1}{\alpha \ln(1+x)} \cdot \lim_{x\to 0} \frac{\ln(1+x)}{x} = \alpha.$$

□

Theorem 8.9. *Let $\alpha \in \mathbb{R}$ and $f(x) = x^\alpha$, $x > 0$. Then f is continuous on $(0, +\infty)$.*

Proof. Since for each $x > 0$, one has $f(x) = e^{\alpha \ln x}$, the statement follows from the continuities of the exponential function and the logarithm (see examples 8.6 and 8.9, respectively) and Theorem 8.4. □

Exercise 8.12. Compute the following limits:

a) $\lim_{x\to 0} (\cos x)^x$; b) $\lim_{x\to +\infty} x(\ln(1+x) - \ln x)$; c) $\lim_{x\to 0} \left(\frac{1+\sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$; d) $\lim_{x\to 0} \frac{1-\cos x}{1-\cos 2x}$; e) $\lim_{x\to 0} \frac{\ln(1+x)+e^x-\cos x}{e^{x^2}-1+\sin x}$; f) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$; g) $\lim_{x\to 0} \frac{\arcsin(x-1)}{x^m - 1}$ for $m \in \mathbb{N}$; h) $\lim_{x\to 0} \frac{1-(\cos mx)^m}{x^2}$ for $m \in \mathbb{N}$; i) $\lim_{x\to 0} \frac{1-(\cos mx)^{\frac{1}{m}}}{x^2}$ for $m \in \mathbb{N}$; k) $\lim_{x\to 1} \frac{(\sin(\pi \cdot 2^x))^2}{\ln(\cos(\pi \cdot 2^x))}$; l) $\lim_{x\to 0} \left(\frac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}$; m) $\lim_{x\to 0} \frac{x^x - a^a}{x-a}$, for $a > 0$; n) $\lim_{x\to 1} (1-x) \log_x 2$.

9 Lecture 9 – Properties of Continuous Functions

9.1 Boundedness of Continuous Functions and Intermediate Value Theorem

For more details see [1, Section 3.18].

Let $-\infty < a < b < +\infty$ be fixes.

Theorem 9.1 (1st Weierstrass theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f is bounded on $[a,b]$.*

Proof. We assume that f is unbounded on $[a,b]$. Then for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $|f(x_n)| \geq n$. Since the sequence $(x_n)_{n\geq 1}$ is bounded (each x_n belongs to the interval $[a,b]$), it has a convergent subsequence $(x_{n_k})_{k\geq 1}$, by the Bolzano-Weierstrass theorem (see Theorem 4.6). So, let $x_{n_k} \to x_\infty$, $k \to \infty$. Using the inequalities $a \leq x_{n_k} \leq b$ for all $k \geq 1$ and Theorem 3.6, we have that $a \leq x_\infty \leq b$. Since the function f is continuous on $[a,b]$, we have that $f(x_{n_k}) \to f(x_\infty)$, $k \to \infty$. But this is impossible because $|f(x_{n_k})| \geq n_k \to +\infty$, $k \to \infty$. So, the function f must be bounded. $\square$

Example 9.1. If $f : (a,b] \to \mathbb{R}$ is a continuous function on $(a,b]$, then the function could be unbounded. Indeed, we set $(a,b] = (0,1]$ and $f(x) = \frac{1}{x}$, $x \in (0,1]$. Then $f \in \mathrm{C}((0,1])$ but $f(x) \to +\infty$, $x \to 0+$.

Corollary 9.1. *Let $f : [a,+\infty) \to \mathbb{R}$ be a continuous function on $[a,+\infty)$ and $f(x) \to p \in \mathbb{R}$, $x \to +\infty$. Then f is bounded on $[a,+\infty)$.*

Proof. By Theorem 7.1 (iii), for $\varepsilon := 1$ there exists $D > a$ such that $|f(x) - p| < \varepsilon = 1$ for all $x \geq D$. Hence $p - 1 < f(x) < p + 1$ for all $x \geq D$, which implies the boundedness of f on $[D,+\infty)$. Next, since the function is continuous on the interval $[a,D]$, we can apply the 1st Weierstrass theorem. Consequently, f is also bounded on $[a,D]$. Hence the function f is bounded on $[a,+\infty)$. $\square$

Exercise 9.1. Prove that the function $f(x) = \left(1 + \frac{1}{x}\right)^x$, $x > 0$, is bounded on $(0,+\infty)$.

(*Hint:* Theorem 9.1 as well as Corollary 9.1 can not be applied to the interval $(0,+\infty)$, since the the point a does not belong to the interval. First find the limits of f as $x \to 0+$ and $x \to +\infty$) and then use the argument from Corollary 9.1.)

Theorem 9.2 (2nd Weierstrass theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f assumes its minimum and maximum values on $[a,b]$, that is, there exist x_* and x^* in $[a,b]$ such that $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a,b]$.*

Proof. We will prove the existence of x^*. The proof is similar for x_*. By the 1st Weierstrass theorem, the function f is bounded on $[a,b]$, that implies that the set $f([a,b]) = \{f(x) : \ x \in [a,b]\}$ is bounded. So, we set $p := \sup f([a,b]) = \sup_{x\in[a,b]} f(x)$, which exists, by Theorem 2.2 (i). According to Theorem 2.1 (i), for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $p - \frac{1}{n} < f(x_n) \leq p$. We apply the Bolzano-Weierstrass theorem (see Theorem 4.6) to the sequence $(x_n)_{n\geq 1}$. Consequently, there exists a convergent subsequence $(x_{n_k})_{k\geq 1}$. We denote its limit by x^*. So, $x_{n_k} \to x^*$, $k \to \infty$. Since $f \in \mathrm{C}([a,b])$, we have that $f(x_{n_k}) \to f(x^*)$, $k \to \infty$. Moreover,

$$p - \frac{1}{n_k} < f(x_{n_k}) \leq p$$

for all $k \geq 1$. Hence, $f(x_{n_k}) \to p$, $k \to \infty$, by the Squeeze theorem (see Theorem 3.7). It implies that $f(x^*) = p$. Consequently, $f(x^*) = \sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x)$, that is, $f(x) \leq f(x^*)$ for all $x \in [a,b]$. $\square$

Exercise 9.2. Prove the existence of the point x_* in the 2nd Weierstrass theorem.

Exercise 9.3. Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function on $[0, +\infty)$ and $f(x) \to +\infty$, $x \to +\infty$. Show that there exists $x_* \in [0, +\infty)$ such that $f(x_*) = \inf\limits_{x \in [0,+\infty)} f(x) = \min\limits_{x \in [0,+\infty)} f(x)$.

Theorem 9.3 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then for any real number y_0 between $f(a)$ and $f(b)$, i.e. $f(a) \leq y_0 \leq f(b)$ or $f(b) \leq y_0 \leq f(a)$, there exists x_0 from $[a, b]$ such that $f(x_0) = y_0$.*

Proof. If $y_0 = f(a)$ or $y_0 = f(b)$, then x_0 equals a or b, respectively. Now we assume that $f(a) < y_0 < f(b)$. The case $f(b) < y_0 < f(a)$ is similar. We set $M := \{x \in [a, b] : f(x) \leq y_0\}$, which is non empty set because $a \in M$. Moreover, it is bounded as a subset of the interval $[a, b]$. Consequently, there exists $\sup M =: x_0$.

We are going to show that $f(x_0) = y_0$. According to Theorem 2.1 (i), for each $n \in \mathbb{N}$ there exists $x_n \in M$ such that $x_0 - \frac{1}{n} < x_n \leq x_0$. Thus, $x_n \to x_0$, $n \to \infty$, by the Squeeze theorem (see Theorem 3.7). Since $x_n \in M$, we have that $f(x_n) \leq y_0$ for all $n \geq 1$. Moreover, $f(x_n) \to f(x_0)$, $n \to \infty$ due to the continuity of f. Thus, using Theorem 3.6, we obtain $f(x_0) \leq y_0$.

Next, for every $x > x_0$ we have that $x \notin M$, since x_0 is the supremum of M. It implies that $f(x) > y_0$. Consequently, $y_0 \leq \lim\limits_{x \to x_0+} f(x) = \lim\limits_{x \to x_0} f(x) = f(x_0)$. Here we have also used the continuity of f and Theorem 7.8. Thus, $y_0 = f(x_0)$. □

Exercise 9.4. Prove that the function $P(x) = x^3 + 7x^2 - 1$, $x \in \mathbb{R}$, has at least one root, that is, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = 0$.

Corollary 9.2. *Let $f, g \in \mathrm{C}([a, b])$ and $f(a) \leq g(a)$, $f(b) \geq g(b)$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.*

Proof. We note that the function $h(x) := f(x) - g(x)$, $x \in [a, b]$, is continuous on $[a, b]$ and satisfies $h(a) \leq 0 \leq h(b)$. So, taking $y_0 := 0$ and applying the intermediate value theorem, we obtain that there exists $x_0 \in [a, b]$ such that $h(x_0) = f(x_0) - g(x_0) = 0$. □

Example 9.2. Let $g : [0, 1] \to [0, 1]$ be a continuous function on $[0, 1]$. Then there exists $x_0 \in [0, 1]$ such that $g(x_0) = x_0$.

To prove the existence of x_0, we take $f(x) = x$, $x \in [0, 1]$, and note that f is continuous on $[0, 1]$ and $f(0) = 0 \leq g(0)$, $f(1) = 1 \geq g(1)$. Thus, by Corollary 9.2, there exists $x_0 \in [0, 1]$ such that $g(x_0) = f(x_0) = x_0$.

Corollary 9.3. *Let $f \in \mathrm{C}([a, b])$. Then its range $f([a, b]) = \{f(x) : \ x \in [a, b]\}$ is an interval.*

Proof. By the 2nd Weierstrass theorem (see Theorem 9.2), there exists $x_*, x^* \in [a, b]$ such that $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a, b]$. Consequently, $f([a, b]) \subset [f(x_*), f(x^*)]$. Next, due to the intermediate value theorem, for each $y_0 \in [f(x_*), f(x^*)]$ there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$. It implies that $y_0 \in f([a, b])$ and, consequently, $[f(x_*), f(x^*)] \subset f([a, b])$. Hence, $f([a, b]) = [f(x_*), f(x^*)]$. □

Exercise 9.5. Let $f : [a, b] \to \mathbb{R}$ strictly increase on $[a, b]$ and for each $y_0 \in [f(a), f(b)]$ there exist $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Prove that $f \in \mathrm{C}([a, b])$.

Exercise 9.6. Let $f, g : [0, 1] \to [0, 1]$ be continuous and f be a surjection. Prove that there exists $x_0 \in [0, 1]$ such that $f(x_0) = g(x_0)$.

9.2 Uniformly Continuous Functions

For more details see [1, Section 3.19].

Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. We recall that f is continuous at point x_0 provided $\forall \varepsilon > 0\ \exists \delta > 0$ such that for each $x \in A$ the inequality $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$ (see Definition 8.3). The choice of δ depends on ε and the point x_0. It turns out to be very useful to know when the δ can be chosen to depend only on ε. Such functions are said to be uniformly continuous on A.

Definition 9.1. A function $f : A \to \mathbb{R}$ is said to be **uniformly continuous on** A, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x', x'' \in A,\ |x' - x''| < \delta :\ |f(x') - f(x'')| < \varepsilon.$$

Remark 9.1. Any uniformly continuous function on A is continuous on A. The converse statement is not true, see Example 9.5 below.

Example 9.3. The function $f(x) = x$, $x \in \mathbb{R}$, is uniformly continuous on $\mathbb{R}$, since for each $\varepsilon > 0$ we can take $\delta := \varepsilon$. Then for all $x', x'' \in \mathbb{R}$ such that $|x' - x''| < \delta$ we have $|f(x') - f(x'')| = |x' - x''| < \delta = \varepsilon$.

Example 9.4. The functions sin and cos are uniformly continuous on $\mathbb{R}$.

The function sin is uniformly continuous on $\mathbb{R}$, since for each $\varepsilon > 0$ we can take $\delta := \varepsilon > 0$. Then for all $x', x'' \in \mathbb{R}$ such that $|x' - x''| < \delta$ we have

$$|\sin x' - \sin x''| = 2\left|\cos \frac{x' + x''}{2}\right| \cdot \left|\sin \frac{x' - x''}{2}\right| \leq 2 \cdot 1 \cdot \frac{|x' - x''|}{2} = |x' - x''| < \delta = \varepsilon,$$

where we have also used Remark 6.3 for the estimation of $\left|\sin \frac{x' - x''}{2}\right|$.

Example 9.5. The function $f(x) = \frac{1}{x}$, $x \in (0, 1]$, is not uniformly continuous on $(0, 1]$.

Indeed, for $\varepsilon := 1$ we have that for all $\delta > 0$ we can take $x' := \frac{1}{n}$ and $x'' := \frac{1}{n+1}$ from $(0, 1]$ such that $|x' - x''| < \delta$ and $\left|\frac{1}{x'} - \frac{1}{x''}\right| = |n - (n + 1)| = 1 = \varepsilon$, where $n \in \mathbb{N}$ and $n > \frac{1}{\delta}$.

Exercise 9.7. Prove that the following functions are uniformly continuous on their domains:
a) $f(x) = \ln x$, $x \in [1, +\infty)$; b) $f(x) = \sqrt{x}$, $x \in [0, +\infty)$; c) $f(x) = x \sin \frac{1}{x}$, $x \in (0, +\infty)$.

Exercise 9.8. Prove that the following functions are not uniformly continuous on their domains:
a) $f(x) = \ln x$, $x \in (0, 1]$; b) $f(x) = \sin(x^2)$, $x \in [0, +\infty)$; c) $f(x) = x \sin x$, $x \in [0, +\infty)$.

Theorem 9.4 (Heine-Cantor theorem)**.** *Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

Proof. Be assume that f is not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists x' and x'' from $[a, b]$ such that $|x' - x''| < \delta$ and $|f(x') - f(x'')| \geq \varepsilon$. So, for each $n \in \mathbb{N}$ taking $\delta := \frac{1}{n}$, we can find x'_n and x''_n from $[a, b]$ such that $|x'_n - x''_n| < \delta = \frac{1}{n}$ and $|f(x'_n) - f(x''_n)| \geq \varepsilon$.

We will consider the obtained sequences $(x'_n)_{n \geq 1}$ and $(x''_n)_{n \geq 1}$. By the Bolzano-Weierstrass theorem (see Theorem 4.6), there exists a subsequence $(x'_{n_k})_{n \geq 1}$ of $(x'_n)_{n \geq 1}$ which converges to some real number $x_\infty \in [a, b]$. Since $|x'_{n_k} - x''_{n_k}| < \frac{1}{n_k}$ for all $k \geq 1$, we have $x''_{n_k} \to x_\infty$. By the continuity of f, $f(x'_{n_k}) \to f(x_\infty)$, $k \to \infty$, and $f(x''_{n_k}) \to f(x_\infty)$, $k \to \infty$. But $|f(x'_{n_k}) - f(x''_{n_k})| \geq \varepsilon > 0$, for all $k \geq 1$, that contradict our assumption. $\square$

10 Lecture 10 – Differentiation

10.1 Definition and Some Examples

Let $A \subset \mathbb{R}$ and $a \in A$. We also assume that there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset A$. Let $f : A \to \mathbb{R}$ be a given function.

Definition 10.1.

- We say that f is **differentiable at** a, or f has a **derivative at** a, if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We will write $f'(a)$ or $\frac{df}{dx}(a)$ for the derivative of f at a, that is,

$$f'(a) := \frac{df}{dx}(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

whenever this limit exists and is finite.

- If for each $a \in A$ the derivative $f'(a)$ exists, then the function f is said to be **differentiable on** A and the function defined by $A \ni x \mapsto f'(x)$ is called the **derivative of** f **on the set** A.

Remark 10.1. Taking $h := x - a$ in Definition 10.1, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Definition 10.2.

- If a finite left-sided limit

$$\lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$$

exists, then this limit is called a **left derivative of** f **at** a and is denoted by $f'_-(a)$ or $\frac{d^- f}{dx}(a)$.

- If a finite right-sided limit

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

exists, then this limit is called a **right derivative of** f **at** a and is denoted by $f'_+(a)$ or $\frac{d^+ f}{dx}(a)$.

Remark 10.2. By Theorem 7.8, a derivative $f'(a)$ exists iff $f'_-(a)$ and $f'_+(a)$ exist and $f'_-(a) = f'_+(a)$.

Example 10.1. For the function $f(x) = x$, $x \in \mathbb{R}$, and any point $a \in \mathbb{R}$ we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1.$$

Thus, $(x)' = 1$, $x \in \mathbb{R}$.

Example 10.2. For the function $f(x) = |x|$, $x \in \mathbb{R}$, and the point $a = 0$ we have

$$f'_-(0) = \lim_{x \to 0-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

and

$$f'_+(0) = \lim_{x \to 0+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0+} \frac{x}{x} = 1.$$

So, $f'_-(0) = -1 \neq f'_+(0) = 1$ and the derivative $f'(0)$ does not exist.

Example 10.3. Let $f(x) = x^2$, $x \in \mathbb{R}$, and $a \in \mathbb{R}$. Then

$$f'(a) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x \to a}(x+a) = 2a.$$

Hence, $(x^2)' = 2x$, $x \in \mathbb{R}$.

Example 10.4. For the function $f(x) = \sqrt[3]{x}$, $x \in \mathbb{R}$, and the point $a = 0$ we have

$$\lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Consequently, the derivative of f at 0 does not exist.

Example 10.5. Let $f(x) = x \sin \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, and $f(0) = 0$. Let also $a = 0$. Then $\frac{f(x)-f(0)}{x-0} = \frac{x \sin \frac{1}{x} - 0}{x - 0} = \sin \frac{1}{x}$ does not have any limit as $x \to 0$ (see Exercise 8.8). Thus, the function f is not differentiable at 0.

Exercise 10.1. Check that $(x|x|)' = 2|x|$, $x \in \mathbb{R}$.

Exercise 10.2. For the function $f(x) = |x^2 - x|$, $x \in \mathbb{R}$, compute $f'(x)$ for each $x \in \mathbb{R} \setminus \{0, 1\}$. Compute left and right derivatives at points 0 and 1.

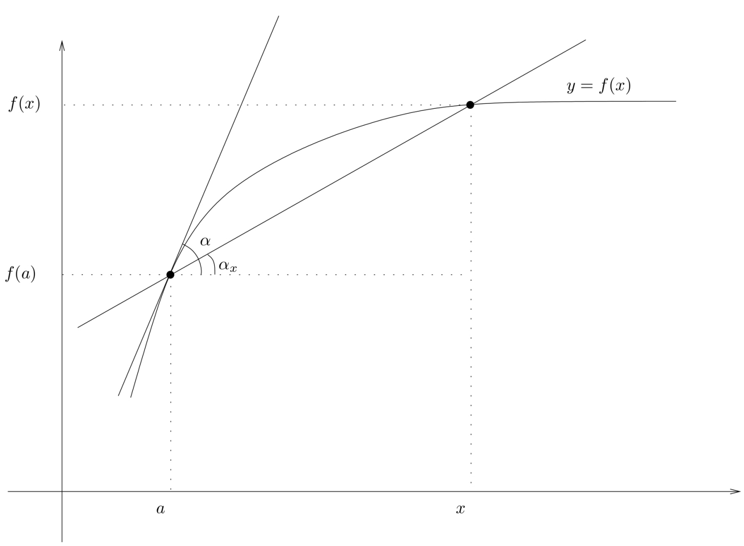
10.2 Interpretation of Derivative

a) **Physical Interpretation**.

Let a point P move on the real line and $s(t)$ is its position at time t. Let t_1, t_2 be two moments of time and $t_1 < t_2$. Then the average velocity over the period of time $[t_1, t_2]$ is the ratio $\frac{s(t_2)-s(t_1)}{t_2-t_1}$, where $s(t_2) - s(t_1)$ is the distance travelled by P during the time $t_2 - t_1$. The instantaneous velocity at t_1 is the limit of the average velocity as t_2 approaches t_1, that is, it is the limit $\lim_{t_2 \to t_1} \frac{s(t_2)-s(t_1)}{t_2-t_1}$. Thus, **instantaneous velocity** $v(t)$ **of the point** P **at time** t **is the derivative of** s **at** t, i.e. $v(t) = s'(t)$.

b) **Geometric interpretation**

Let a function $f : (a, b) \to \mathbb{R}$ be differentiable at a.



The slope of the secant line through $(a, f(a))$ and $(x, f(x))$ is

$$\tan \alpha_x = \frac{f(x) - f(a)}{x - a}.$$

If x approaches a, the secant line through $(a, f(a))$ and $(x, f(x))$ approaches the tangent line through $(a, f(a))$. Hence, the **derivative $f'(a)$ is the slope of the tangent line through the point** $(a, f(a))$, that is,

$$\tan \alpha = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

So, the linear function whose graph is the tangent line through $(a, f(a))$ can be given by the equation

$$f'(a) = \tan \alpha = \frac{y - f(a)}{x - a}, \quad x \in \mathbb{R},$$

that is,

$$y = f(a) + f'(a)(x - a), \quad x \in \mathbb{R}.$$

10.3 Properties of Derivatives

Theorem 10.1. *If a function $f : A \to \mathbb{R}$ is differentiable at a, then there exists a function $\varphi : A \to \mathbb{R}$ such that*

$$f(x) = f(a) + f'(a)(x - a) + \varphi(x)(x - a), \quad x \in A$$

and $\varphi(x) \to 0$, $x \to a$.

Proof. We take $\varphi(x) := \frac{f(x)-f(a)}{x-a} - f'(a)$, $x \in A \setminus \{a\}$ and $\varphi(a) = 0$. Then the statement easily follows from Definition 10.1. □

Exercise 10.3. If there exists $L \in \mathbb{R}$ such that $f(x) = f(a) + L(x - a) + \varphi(x)(x - a)$, $x \in A$, for some function $\varphi : A \to \mathbb{R}$ satisfying $\varphi(x) \to 0$, $x \to a$, then f is differentiable at a and $f'(a) = L$. Prove this statement.

Theorem 10.2. *If f is differentiable at a point a, then f is continuous at a.*

Proof. Using theorems 10.1 and 8.1, we obtain

$$\lim_{x \to a} f(x) = \lim_{x \to a} \big(f(a) + f'(a)(x - a) + \varphi(x)(x - a) \big) = f(a).$$

□

Exercise 10.4. Let f has a derivative $f'(a)$ at a point a. Express through $f(a)$ and $f'(a)$ the following limits:
a) $\lim\limits_{h \to 0} \frac{f(a-h)-f(a)}{h}$; b) $\lim\limits_{h \to 0} \frac{f(a+2h)-f(a)}{h}$; c) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a-h)}{h}$; d) $\lim\limits_{n \to \infty} n \left(f\left(a + \frac{1}{n}\right) - f(a) \right)$;
e) $\lim\limits_{n \to \infty} n \left(f\left(\frac{n+1}{n} a\right) - f(a) \right)$; f) $\lim\limits_{x \to 1} \frac{f(xa)-f(a)}{x-1}$; g) $\lim\limits_{n \to \infty} \left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)} \right)^n$, $f(a) \neq 0$.

Exercise 10.5. Prove that f is continuous at a point a if $f'_-(a)$ and $f'_+(a)$ exist.

Theorem 10.3 (Differentiation rules)**.** *Let functions $f, g : A \to \mathbb{R}$ have derivatives $f'(a)$ and $g'(a)$ at a point a. Then*

1) for each $c \in \mathbb{R}$ the function cf has a derivative at a and $(cf)'(a) = cf'(a)$;

2) the function $f + g$ has a derivative at a and $(f + g)'(a) = f'(a) + g'(a)$;

3) the function $f \cdot g$ has a derivative at a and $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$;

4) if additionally $g(a) \neq 0$, then the function $\frac{f}{g}$ has a derivative at a and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2}$.

Proof. Proof of 2). By the definition of derivative and Theorem 8.1 b), we have

$$(f+g)'(a) \overset{\text{Def. } 10.1}{=} \lim_{x\to a} \frac{f(x)+g(x)-(f(a)+g(a))}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right)$$
$$\overset{\text{Th. } 8.1}{=} \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} \overset{\text{Def. } 10.1}{=} f'(a) + g'(a).$$

Proof of 3). We compute

$$(f\cdot g)'(a) \overset{\text{Def. } 10.1}{=} \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}$$
$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}\right) \overset{\text{Th. } 8.1}{=} \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a} g(x)$$
$$+ f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a} \overset{(\text{Def. } 10.1+\text{Th. } 10.2)}{=} f'(a)g(a)+f(a)g'(a).$$

Proof of 4). Since $g(a) \neq 0$, we have that $g(x) \neq 0$ in some neighbourhood of the point a, by theorems 10.2 and 8.2. Thus,

$$\frac{1}{x-a}\left(\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}\right) = \frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{g(x)g(a)(x-a)}$$
$$= \frac{1}{g(x)g(a)}\left(\frac{f(x)-f(a)}{x-a}g(a) - f(a)\frac{g(x)-g(a)}{x-a}\right).$$

Thus, the needed equality follows from the latter relation and theorems 10.2 and 8.1. □

Theorem 10.4 (Chain rule)**.** *Let a function $f : A \to \mathbb{R}$ have a derivative $f'(a)$ at a point $a \in A$. Let $f(A) \subset B$ and a function $g : B \to \mathbb{R}$ have a derivative $g'(b)$ at the point $b = f(a)$. Then the composition $g \circ f = g(f)$ has a derivative at the point a and*

$$(g\circ f)'(a) = (g(f))'(a) = g'(f(a))f'(a).$$

Proof. By Theorem 10.1,

$$g(y) - g(b) = g'(b)(y-b) + \varphi(y)(y-b),$$

for some function $\varphi : B \to \mathbb{R}$ satisfying $\varphi(y) \to 0$, $y \to b$. Taking $y := f(x)$ and dividing the latter equality by $x - a$, we obtain

$$\frac{g(f(x))-g(f(a))}{x-a} = g'(b)\frac{f(x)-f(a)}{x-a} + \varphi(f(x))\frac{f(x)-f(a)}{x-a}$$

By Theorem 10.2, $f(x) \to f(a) = b$, $x \to a$, and, consequently, $\varphi(f(x)) \to 0$, $x \to a$. So,

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = g'(b)f'(a) + 0f'(a) = g'(f(a))f'(a).$$

□

Example 10.6. Let $\alpha \in \mathbb{R}$. Then $(x^\alpha)' = \alpha x^{\alpha-1}$, $x > 0$.

Indeed, using Remark 10.1 and Theorem 8.8, we obtain for $x > 0$

$$\frac{(x+h)^\alpha - x^\alpha}{h} = x^{\alpha-1} \frac{\left(1 + \frac{h}{x}\right)^\alpha - 1}{\frac{h}{x}} \to x^{\alpha-1} \cdot \alpha, \quad h \to 0.$$

Exercise 10.6. a) Let $n \in \mathbb{N}$. Show that $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$.
b) Let $m \in \mathbb{Z}$. Show that $(x^m)' = mx^{m-1}$ for all $x \in \mathbb{R} \setminus \{0\}$.

Example 10.7. Let $a > 0$. Then $(a^x)' = a^x \ln a$ for all $x \in \mathbb{R}$. In particular, if $a = e$, then $(e^x)' = e^x$ for all $x \in \mathbb{R}$.

Indeed, using Remark 10.1 and Theorem 8.7, we have for $x \in \mathbb{R}$

$$\frac{a^{x+h} - a^x}{h} = a^x \frac{a^h - 1}{h} \to a^x \cdot \ln a, \quad h \to 0.$$

Example 10.8. a) $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ for all $x \in \mathbb{R}$;
b) $(\tan x)' = \frac{1}{\cos^2 x}$ for all $x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi k : \ k \in \mathbb{Z}\right\}$;
c) $(\cot x)' = -\frac{1}{\sin^2 x}$ for all $x \in \mathbb{R} \setminus \{\pi k : \ k \in \mathbb{Z}\}$.

Let us check the equalities in a). For every $x \in \mathbb{R}$ we have

$$\frac{\sin(x+h) - \sin x}{h} = \frac{2}{h} \sin\frac{h}{2} \cos\left(x + \frac{h}{2}\right) = \frac{\sin\frac{h}{2}}{\frac{h}{2}} \cos\left(x + \frac{h}{2}\right) \to \cos x, \quad h \to 0.$$

Thus, $(\sin x)' = \cos x$, $x \in \mathbb{R}$.

Similarly,

$$\frac{\cos(x+h) - \cos x}{h} = -\frac{2}{h} \sin\frac{h}{2} \sin\left(x + \frac{h}{2}\right) = -\frac{\sin\frac{h}{2}}{\frac{h}{2}} \sin\left(x + \frac{h}{2}\right) \to -\sin x, \quad h \to 0.$$

Hence, $(\cos x)' = -\sin x$, $x \in \mathbb{R}$.

In order to compute $(\tan x)'$, we will use Theorem 10.3 4). So, for every $x \in \mathbb{R}$ such that $\cos x \neq 0$ we have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

and for every $x \in \mathbb{R}$ such that $\sin x \neq 0$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \cdot \sin x - \cos x \cdot (\sin x)'}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Exercise 10.7. Compute derivatives of the following functions:
a) $y = \frac{2x}{1-x^2}$; b) $y = \sqrt[3]{\frac{1+x^2}{1-x^2}}$; c) $y = e^{-x^2 \sin x}$; d) $y = \frac{\sin^2 x}{\sin x^2}$; e) $y = e^x \left(1 + \cot\frac{x}{2}\right)$.

Exercise 10.8. Let $f(x) = x^2$, $x \leq 1$, and $f(x) = ax + b$, $x > 1$. For which $a, b \in \mathbb{R}$ the function f: a) is continuous on $\mathbb{R}$; b) is differentiable on $\mathbb{R}$? Compute f'.

Exercise 10.9. Show that
a) $(\sinh x)' = \cosh x$, $x \in \mathbb{R}$; b) $(\cosh x)' = \sinh x$, $x \in \mathbb{R}$;
c) $(\tanh x)' = \frac{1}{\cosh^2 x}$, $x \in \mathbb{R}$; d) $(\coth x)' = -\frac{1}{\sinh^2 x}$, $x \in \mathbb{R} \setminus \{0\}$.

Exercise 10.10. Let $f(x) = \frac{1}{x^3} e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f'(0) = 0$.

11 Lecture 11 – Derivatives of Inverse Functions and some Theorems

11.1 Derivative of Inverse Function

Theorem 11.1 (Differentiation of inverse function)**.** *Let $-\infty \le a < b \le +\infty$ and a function $f : (a, b) \to \mathbb{R}$ satisfy the following properties*

1) f is continuous on (a, b);

2) f strictly increases on (a, b).

Let $(c, d) := f((a, b)) = \{f(x) : x \in (a, b)\}$, where $-\infty \le c < d \le +\infty$. Let also $g : (c, d) \to (a, b)$ be the inverse function to f.

If there exists a derivative $f'(x_0) \ne 0$ at a point $x_0 \in (a, b)$, then the function g has a derivative $g'(y_0)$ at the point $y_0 = f(x_0)$. Moreover,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Remark 11.1. If a function $f : (a, b) \to \mathbb{R}$ is continuous and strictly increasing, then, by Theorem 8.5, the range $f((a, b))$ of f is an interval and there exists the inverse function g to f which is also continuous and strictly increasing.

Proof of Theorem 11.1. Since the function g is strictly increasing (see Remark 11.1), we have that $g(y) \ne g(y_0)$ for $y \ne y_0$. Using the definition of inverse function and Theorem 10.1, we obtain

$$y - y_0 = f(g(y)) - f(g(y_0)) = f'(g(y_0))(g(y) - g(y_0)) + \varphi(g(y))(g(y) - g(y_0)),$$

where $\varphi(g(y)) \to 0$ as $g(y) \to g(y_0)$. Since g is continuous on (c, d), one has $g(y) \to g(y_0)$, $y \to y_0$. Thus, $\varphi(g(y)) \to 0$, $y \to y_0$. Consequently,

$$\begin{aligned}\frac{g(y) - g(y_0)}{y - y_0} &= \frac{g(y) - g(y_0)}{f'(g(y_0))(g(y) - g(y_0)) + \varphi(g(y))(g(y) - g(y_0))} \\ &= \frac{1}{f'(g(y_0)) + \varphi(g(y))} \to \frac{1}{f'(g(y_0))}, \quad y \to y_0.\end{aligned}$$

□

Remark 11.2. Let us assume that, in Theorem 11.1, the function f has a derivative $f'(x) \ne 0$ at each point $x \in (a, b)$. Then for each $y \in (c, d)$ there exists the derivative $g'(y)$ and one can get a relationship between f' and g' using the equalities $g(f(x)) = x$, $x \in (a, b)$, and $f(g(y)) = y$, $y \in (c, d)$. Indeed, by the chain rule (see Theorem 10.4), $g'(f(x))f'(x) = 1$, $x \in (a, b)$, and $f'(g(y))g'(y) = 1$, $y \in (c, d)$.

Example 11.1. Let $\alpha > 0$, $\alpha \ne 1$. Then $(\log_\alpha x)' = \frac{1}{x \ln \alpha}$ for all $x > 0$. In particular, $(\ln x)' = \frac{1}{x}$ for all $x > 0$.

We will consider the case $\alpha > 1$. To compute the derivative $(\log_\alpha x)'$, we are going to use Theorem 11.1. So, we set $f(x) := \alpha^x$, $x \in (a, b) := \mathbb{R}$ and $(c, d) := (0, +\infty)$. Then f is continuous and strictly increasing on $\mathbb{R}$. Moreover, $f'(x) = \alpha^x \ln \alpha \ne 0$, $x \in \mathbb{R}$, by Example 10.7. So, applying

Theorem 11.1, to the function $g(y) = \log_\alpha y$, $y \in (c,d) = (0,+\infty)$, which is inverse to f, we get for $y_0 \in (0,+\infty)$

$$g'(y_0) = (\log_\alpha y_0)' = \frac{1}{f'(x_0)} = \frac{1}{\alpha^{x_0} \ln \alpha} = \frac{1}{y_0 \ln \alpha},$$

where $y_0 = f(x_0) = \alpha^{x_0}$.

Exercise 11.1. Show that $(\log_\alpha x)' = \frac{1}{x \ln \alpha}$, $x > 0$, for $0 < \alpha < 1$.

Example 11.2. For all $x \in \mathbb{R}$ $(\arctan x)' = \frac{1}{1+x^2}$.

Again we are going to use Theorem 11.1. We set $f(x) := \tan x$, $x \in (a,b) := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(c,d) := \mathbb{R}$. By Example 8.5, f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, it is strictly increasing and $f'(x) = \frac{1}{\cos^2 x} \neq 0$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, applying Theorem 11.1 to $g(y) = \arctan y$, $y \in \mathbb{R}$, we have for each $y_0 \in \mathbb{R}$

$$g'(y_0) = (\arctan y_0)' = \frac{1}{f'(x_0)} = \cos^2 x_0 = \frac{1}{1 + \tan^2 x_0} = \frac{1}{1 + y_0^2},$$

where $y_0 = f(x_0) = \tan x_0$.

Exercise 11.2. For all $x \in \mathbb{R}$ $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

Example 11.3. For each $x \in (-1,1)$ $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

We set $f(x) := \sin x$, $x \in (a,b) := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(c,d) := (-1,1)$. By Example 8.5, f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, f is strictly increasing and $f'(x) = \cos x \neq 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by Example 10.8. Thus, applying Theorem 11.1 to the function $g(x) = \arcsin y$, $y \in (-1,1)$, we obtain for $y_0 \in (-1,1)$

$$g'(y_0) = (\arcsin y_0)' = \frac{1}{f'(x_0)} = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1 - \sin^2 x_0}} = \frac{1}{\sqrt{1 - y_0^2}},$$

where $y_0 = f(x_0) = \sin x_0$.

Exercise 11.3. Show that for each $x \in (-1,1)$ $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.

Example 11.4. Compute the derivative of the function $f(x) = x^x$, $x > 0$.

Solution. For $x > 0$ we have $(x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x \ln x}\right)' = e^{x \ln x}(x \ln x)' = x^x \left((x)' \ln x + x(\ln x)'\right) = x^x \left(\ln x + x\frac{1}{x}\right) = x^x (\ln x + 1)$.

11.2 Some Theorems

Theorem 11.2 (Fermat theorem)**.** *Let $f : (a,b) \to \mathbb{R}$, $x_0 \in (a,b)$ and $f(x_0) = \max\limits_{x \in (a,b)} f(x)$ or $f(x_0) = \min\limits_{x \in (a,b)} f(x)$. If f has a derivative at the point x_0, then $f'(x_0) = 0$.*

Proof. We assume that $f(x_0) = \max\limits_{x \in (a,b)} f(x)$. Then for each $x \in (a,b)$ $f(x) \leq f(x_0)$. Thus, by Remark 10.2, we have

$$f'(x_0) = f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

and, similarly,

$$f'(x_0) = f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

This implies that $f'(x_0) = 0$.

The case $f(x_0) = \min\limits_{x \in (a,b)} f(x)$ is similar. □

Remark 11.3. In the Fermat theorem, the assumption $a < x_0 < b$ is essential. Indeed, the statement is not valid for the function $f(x) = x$, $x \in [0, 1]$. In that case, $x_0 = 1$, but $f'(x_0) = 1$.

Theorem 11.3 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ satisfies the following properties*

1) f is continuous on $[a, b]$;

2) for each $x \in (a, b)$ the derivative $f'(x)$ exists;

3) $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If for every $x \in [a, b]$ $f(x) = f(a)$, then f is a constant function. Consequently, $f'(c) = 0$ for all $c \in (a, b)$.

We now assume that

$$\exists x \in [a, b] \quad \text{such that} \quad f(x) \neq f(a). \tag{9}$$

According to the assumption 1) and the 2nd Weierstrass theorem (see Theorem 9.2), there exist $x_*, x^* \in [a, b]$ such that $f(x_*) = \min_{x \in [a,b]} f(x)$ and $f(x^*) = \max_{x \in [a,b]} f(x)$. Using assumptions (9) and 3), we have that $f(x_*) \neq f(a)$ or $f(x^*) \neq f(a)$. We consider the case $f(x^*) \neq f(a)$. In this case, we have $x^* \neq a$ and $x^* \neq b$, which implies that $x^* \in (a, b)$. Hence, the function f and the point $x_0 = x^*$ satisfy all assumptions of the Fermat theorem (see Theorem 11.2). Consequently, $f'(x^*) = 0$. We take $c := x^*$.

The case $f(x_*) \neq f(a)$ can be considered similarly. □

Exercise 11.4. Let a function $f \in \mathrm{C}([a, b])$ have the derivative $f'(x) \neq 0$ for all $x \in (a, b)$. Then $f(a) \neq f(b)$.

Theorem 11.4 (Lagrange (mean value) theorem)**.** *We assume that a function $f : [a, b] \to \mathbb{R}$ satisfies the following properties*

1) f is continuous on $[a, b]$;

2) f is differentiable on (a, b), that is, f has a derivative $f'(x)$ for all $x \in (a, b)$.

Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof. We take

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a), \quad x \in [a, b],$$

and note that g satisfies the assumptions of Rolle's theorem (see Theorem 11.3). Moreover,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \quad x \in (a, b).$$

By Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. It implies that $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Consequently, $f(b) - f(a) = f'(c)(b - a)$. □

Exercise 11.5. Let a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

Theorem 11.5 (Cauchy theorem). *Let functions $f, g : [a, b] \to \mathbb{R}$ satisfy the following conditions*

1) f, g are continuous on $[a, b]$;

2) f, g are differentiable on (a, b);

3) for every $x \in (a, b)$ $g'(x) \neq 0$.

Then there exists $c \in (a, b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof. We first note that $g(a) \neq g(b)$. Otherwise, if $g(a) = g(b)$, then there exists $c \in (a, b)$ such that $g'(c) = 0$, by Rolle's theorem. But this contradicts assumption 3).

So, we can set

$$h(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)), \quad x \in [a, b].$$

Then the function h satisfies the assumptions of Rolle's theorem. Consequently, there exists $c \in (a, b)$ such that $h'(c) = 0$. Thus, $f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$. $\square$

12 Lecture 12 – Application of Derivatives

12.1 Applications of Lagrange Theorem

Corollary 12.1. *Let a function $f : (a, b) \to \mathbb{R}$ have the derivative f' on (a, b) and for each $x \in (a, b)$ $f'(x) = 0$. Then there exists $L \in \mathbb{R}$ such that $f(x) = L$ for all $x \in (a, b)$.*

Proof. Let $x_0 \in (a, b)$ be an arbitrary fixed point and $x \neq x_0$. Applying the Lagrange theorem to the interval with the ends x_0 and x, we obtain

$$f(x) - f(x_0) = f'(c)(x - x_0) = 0.$$

Thus, we can set $L := f(x_0)$. □

Corollary 12.2. *Let functions $f, g : (a, b) \to \mathbb{R}$ have the derivatives f', g' on (a, b) and for each $x \in (a, b)$ $f'(x) = g'(x)$. Then there exists $L \in \mathbb{R}$ such that $f(x) = g(x) + L$ for all $x \in (a, b)$.*

Proof. Applying Corollary 12.1 to the function $f - g$, we obtain that there exists a constant L such that $f(x) - g(x) = L$, $x \in (a, b)$. □

Corollary 12.3. *Let a function $f : (a, b) \to \mathbb{R}$ have the derivative f' on (a, b) and for each $x \in (a, b)$ $f'(x) = M$, where M is some real number. Then there exists $L \in \mathbb{R}$ such that $f(x) = Mx + L$ for all $x \in (a, b)$.*

Proof. Applying Corollary 12.2 to the functions f and $g(x) = Mx$, $x \in (a, b)$, we obtain the statement. □

Exercise 12.1. Let a, b be a fixed numbers. Identify all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = ax + b$, $x \in \mathbb{R}$.

Exercise 12.2. Identify all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$, $x \in \mathbb{R}$.
(*Hint:* Note that $(f(x)e^{-x})' = (f'(x) - f(x))e^{-x}$, $x \in \mathbb{R}$)

Exercise 12.3. Let functions $f, g : (a, b) \to (0, +\infty)$ be differentiable on (a, b) and for every $x \in (a, b)$ $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$. Prove that there exists $L > 0$ such that $f(x) = Lg(x)$ for all $x \in (a, b)$.
(*Hint:* Consider the functions $\ln f$ and $\ln g$)

12.2 Proofs of Inequalities

In this section, we are going to prove a couple of inequalities which are often used in mathematics.

Example 12.1. We prove that for all $x_1, x_2 \in \mathbb{R}$
a) $|\sin x_1 - \sin x_2| \leq |x_1 - x_2|$; b) $|\cos x_1 - \cos x_2| \leq |x_1 - x_2|$; c) $|\arctan x_1 - \arctan x_2| \leq |x_1 - x_2|$.

The proof of these inequalities are similar. So, we will prove only a). We assume that $x_1 < x_2$. Then applying the Lagrange theorem to the function $f(x) = \sin x$, $x \in [x_1, x_2]$, we have that there exists $c \in (x_1, x_2)$ such that

$$|\sin x_2 - \sin x_1| = |\cos c| \cdot |x_2 - x_1| \leq |x_2 - x_1|,$$

since $|\cos c| \leq 1$.

Exercise 12.4. Prove b) and c) in Example 12.1.

Exercise 12.5. Prove that
a) $|\sqrt{x_1} - \sqrt{x_2}| \le \frac{1}{2}|x_1 - x_2|$ for all $x_1, x_2 \in [1, +\infty)$;
b) $|\sqrt{u^2+v^2} - \sqrt{u^2+w^2}| \le |v-w|$ for all $u, v, w \in \mathbb{R}$. (*Hint:* Consider the function $f(t) = \sqrt{u^2+t^2}$, $t \in \mathbb{R}$)

Example 12.2. We prove that
a) $e^x \ge 1 + x$ for all $x \in \mathbb{R}$, where $e^x = 1 + x$ only if $x = 0$; b) $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$.

We prove a). We first assume that $x > 0$. Then applying the Lagrange theorem to the function $f(u) = e^u$, $u \in [0, x]$, we obtain that there exists $c \in (0, x)$ such that $e^x - e^0 = e^c \cdot (x - 0)$. Since $e^c > 1$ for $c > 0$, we obtain $e^x - 1 > x$ for all $x > 0$. Next let $x < 0$. Then we can apply the Lagrange theorem to the function $f(u) = e^u$, $u \in [x, 0]$. So, we obtain that there exists $c \in (x, 0)$ such that $e^0 - e^x = e^c \cdot (0 - x)$. Since $e^c < 1$ for $c < 0$, we get $1 - e^x < -x$.

In order to prove b), we apply the Cauchy theorem to the functions $f(u) = e^u$, $g(u) = 1 + u + \frac{u^2}{2}$, $u \in [0, x]$. Hence, there exists $c \in (0, x)$ such that

$$\frac{e^x - e^0}{1 + x + \frac{x^2}{2} - 1} = \frac{e^c}{1 + c},$$

Using a), we have $e^x - 1 > x + \frac{x^2}{2}$.

Exercise 12.6. Prove that $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ for all $x > 0$ and $n \in \mathbb{N}$.
(*Hint:* Use Example 12.2 and mathematical induction)

Exercise 12.7. Prove that $\frac{x}{1+x} \le \ln(1 + x) \le x$ for all $x > -1$.

Exercise 12.8 (Generalised Bernoulli inequality)**.** For each $\alpha > 1$, prove that $(1 + x)^\alpha \ge 1 + \alpha x$ for all $x > -1$. Moreover, $(1 + x)^\alpha = 1 + \alpha x$ iff $x = 0$.

Exercise 12.9. Prove that
a) $x - \frac{x^3}{3!} \le \sin x \le x$ for all $x \ge 0$;
b) $1 - \frac{x^2}{2} \le \cos x \le 1$ for all $x \ge 0$.

12.3 Investigation of Monotonicity of Functions

Theorem 12.1. *Let $-\infty \le a < b \le +\infty$ and a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b).*

(i) The function f increases on (a, b) iff $f'(x) \ge 0$ for all $x \in (a, b)$.

(ii) The function f decreases on (a, b) iff $f'(x) \le 0$ for all $x \in (a, b)$.

Proof. We prove (i). Let first $f'(x) \ge 0$ for all $x \in (a, b)$. We take $x_1, x_2 \in (a, b)$ and $x_1 < x_2$. Then applying the Lagrange theorem to the function f on the interval $[x_1, x_2]$, we have that there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \ge 0. \tag{10}$$

Next, let f increases on (a, b). Then for each $x_0 \in (a, b)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \ge 0.$$

Here we used the definition of derivative, Remark 10.2 and the fact that $f(x) \ge f(x_0)$ for $x > x_0$.

In order to prove (ii), apply (i) of the theorem to the function $g(x) = -f(x)$, $x \in (a, b)$. □

Remark 12.1. a) If $f'(x) > 0$ for all $x \in (a,b)$, then the function f is strictly increasing.

b) If $f'(x) < 0$ for all $x \in (a,b)$, then the function f is strictly decreasing.
Indeed, a) immediately follows from (10), where we have the strict inequality.

We note that the inverse statements of Remark 12.1 is not valid. Indeed, the function $f(x) = x^3$, $x \in \mathbb{R}$, strictly increases but its derivative $f'(x) = 3x^2$, $x \in \mathbb{R}$, equals 0 at $x = 0$.

We formulate more general statement about strictly monotone functions.

Theorem 12.2. *Let $-\infty \leq a < b \leq +\infty$ and a function $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b).*

(i) The function f strictly increases on (a,b) iff $f'(x) \geq 0$ for all $x \in (a,b)$ and there exists no interval $(\alpha,\beta) \subset (a,b)$ such that $f'(x) = 0$ for all $x \in (\alpha,\beta)$.

(ii) The function f strictly decreases on (a,b) iff $f'(x) \leq 0$ for all $x \in (a,b)$ and there exists no interval $(\alpha,\beta) \subset (a,b)$ such that $f'(x) = 0$ for all $x \in (\alpha,\beta)$.

Example 12.3. By Theorem 12.2, the function $f(x) = x^2 + bx + c$, $x \in \mathbb{R}$, strictly decreases on $\left(-\infty, -\frac{b}{2}\right]$ and strictly increases on $\left[-\frac{b}{2}, +\infty\right)$, since $f'(x) = 2x + b < 0$ for $x < -\frac{b}{2}$ and $f'(x) = 2x + b > 0$ for $x > -\frac{b}{2}$

Example 12.4. By Theorem 12.2, the function $f(x) = e^x$, $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$, since $f'(x) = e^x > 0$, $x \in \mathbb{R}$.

Example 12.5. By Theorem 12.2, the function $f(x) = x + \sin x$, $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$, since $f'(x) = 1 + \cos x > 0$ for all $x \in \mathbb{R} \setminus \{x : \cos x = -1\} = \mathbb{R} \setminus \{(2k+1)\pi : k \in \mathbb{Z}\}$.

Example 12.6. The function $f(x) = \frac{\ln x}{x}$, $x > 0$, strictly increases on $(0, e]$ and strictly decreases on $[e, +\infty)$ according to Theorem 12.2. Indeed, its derivative $f'(x) = \frac{1-\ln x}{x^2}$, $x > 0$, is strictly positive on $(0, e)$ and strictly negative on $(e, +\infty)$.

Example 12.7. The function $f(x) = x^x$, $x > 0$, is strictly increasing on $\left[\frac{1}{e}, +\infty\right)$ and strictly decreasing on $\left(-\infty, \frac{1}{e}\right]$ according to Theorem 12.2. Indeed, its derivative $f'(x) = x^x(1 + \ln x)$, $x > 0$, is strictly positive on $\left(\frac{1}{e}, +\infty\right)$ and strictly negative on $\left(-\infty, \frac{1}{e}\right)$. For the computation of the derivative see Example 11.4.

Exercise 12.10. Identify intervals on which the following functions are monotone.
a) $f(x) = x^2 - x$, $x \in \mathbb{R}$; b) $f(x) = \frac{x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{1}{x^3} - \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$;
d) $f(x) = x + \sqrt{|1 - x^2|}$, $x \in \mathbb{R}$.

Exercise 12.11. Identify $a \in \mathbb{R}$ for which the function $f(x) = x + a \sin x$, $x \in \mathbb{R}$, is increasing on $\mathbb{R}$.

13 Lecture 13 – L'Hospital's Rule and Taylor's Theorem

13.1 L'Hospital's Rule

Theorem 13.1 (L'Hospital's Rule)**.** *Let $a \in \mathbb{R}$ or $a = -\infty$ and functions $f, g : (a, b) \to \mathbb{R}$ satisfy the following properties*

1) f, g are differentiable on (a, b);

2) $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$ or $\lim\limits_{x\to a+} |g(x)| = +\infty$;

3) $g'(x) \neq 0$ for all $x \in (a, b)$;

4) there exists $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} =: L \in \mathbb{R}$.

Then there exists $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = L$.

Proof. We will only give a proof for the case $a \in \mathbb{R}$ and $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$. For the general case see e.g. [1, p.242-244].

We first extend the functions f and g to the interval $[a, b)$, setting $f(a) = g(a) := 0$. According to assumption 2), f and g are continuous at the point a. Since f, g are differentiable on (a, b), they are continuous also at each point of (a, b), by Theorem 10.2. Thus, f, g are continuous on $[a, b)$. Next, we note that $g(x) \neq 0$ for all $x \in (a, b)$. Indeed, if $g(x_0) = 0$ for some $x_0 \in (a, b)$, then applying Rolle's theorem (see Theorem 11.3) to the function $g : [a, x_0] \to \mathbb{R}$, we obtain that there exists $c \in (a, x_0)$ such that $g'(c) = 0$, that is impossible by assumption 3).

Next, to show that $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = L$, we are going to use Theorem 7.7. Let $\varepsilon > 0$ be fixed. By Theorem 7.7 and assumption 4),

$$\exists \delta > 0 \;\; \forall x \in (a, a + \delta) : \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Applying the Cauchy theorem (see Theorem 11.5) to the functions $f, g : [a, x] \to \mathbb{R}$, we have for all $x \in (a, a + \delta)$

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon,$$

where $c \in (a, x) \subset (a, a + \delta)$. □

Remark 13.1. A similar statement is true for the left-sided limit as x goes to b.

Example 13.1. Using L'Hospital's Rule, we compute the following limits:

a) $\lim\limits_{x\to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$;

b) $\lim\limits_{x\to 0+} x \ln x \overset{0\cdot\infty}{=} \lim\limits_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to 0+} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim\limits_{x\to 0+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = -\lim\limits_{x\to 0+} x = 0$;

c) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}} \overset{1^\infty}{=} \lim\limits_{x\to 0} e^{\frac{1}{x^2}\ln\cos x}$.
We compute

$$\lim_{x\to 0} \frac{1}{x^2} \ln\cos x = \lim_{x\to 0} \frac{\ln\cos x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{(\ln\cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{-\frac{\sin x}{\cos x}}{2x} = -\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x\cos x} = -\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = -\frac{1}{2}.$$

Thus, by the continuity of the function $f(x) = e^x$, $x \in \mathbb{R}$, we have
$\lim\limits_{x\to 0} e^{\frac{1}{x^2}\ln\cos x} = e^{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\cos x} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

See [1, p.245-248] for more examples of the application of L'Hospital's Rule.

Exercise 13.1. Using L'Hospital's Rule, show that
a) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = 1$; b) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{\sin x} = 1$; c) $\lim\limits_{x\to e}\frac{(\ln x)^\alpha - \left(\frac{x}{e}\right)^\beta}{x-e} = \frac{\alpha-\beta}{e}$, where α, β are some real numbers;
d) $\lim\limits_{x\to 1}\frac{\left(\frac{4}{\pi}\arctan x\right)^\alpha - 1}{\ln x} = \frac{2\alpha}{\pi}$, $\alpha \in \mathbb{R}$; e) $\lim\limits_{x\to 0+}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = e^{-\frac{1}{2}}$; f) $\lim\limits_{x\to+\infty}\frac{x}{2^x} = 0$;
g) $\lim\limits_{x\to+\infty}\frac{\ln x}{x^\varepsilon} = 0$ for all $\varepsilon > 0$; h) $\lim\limits_{x\to+0} x^\varepsilon \ln x = 0$ for all $\varepsilon > 0$; i) $\lim\limits_{x\to+0}(\ln(1+x))^x = 1$.

Exercise 13.2. Compute the following limits:
a) $\lim\limits_{x\to 0}\frac{\ln(1+x)-x}{x^2}$; b) $\lim\limits_{x\to 0}\frac{e^x - e^{\sin x}}{x-\sin x}$; c) $\lim\limits_{x\to+\infty}\left(x\left(\frac{\pi}{2} - \arctan x\right)\right)$; d) $\lim\limits_{x\to+\infty}\frac{\ln(x+1)-\ln(x-1)}{\sqrt{x^2+1}-\sqrt{x^2-1}}$;
e) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x}\right)^x$; f) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x^2}\right)^x$; g) $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e}{x}$; h) $\lim\limits_{x\to+\infty}\frac{x^{\ln x}}{(\ln x)^x}$.

13.2 Higher Order Derivatives

We assume that a function $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b). We denote its derivative f' by g, that is $g(x) = f'(x)$, $x \in (a,b)$.

Definition 13.1. If there exists a derivative $g'(x_0)$ of the function g at a point x_0, then this derivative is called the **second derivative of f at the point** x_0 and is denoted by $f''(x_0)$ or $\frac{d^2 f}{dx^2}(x_0)$.

Let the n-th derivative $f^{(n)}$ be defined on (a,b). Then the $(n+1)$**-th derivative of f at** $x_0 \in (a,b)$ is defined as $f^{(n+1)}(x_0) = \frac{d(f^{(n)})}{dx}(x_0)$, if it exists.

Example 13.2. Let $a > 0$. Then for each $x \in \mathbb{R}$ we obtain $(a^x)' = a^x \ln a$, $(a^x)'' = a^x \ln^2 a$, $(a^x)''' = a^x \ln^3 a$, ..., $(a^x)^{(n)} = a^x \ln^n a$. In particular, $(e^x)^{(n)} = e^x$, $x \in \mathbb{R}$.

Exercise 13.3. Let $\alpha \in \mathbb{R}$. Show that $(x^\alpha)^n = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)x^{\alpha-n}$ for all $x > 0$ and $n \in \mathbb{N}$.

Example 13.3. Let $\alpha \in \mathbb{R}$. Then $((1+x)^\alpha)^{(n)} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)(1+x)^{\alpha-n}$ for all $x > -1$ and $n \in \mathbb{N}$.

Indeed, $((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1}$, $((1+x)^\alpha)'' = \left(\alpha(1+x)^{\alpha-1}\right)' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ and so on.

Exercise 13.4. Show that $(\ln(1+x))^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all $x > -1$ and $n \in \mathbb{N}$.

Example 13.4. For each $x \in \mathbb{R}$ $(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$ and $(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$.

Indeed, $(\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$, $(\sin x)'' = (\cos x)' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right)$, $(\sin x)''' = (-\sin x)' = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right)$ and so on. The same computation for $(\cos x)^{(n)}$.

Exercise 13.5. Compute the n-th derivative of the following functions:
a) $f(x) = 2^{x-1}$, $x \in \mathbb{R}$; b) $f(x) = \sqrt{1+x}$, $x > -1$; c) $f(x) = \arctan x$, $x \in \mathbb{R}$.

Theorem 13.2. *Let functions $f, g : (a,b) \to \mathbb{R}$ have n-th derivatives on (a,b). Then the following equalities are true.*

1) for all $k \in \{1, \ldots, n\}$ $(f^{(n-k)})^{(k)} = (f^{(k)})^{(n-k)} = f^{(n)}$, *where* $f^{(0)} = f$;

2) for all $c \in \mathbb{R}$ $(cf)^{(n)} = cf^{(n)}$;

3) $(f+g)^{(n)} = f^{(n)} + g^{(n)}$.

Theorem 13.3 (Leibniz Formula)**.** *For a number* $n \in \mathbb{N}$ *let* $g, f : (a,b) \to \mathbb{R}$ *have* n*-th derivatives on* (a,b)*. Then* $f \cdot g$ *has the* n*-th derivative on* (a,b) *and*

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

Exercise 13.6. Compute the following derivatives:
a) $(x^2 e^x)^{(n)}$, $x \in \mathbb{R}$; b) $(x^3 \sin x)^{(n)}$, $x \in \mathbb{R}$; c) $(x^n \ln x)^{(n)}$, $x > 0$.

13.3 Taylor's Formula

13.3.1 Taylor's Formula for a Polynomial

Let $n \in \mathbb{N}$ and $\{a_0, a_1, a_2, \ldots, a_n\} \subset \mathbb{R}$. For any point $x_0 \in \mathbb{R}$ a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \quad x \in \mathbb{R},$$

can be written in the form

$$P(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \ldots + b_n(x - x_0)^n, \quad x \in \mathbb{R}, \tag{11}$$

where $\{b_0, b_1, b_2, \ldots, b_n\}$ are some real numbers, which can be computed by the following way. Inserting $x = x_0$ into (11), we obtain $b_0 = P(x_0)$. Next we compute P'. So,

$$P'(x) = b_1 + 2b_2(x - x_0) + 3b_3(x - x_0)^2 + \ldots + nb_n(x - x_0)^{n-1}, \quad x \in \mathbb{R}. \tag{12}$$

Inserting $x = x_0$ into (12), we get $b_1 = P'(x_0)$. Next, we compute the second derivative of P

$$P''(x) = 2b_2 + 3 \cdot 2 \cdot b_3(x - x_0) + \ldots + n(n-1)b_n(x - x_0)^{n-2}, \quad x \in \mathbb{R}. \tag{13}$$

Inserting $x = x_0$ into (13), we obtain $b_2 = \frac{P''(x_0)}{2}$. Similarly, we obtain

$$b_k = \frac{P^{(k)}(x_0)}{k!}, \quad k \geq 0.$$

Thus, for each $x \in \mathbb{R}$

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x - x_0) + \frac{P''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{14}$$

We see that any polynomial can be completely defined only by its value and values of its derivatives at a point x_0. Formula (14) does not hold if P is not a polynomial, but it turns out that values of a function are close to the right hand side of (14) if x is close to x_0.

13.3.2 Taylor's Formula with Peano Remainder Term

Let $f, g : A \to \mathbb{R}$ be some functions and x_0 be a limit point of A. If $\frac{f(x)}{g(x)} \to 0$, $x \to x_0$, then we will write $f(x) = o(g(x))$, $x \to 0$, or $f = o(g)$, $x \to x_0$.

Exercise 13.7. Show that
a) $x = o(1)$, $x \to 0$; b) $x^3 = o(2^x)$, $x \to +\infty$; c) $\ln x = o(\sqrt{x})$, $x \to +\infty$; d) $x - \sin x = o(x)$, $x \to 0$.

Theorem 13.4. *Let $n \in \mathbb{N}$ and let a function $f : (a,b) \to \mathbb{R}$ and a point $x_0 \in (a,b)$ satisfy the following conditions:*

1) there exists $f^{(n-1)}(x)$ for all $x \in (a,b)$;

2) there exists $f^{(n)}(x_0)$.

Then
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o((x-x_0)^n), \quad x \to x_0. \tag{15}$$
The term $o((x-x_0)^n)$ is called the **Peano remainder term**.

Proof. We recall that $0! = 1$ and set
$$R_n(x) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \quad x \in (a,b).$$
According to assumptions 1) and 2), there exists $R^{(n-1)}(x)$ for all $x \in (a,b)$ and $R^{(n)}(x_0)$. Moreover it is easy to see that
$$R_n(x_0) = R_n'(x_0) = R_n''(x_0) = \ldots = R_n^{(n)}(x_0) = 0.$$
Assuming $x > x_0$ and applying the Lagrange theorem (see Theorem 11.4), we have
$$\begin{aligned}\left|\frac{R_n(x)}{(x-x_0)^n}\right| &= \left|\frac{R_n(x) - R_n(x_0)}{(x-x_0)^n}\right| = \left|\frac{R_n'(c_1)(x-x_0)}{(x-x_0)^n}\right| = \left|\frac{R_n'(c_1) - R_n'(x_0)}{(x-x_0)^{n-1}}\right| \\ &= \left|\frac{R_n''(c_2)(c_1-x_0)}{(x-x_0)^{n-1}}\right| \le \left|\frac{R_n''(c_2)}{(x-x_0)^{n-2}}\right| = \left|\frac{R_n''(c_2) - R_n''(x_0)}{(x-x_0)^{n-2}}\right| = \left|\frac{R_n'''(c_3)(c_2-x_0)}{(x-x_0)^{n-2}}\right| \le \ldots \\ &\le \left|\frac{R_n^{(n-1)}(c_{n-1}) - R_n^{(n-1)}(x_0)}{x-x_0}\right| \to \left|R_n^{(n)}(x_0)\right| = 0, \quad x \to x_0+,\end{aligned}$$
where $x_0 < c_{n-1} < c_{n-2} < \ldots < c_2 < c_1 < x$. Moreover $c_{n-1} \to x_0$ as $x \to x_0+$.

One can similarly obtain that $\left|\frac{R_n(x)}{(x-x_0)^n}\right| \to 0$, $x \to x_0-$. Consequently,
$$R_n(x) = o((x-x_0)^n), \quad x \to x_0,$$
by Theorem 7.8. □

Example 13.5. For every $n \in \mathbb{N}$
$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + o(x^n), \quad x \to 0.$$
The formula follows from Theorem 13.4 applying to $f(x) = e^x$, $x \in \mathbb{R}$, and the fact that $f^{(k)}(0) = e^0 = 1$ (see Example 13.2).

Example 13.6. For all $n \in \mathbb{N}$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n), \quad x \to 0.$$

The formula follows from Theorem 13.4 applying to $f(x) = \ln(1+x)$, $x > -1$, and the fact that $f^{(k)}(0) = \frac{(-1)^{k-1}(k-1)!}{(1+0)^k} = (-1)^{k-1}(k-1)!$ (see Example 13.4).

Example 13.7. For each $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)x^n}{n!} + o(x^n), \quad x \to 0.$$

The formula follows from Theorem 13.4 applying to $f(x) = (1+x)^\alpha$, $x > -1$, and the fact that $f^{(k)}(0) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)(1+0)^{\alpha-k} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)$ (see Example 13.3).

Exercise 13.8. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$$

Exercise 13.9. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0,$$
$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$$

Exercise 13.10. Use Taylor's formula to compute the limits:
a) $\lim\limits_{x\to 0} \frac{e^x - 1 - x}{x^2}$; b) $\lim\limits_{x\to 0} \frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}}$; c) $\lim\limits_{x\to 0} \frac{\ln(1+x+x^2) + \ln(1-x-x^2)}{x \sin x}$; d) $\lim\limits_{x\to 0} \frac{\cos(xe^x) - \cos(xe^{-x})}{x^3}$.

14 Lecture 14 – Local Extrema of Function

14.1 Taylor's Formula with Lagrangian Remainder Term

Theorem 14.1. *Let $n \in \mathbb{N} \cup \{0\}$ and $f : (a, b) \to \mathbb{R}$. We assume that there exists $f^{(n+1)}(x)$ for all $x \in (a, b)$. Then for each $x, x_0 \in (a, b)$ there exists a point ξ between x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}. \tag{16}$$

The term $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ is called the **Lagrangian remainder term***.*

Proof. If $x = x_0$, then formula (16) holds. We assume that $x_0 < x$ and consider a new function

$$g(z) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x - z)^k - \frac{L}{(n + 1)!}(x - z)^{n+1}, \quad z \in [x_0, x],$$

where the number L is chosen such that $g(x_0) = 0$. We note that the function g is continuous on $[x_0, x]$ and has a derivative

$$g'(z) = -\frac{f^{(n+1)}(z)}{n!}(x - z)^n + \frac{L}{n!}(x - z)^n$$

Moreover, $g(x) = 0$. By Rolle's theorem (see Theorem 11.3), there exists $\xi \in (x_0, x)$ such that $g'(\xi) = 0$, that is,

$$g'(\xi) = -\frac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n + \frac{L}{n!}(x - \xi)^n = 0.$$

Consequently, we have $L = f^{(n+1)}(\xi)$.

The case $x < x_0$ is similar. □

Remark 14.1. Formula (16) is a generalisation of the Lagrange theorem, which can be obtained taking $n = 0$.

Example 14.1. Let $f(x) = e^x$, $x \in \mathbb{R}$, and $x_0 = 0$. Then for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ there exists ξ between 0 and x such that

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^\xi}{(n + 1)!}x^{n+1}. \tag{17}$$

This formula follows from Theorem 14.1 and Example 13.2, since $f^{(k)}(0) = e^0 = 1$.

Remark 14.2. Formula (17) allows to obtain an approximate value of e^x, computing the value of the polynomial $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$. Moreover, the error is equal $\frac{e^\xi}{(n+1)!}x^{n+1}$. For instance, for $x \in [0, 3]$ and $n = 12$ we have

$$\left| \frac{e^\xi}{(n + 1)!}x^{n+1} \right| < \frac{e^3 3^{13}}{13!} < \frac{1}{1000}.$$

14.2 Local Extrema of Function

Let $f : (a, b) \to \mathbb{R}$ be a given function.

Definition 14.1.
- A point x_0 is called a **point of local maximum** (**local minimum**) **of** f, if there exists $\delta > 0$ such that $B(x_0, \delta) = (x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$) for all $x \in B(x_0, \delta)$.

  If x_0 is a point of local minimum or local maximum of f, then it is called a **point of local extrema of** f.

- A point x_0 is called a **point of strict local maximum** (**strict local minimum**) **of** f, if there exists $\delta > 0$ such that $B(x_0, \delta) \subset (a, b)$ and $f(x) < f(x_0)$ (resp. $f(x) > f(x_0)$) for all $x \in B(x_0, \delta) \setminus \{x_0\}$.

  If x_0 is a point of strict local minimum or strict local maximum of f, then it is called a **point of strict local extrema of** f.

Example 14.2. For the function $f(x) = x^2$, $x \in \mathbb{R}$, the point $x_0 = 0$ is a point of strict local minimum of f and f takes the smallest value at this point.

Example 14.3. For the function $f(x) = x$, $x \in [0, 1]$, the points $x_* = 0$ and $x^* = 1$ are points at which the function takes the smallest and the largest values, respectively. But they are not points of local extrema.

Theorem 14.2. *If x_0 is a point of local extrema of f and f has a derivative at x_0, then $f'(x_0) = 0$.*

Proof. Let x_0 be a point of local maximum. Then by Definition 14.1, there exists $\delta > 0$ such that $B(x_0, \delta) \subset (a, b)$ and $f(x) \leq f(x_0)$ for all $x \in B(x_0, \delta)$. In particular, $f(x_0) = \max\limits_{x \in B(x_0, \delta)} f(x)$. Applying the Fermat theorem (see Theorem 11.2) to the function f defined on $(x_0 - \delta, x_0 + \delta)$, we obtain $f'(x_0) = 0$. □

Remark 14.3. Theorem 14.2 gives only a necessary condition of local extrema. If $f'(x_0) = 0$ at some point $x_0 \in (a, b)$, then it does not imply that x_0 is a point of local extrema. For instance, for the function $f(x) = x^3$, $x \in \mathbb{R}$, the point $x_0 = 0$ is not a point of a local extrema while $f'(0) = 0$.

Remark 14.4. A point at which derivative does not exist can also be a point of local extrema. For example, for the function $f(x) = |x|$, $x \in \mathbb{R}$, the point $x_0 = 0$ is a point of local minimum but the derivative at $x_0 = 0$ does not exist (see Example 10.2).

Definition 14.2. A point $x_0 \in (a, b)$ is said to be a **critical point** or **stationary point of** f, if $f'(x_0) = 0$.

Remark 14.5. Point of local extrema of f belong to the set of all critical points of f and points where the derivative of f does not exist.

Theorem 14.3. *Let x_0 be a critical point of f and the function f be differentiable on some neighbourhood of the point x_0.*

a) If for some $\delta > 0$ $f'(x) > 0$ for all $x \in (x_0 - \delta, x_0)$ and $f'(x) < 0$ for all $x \in (x_0, x_0 + \delta)$, then x_0 is a point of strict local maximum of f.

b) If for some $\delta > 0$ $f'(x) < 0$ for all $x \in (x_0 - \delta, x_0)$ and $f'(x) > 0$ for all $x \in (x_0, x_0 + \delta)$, then x_0 is a point of strict local minimum of f.

Proof. We will only prove a). Since $f'(x) > 0$ for all $x \in (x_0 - \delta, x_0)$, the function f strictly increases on $(x_0 - \delta, x_0]$, by Remark 12.1. Hence, $f(x) < f(x_0)$ for all $x \in (x_0 - \delta, x_0)$. Similarly, $f(x_0) > f(x)$ for all $x \in (x_0, x_0 + \delta)$, since the function f strictly decreases on $[x_0, x_0 + \delta)$ due to $f'(x) < 0$, $x \in (x_0, x_0 + \delta)$. Thus, x_0 is a point of strict local maximum. □

Example 14.4. For the function $f(x) = x^3 - 3x$, $x \in \mathbb{R}$, the points 1 and -1 are critical points of f, since the derivative $f(x) = 3x^2 - 3$, $x \in \mathbb{R}$, equals zero at those points. The point -1 is a point of strict local maximum because the derivative changes its sign from "+" to "−", passing through -1. The point 1 is a point of strict local minimum because the derivative changes its sign from "−" to "+", passing through 1.

Exercise 14.1. Find points of local extrema of the following functions:
a) $f(x) = x^2 e^x$, $x \in \mathbb{R}$; b) $f(x) = x + \frac{1}{x}$, $x > 0$; c) $f(x) = x^x$, $x > 0$; d) $f(x) = |x|e^{-x^2}$, $x \in \mathbb{R}$.

Theorem 14.4. *Let a function $f : (a, b) \to \mathbb{R}$ and a point $x_0 \in (a, b)$ satisfy the following properties:*

1) there exists $\delta > 0$ such that f is differentiable on $(x_0 - \delta, x_0 + \delta)$;

2) $f'(x_0) = 0$;

3) there exists $f''(x_0)$ and $f''(x_0) \neq 0$.

If $f''(x_0) < 0$, then x_0 is a point of strict local maximum. If $f''(x_0) > 0$, then x_0 is a point of strict local minimum.

Proof. We write for the function f and the point x_0 the Taylor formula (see Theorem 13.4). So,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + o((x - x_0)^2), \quad x \to x_0.$$

Hence, for $x \neq x_0$ we have

$$f(x) - f(x_0) = (x - x_0)^2 \left(\frac{f''(x_0)}{2!} + \frac{o((x - x_0)^2)}{(x - x_0)^2} \right)$$

and, hence, $f(x) - f(x_0)$ has the same sign as $f''(x_0)$ on some neighbourhood of x_0, since $\frac{o((x-x_0)^2)}{(x-x_0)^2} \to 0$, $x \to x_0$. □

Example 14.5. For the function $f(x) = x^2 - x$, $x \in \mathbb{R}$, the point $\frac{1}{2}$ is a point of strict local minimum, since $f'\left(\frac{1}{2}\right) = 0$ and $f''\left(\frac{1}{2}\right) = 2 < 0$.

Theorem 14.5. *Let $f : (a, b) \to \mathbb{R}$, a point x_0 belong to (a, b) and $m \in \mathbb{N}$, $m \geq 2$. We also assume that the following conditions hold:*

1) there exists $\delta > 0$ such that $f^{(m-1)}(x)$ exists for all $x \in (x_0 - \delta, x_0 + \delta)$;

2) $f'(x_0) = f''(x_0) = \ldots = f^{(m-1)}(x_0) = 0$;

3) there exists $f^{(m)}(x_0)$ and $f^{(m)}(x_0) \neq 0$.

If m is even and $f^{(m)}(x_0) < 0$, then x_0 is a point of local maximum.
If m is even and $f^{(m)}(x_0) > 0$, then x_0 is a point of local minimum.
If m is odd, then x_0 is not a point of local extrema.

Proof. The proof of Theorem 14.5 is similar to the proof of Theorem 14.4. □

Exercise 14.2. Prove Theorem 14.5.

Exercise 14.3. Find points of local extrema of the following functions:

a) $f(x) = x^4(1-x)^3$, $x \in \mathbb{R}$; b) $f(x) = \frac{x^2}{2} - \frac{1}{4} + \frac{9}{4(2x^2+1)}$, $x \in \mathbb{R}$; c) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$, $x \in \mathbb{R}$.

14.3 Convex and Concave Functions

Let $-\infty \leq a < b \leq +\infty$.

Definition 14.3.

- A function $f : (a,b) \to \mathbb{R}$ is said to be a **convex function on** (a,b), if for each $x_1, x_2 \in (a,b)$ and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2).$$

- A function $f : (a,b) \to \mathbb{R}$ is said to be a **concave function on** (a,b), if for each $x_1, x_2 \in (a,b)$ and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2).$$

Definition 14.4.

- A function $f : (a,b) \to \mathbb{R}$ is said to be a **strictly convex function on** (a,b), if for each $x_1, x_2 \in (a,b)$, $x_1 \neq x_2$, and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2).$$

- A function $f : (a,b) \to \mathbb{R}$ is said to be a **strictly concave function on** (a,b), if for each $x_1, x_2 \in (a,b)$, $x_1 \neq x_2$, and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) > \alpha f(x_1) + (1-\alpha)f(x_2).$$

Example 14.6. Let $M, L \in \mathbb{R}$. The function $f(x) = Mx + L$, $x \in \mathbb{R}$, is both convex and concave on $\mathbb{R}$. Indeed, for each $x_1, x_2 \in \mathbb{R}$ and $\alpha \in (0,1)$ we have

$$f(\alpha x_1+(1-\alpha)x_2) = M(\alpha x_1+(1-\alpha)x_2)+L = \alpha(Mx_1+L)+(1-\alpha)(Mx_2+L) = \alpha f(x_1)+(1-\alpha)f(x_2).$$

Example 14.7. The function $f(x) = |x|$, $x \in \mathbb{R}$, is convex on $\mathbb{R}$. Indeed, for each $x_1, x_2 \in (a,b)$ and $\alpha \in (0,1)$

$$f(\alpha x_1 + (1-\alpha)x_2) = |\alpha x_1 + (1-\alpha)x_2| \leq \alpha|x_1| + (1-\alpha)|x_2| = \alpha f(x_1) + (1-\alpha)f(x_2),$$

by the triangular inequality (see Theorem 2.5).

Example 14.8. The function $f(x) = x^2$, $x \in \mathbb{R}$, is strictly convex on $\mathbb{R}$. To prove this, we fix $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, $\alpha \in (0,1)$ and use the inequality $2x_1x_2 < x_1^2 + x_2^2$ which trivially follows from $(x_1 - x_2)^2 > 0$. Thus,

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= (\alpha x_1 + (1-\alpha)x_2)^2 = \alpha^2 x_1^2 + 2\alpha(1-\alpha)x_1x_2 + (1-\alpha)^2 x_2^2 \\ &< \alpha^2 x_1^2 + \alpha(1-\alpha)(x_1^2 + x_2^2) + (1-\alpha)^2 x_2^2 = \alpha x_1^2 + (1-\alpha)x_2^2 = \alpha f(x_1) + (1-\alpha)f(x_2). \end{aligned}$$

Theorem 14.6. *Let a function $f : (a, b) \to \mathbb{R}$ has the derivative $f'(x)$ for all $x \in (a, b)$.*

(i) The function f is convex (strictly convex) on (a, b), if f' increases (strictly increases) on (a, b).

(ii) The function f is concave (strictly concave) on (a, b), if f' decreases (strictly decreases) on (a, b).

Combining theorems 14.6, 12.1 and 12.2 we obtain the following statement.

Theorem 14.7. *Let a function $f : (a, b) \to \mathbb{R}$ have the second derivative $f''(x)$ for all $x \in (a, b)$.*

(i) The function f is convex (concave) on (a, b) iff $f''(x) \geq 0$ (resp. $f''(x) \leq 0$) for all $x \in (a, b)$.

(ii) The function f is strictly convex (strictly concave) on (a, b) iff $f''(x) \geq 0$ (resp. $f''(x) \leq 0$) for all $x \in (a, b)$ and there is no interval $(\alpha, \beta) \subset (a, b)$ such that $f''(x) = 0$ for all $x \in (\alpha, \beta)$.

Exercise 14.4. Identify intervals on which the following functions are convex or concave:
a) $f(x) = e^x$, $x \in \mathbb{R}$; b) $f(x) = \ln x$, $x > 0$; c) $f(x) = \sin x$, $x \in \mathbb{R}$; d) $f(x) = \arctan x$, $x \in \mathbb{R}$;
e) $f(x) = x^\alpha$, $x > 0$, $\alpha \in \mathbb{R}$.

Theorem 14.8 (Jensen's inequality)**.** *Let $f : (a, b) \to \mathbb{R}$ be a convex function. Then for each $n \geq 2$, $x_1, \ldots, x_n \in (a, b)$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$, $\alpha_1 + \ldots + \alpha_n = 1$, the inequality*

$$f(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \ldots + \alpha_n f(x_n) \tag{18}$$

holds.

Proof. We are going to use the mathematical induction to prove the theorem. For $n = 2$ inequality (18) is true due to the convexity of f.

Next, we assume that inequality (18) holds for some $n \geq 2$ and each $x_1, \ldots, x_n \in (a, b)$ and each $\alpha_1, \ldots, \alpha_n \in [0, 1]$, $\alpha_1 + \ldots + \alpha_n = 1$, and prove (18) for $n + 1$ and $x_1, \ldots, x_{n+1} \in (a, b)$, $\alpha_1, \ldots, \alpha_{n+1} \in [0, 1]$, $\alpha_1 + \ldots + \alpha_{n+1} = 1$. We remark that there exists k such that $\alpha_k < 1$. So, let $\alpha_{n+1} < 1$. Then, by Definition 14.3 and the induction assumption,

$$\begin{aligned} f\left(\sum_{k=1}^{n+1} \alpha_k x_k\right) &= f\left(\alpha_{n+1} x_{n+1} + \sum_{k=1}^{n} \alpha_k x_k\right) \leq \alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1}) f\left(\sum_{k=1}^{n} \frac{\alpha_k}{1 - \alpha_{n+1}} x_k\right) \\ &\leq \alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1}) \sum_{k=1}^{n} \frac{\alpha_k}{1 - \alpha_{n+1}} f(x_k) = \sum_{k=1}^{n+1} \alpha_k f(x_k). \end{aligned}$$

□

Example 14.9. The function $f(x) = -\ln x$, $x > 0$, is convex on $(0, +\infty)$, since $f''(x) = \frac{1}{x^2} > 0$, $x > 0$ (see Theorem 14.7 (i)). Applying (18) to f, for each $n \geq 2$, $x_1, \ldots, x_n \in (0, +\infty)$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$, $\alpha_1 + \ldots + \alpha_n = 1$, we have

$$\ln\left(\sum_{k=1}^{n} \alpha_k x_k\right) \geq \sum_{k=1}^{n} \alpha_k \ln x_k.$$

This implies

$$\prod_{k=1}^{n} x_k^{\alpha_k} \leq \sum_{k=1}^{n} \alpha_k x_k \tag{19}$$

for all $n \geq 2$, $x_1, \ldots, x_n \in (0, +\infty)$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$, $\alpha_1 + \ldots + \alpha_n = 1$. In particular, taking $\alpha_1 = \ldots = \alpha_n = \frac{1}{n}$, we get

$$\sqrt[n]{\prod_{k=1}^{n} x_k} \leq \frac{1}{n} \sum_{k=1}^{n} x_k$$

for all $n \geq 2$, $x_1, \ldots, x_n \in (0, +\infty)$, which is the **inequality of arithmetic and geometric means**.

Exercise 14.5 (Young's inequality)**.** Let $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for all $x, y \in (0, +\infty)$.

(*Hint:* Use inequality (19))

15 Lecture 15 – Antiderivative and Indefinite Integral

15.1 Definitions and Elementary Properties

In this section, J denotes one of the following intervals $[a,b]$, $[a,b)$, $(a,b]$, (a,b), $(-\infty,a]$, $(-\infty,a)$, $[b,+\infty)$, $(b,+\infty)$ or $(-\infty,+\infty)$. Moreover, for a function $f:[a,b]\to\mathbb{R}$, we set $f'(a):=f'_+(a)$ and $f'(b):=f'_-(b)$.

Definition 15.1. A function $F:J\to\mathbb{R}$ is said to be an **antiderivative** or a **primitive function** of a function $f:J\to\mathbb{R}$, if for each $x\in J$ there exists $F'(x)$ and $F'(x)=f(x)$.

Example 15.1. An antiderivative of the function $f(x)=x$, $x\in\mathbb{R}$, is the function $F(x)=\frac{1}{2}x^2$, $x\in\mathbb{R}$, since $\left(\frac{1}{2}x^2\right)'=x$ for all $x\in\mathbb{R}$.

The function $G(x)=\frac{1}{2}x^2+1$ is also an antiderivative of f because $\left(\frac{1}{2}x^2+1\right)'=x$ for all $x\in\mathbb{R}$.

Example 15.2. An antiderivative of the function $f(x)=\begin{cases}0, & x<0,\\ x, & x\geq 0,\end{cases}$ is the function $F(x)=\begin{cases}0, & x<0,\\ \frac{x^2}{2}, & x\geq 0.\end{cases}$ Indeed, for each $x<0$, $F'(x)=0$ and for each $x>0$ $F'(x)=x$. Moreover, $F'_-(0)=0$, $F'_+(0)=0$ and, thus, $F'(0)=0$, by Remark 10.2.

Remark 15.1. We note *if f has an antiderivative, then it is not unique.* Indeed, if F is an antiderivative of f, then for any constant $C\in\mathbb{R}$ the function $F+C$ is also an andiderivative of f because for each $x\in J$ $(F(x)+C)'=F'(x)=f(x)$. Moreover, if F and G are antiderivatives of f, then there exists a constant $C\in\mathbb{R}$ such that $F=G+C$, by Corollary 12.2.

Definition 15.2. The indefinite integral of a function $f:J\to\mathbb{R}$ is the expression $F(x)+C$, $x\in J$, where F is an antiderivative of f and C denotes an arbitrary constant. The indefinite integral of a function f is denoted by $\int f(x)dx$, $x\in J$.

Exercise 15.1. Find antiderivatives of the following functions:
a) $f(x)=|x|$, $x\in\mathbb{R}$; b) $f(x)=\max\{1,x^2\}$, $x\in\mathbb{R}$; c) $f(x)=|\sin x|$, $x\in\mathbb{R}$;
d) $f(x)=\sin x+|\sin x|$, $x\in\mathbb{R}$.

Exercise 15.2. Let a function $f:\mathbb{R}\to\mathbb{R}$ has an antiderivative $F:\mathbb{R}\to\mathbb{R}$. Find f, if for each $x\in\mathbb{R}$:
a) $F(x)=f(x)$; b) $F(x)=\frac{1}{2}f(x)$; c) $F(x)=f(x)+1$; d) $2xF(x)=f(x)$.

Theorem 15.1 (Properties of indefinite integral)**.** *Indefinite integral satisfies the following properties:*

1) $\frac{d}{dx}\int f(x)dx=f(x)$, $x\in J$;

2) $\int f'(x)dx=f(x)+C$, $x\in J$;

3) $\int(af(x))dx=a\int f(x)dx$, $x\in J$, *for all* $a\in\mathbb{R}$, $a\neq 0$;

4) $\int(f(x)+g(x))dx=\int f(x)dx+\int g(x)dx$, $x\in J$.

From definitions 15.2 and 15.1 we have that $F'(x)=f(x)$, $x\in J$, provided $\int f(x)dx=F(x)+C$, $x\in J$. Using this relationship, we can get the following list of important indefinite integrals.

- $\int x^\alpha dx=\frac{x^{\alpha+1}}{\alpha+1}+C$, $x\in(0,+\infty)$, for all $\alpha\in\mathbb{R}\setminus\{-1\}$;
  $\int x^n dx=\frac{x^n}{n+1}+C$, $x\in\mathbb{R}$, for all $n\in\mathbb{N}\cup\{0\}$;

- $\int \frac{1}{x}dx = \ln|x| + C$ on each interval $(-\infty, 0)$ and $(0, +\infty)$;
- $\int a^x dx = \frac{a^x}{\ln a} + C$, $x \in \mathbb{R}$ for all $a > 0$, $a \neq 1$;
- $\int e^x dx = e^x + C$, $x \in \mathbb{R}$;
- $\int \cos x dx = \sin x + C$, $x \in \mathbb{R}$;
- $\int \sin x dx = -\cos x + C$, $x \in \mathbb{R}$;
- $\int \frac{dx}{\cos^2 x} = \tan x + C$ on each interval $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, $n \in \mathbb{Z}$;
- $\int \frac{dx}{\sin^2 x} = -\cot x + C$ on each interval $(n\pi, \pi + n\pi)$, $n \in \mathbb{Z}$;
- $\int \frac{dx}{1+x^2} = \arctan x + C$, $x \in \mathbb{R}$;
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$, $x \in (-1, 1)$.

15.2 Computation of Indefinite Integrals

An elementary function is the compositions of rational, exponential, trigonometric functions and their inverse functions. A function is called elementary integrable if it has an elementary antiderivative. “Most” functions are not elementary integrable. For example, antiderivatives of $f_1(x) = e^{-x^2}$, $x \in \mathbb{R}$; $f_2(x) = \frac{e^x}{x}$, $x > 0$; $f_3(x) = \frac{\sin x}{x}$, $x > 0$; $f_4(x) = \sin x^2$, $x \in \mathbb{R}$; $f_5(x) = \cos x^2$, $x \in \mathbb{R}$, cannot be expressed as elementary functions.

In the following subsections, we will consider some approaches which allow to compute antiderivatives of some classes of functions.

15.2.1 Substitution rule

Definition 15.3. The **differential** $df(x)$ **of a differentiable function** f is defined by $df(x) = f'(x)dx$.

According to Definition 15.3, we set $\int f(x)d\varphi(x) := \int f(x)\varphi'(x)dx$.

Theorem 15.2. *Let a function $f : J_1 \to \mathbb{R}$ be continuous on J_1, $g : J \to J_1$ be continuously differentiable on J (i.e. g has the continuous derivative on J) and let $\int f(t)dx = F(t) + C$, $t \in J_1$. Then $\int f(g(x))g'(x)dx = \int f(g(x))dg(x) = F(g(x)) + C$, $x \in J$.*

Proof. Indeed, $(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x)$, $x \in J$, by the chain rule. □

Example 15.3. Compute $\int \sin 5x dx$, $x \in \mathbb{R}$.

Solution. According to Theorem 15.2, we have

$$\int \sin 5x dx = \frac{1}{5}\int \sin 5x d(5x) = |5x = t| = \frac{1}{5}\int \sin t dt = -\frac{1}{5}\cos t + C = -\frac{1}{5}\cos 5x + C, \quad x \in \mathbb{R}.$$

Example 15.4. Compute $\int 2xe^{x^2} dx$, $x \in \mathbb{R}$.

Solution. By Theorem 15.2, we obtain

$$\int 2xe^{x^2} dx = \int e^{x^2} dx^2 = \left|x^2 = t\right| = \int e^t dt = e^t + C = e^{x^2} + C, \quad x \in \mathbb{R}.$$

Exercise 15.3. Compute the following indefinite integrals:
a) $\int \sin^2 x dx$, $x \in \mathbb{R}$; b) $\int \sin 2x \sin 3x dx$, $x \in \mathbb{R}$; c) $\int \sin^3 x dx$, $x \in \mathbb{R}$; d) $\int \frac{dx}{\sin x \cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$;
e) $\int x \cos x^2 dx$, $x \in \mathbb{R}$; f) $\int \frac{dx}{1-x}$ on $(-\infty, 1)$ and $(1, +\infty)$.

Theorem 15.3. *Let a function $f : J \to \mathbb{R}$ be continuous, $\varphi : J_0 \to J$ be continuously differentiable on J_0 and let φ have an inverse function φ^{-1}. Let also G be an antiderivative for the function $g(t) = f(\varphi(t))\varphi'(t)$, $t \in J_0$. Then*

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt = G(t) + C = G(\varphi^{-1}(x)) + C, \quad x \in J.$$

Proof. Let F be an antiderivative of f on J. Then according to the chain rule, we have

$$(F(\varphi(t)))' = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi(t) \quad t \in J_0.$$

Thus, there exists a constant C such that $G(t) = F(\varphi(t)) + C$, $t \in J_0$, or $G(\varphi^{-1}(x)) = F(x) + C$, $x \in J$. □

Example 15.5. Compute $\int \sqrt{1-x^2}dx$, $x \in [-1, 1]$.
Solution. Using Theorem 15.3, we have

$$\int \sqrt{1-x^2}dx = \left| \begin{array}{c} x = \sin t \\ dx = d\sin t = \cos t dt \end{array} \right| = \int \cos^2 t dt = \int \frac{1+\cos 2t}{2}dt$$
$$= \frac{1}{2}t + \frac{1}{4}\sin 2t + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C.$$

Here, we have used that $t = \arcsin x$ and $\sin 2t = 2\sin t \cos t = 2\sin t\sqrt{1-\cos^2 t} = x\sqrt{1-x^2}$, for $x = \sin t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exercise 15.4. Compute the following indefinite integrals:
a) $\int \frac{dx}{\cos x}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; b) $\int \frac{1}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$; c) $\int \frac{e^{\frac{1}{x}}dx}{x^2}$, $x > 0$; d) $\int \frac{(2x+1)dx}{\sqrt[3]{1+x+x^2}}$, $x \in \mathbb{R}$;
e) $\int \sqrt{1-3x}dx$, $x < \frac{1}{3}$; f) $\int \frac{\sqrt{\tan x}}{\cos^2 x}dx$, $x \in \left(0, \frac{\pi}{2}\right)$; g) $\int \frac{dx}{x \ln x}$, $x > 0$; h) $\int \cos^2 x \sin^3 x dx$, $x \in \mathbb{R}$;
i) $\int \frac{dx}{x^2+x+1}$, $x \in \mathbb{R}$; j) $\int \frac{dx}{1-x^2}$ on $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.

15.2.2 Integration by Parts Formula

Theorem 15.4. *Let $u, v : J \to \mathbb{R}$ be differentiable on J and the function uv' has an antiderivative on J. Then the function $u'v$ also has an antiderivative on J and the following equality*

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx, \quad x \in J, \tag{20}$$

holds.

Proof. The function uv is antiderivative of the function $u'v + uv'$ on J, by Theorem 10.3 3). Thus,

$$\int (u'(x)v(x) + u(x)v'(x))dx = u(x)v(x) + C,$$

which implies equality (20). □

Remark 15.2. According to Definition 15.3, the integration by parts formula (20) can be written as follows

$$\int v(x)du(x) = u(x)v(x) - \int u(x)dv(x), \quad x \in J.$$

Example 15.6. Compute $\int x \sin x dx$, $x \in \mathbb{R}$.

Solution. Using Theorem 15.4 and Remark 15.2, we have

$$\int x \sin x dx = -\int x d\cos x = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C, \quad x \in \mathbb{R}.$$

Example 15.7. Compute $\int \ln x dx$, $x > 0$.

Solution. Using Theorem 15.4 and Remark 15.2, we get

$$\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - \int dx = x\ln x - x + C, \quad x > 0.$$

Exercise 15.5. Compute $\int e^x \sin x dx$, $x \in \mathbb{R}$.

Solution. Applying Theorem 15.4 and Remark 15.2, we obtain

$$\begin{aligned}\int e^x \sin x dx &= \int \sin x de^x = e^x \sin x - \int e^x d\sin x = e^x\sin x - \int e^x \cos x dx \\ &= e^x \sin x - \int \cos x de^x = e^x \sin x - e^x\cos x + \int e^x d\cos x \\ &= e^x(\sin x - \cos x) - \int e^x \sin x dx \quad x \in \mathbb{R}.\end{aligned}$$

Thus, $\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) + C$, $x \in \mathbb{R}$.

Exercise 15.6. Compute the following indefinite integrals:
a) $\int x \sin x dx$, $x \in \mathbb{R}$; b) $\int x^2 \sin x dx$, $x \in \mathbb{R}$; c) $\int (\ln x)^2 dx$, $x > 0$; d) $\int \ln(x^2+x+1)dx$, $x \in \mathbb{R}$.

Exercise 15.7. Find a mistake in the following reasoning.

Using Theorem 15.4 and Remark 15.2, we have

$$\int \frac{dx}{x} = x \cdot \frac{1}{x} - \int x d\frac{1}{x} = 1 - \int x \cdot \left(-\frac{1}{x^2}\right) dx = 1 + \int \frac{dx}{x}, \quad x > 0.$$

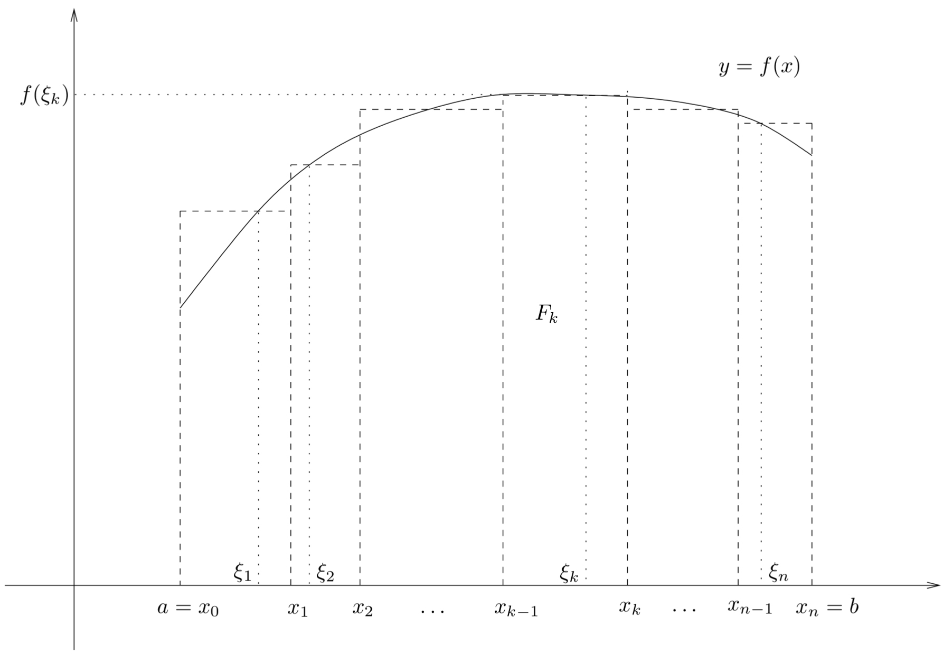
Thus, $0 = 1$!

16 Lecture 16 – Riemann Integral

16.1 Area of the Region under the Graph of Function

We consider the following problem. Let $f : [a, b] \to \mathbb{R}$ be non-negative continuous function. We want to compute the area of the region under the graph of f, that is, the area of the set

$$F := \{(x, y) : \; y \in [0, f(x)], \; x \in [a, b]\}.$$



For this, we divide the interval $[a, b]$ into smaller subintervals $[x_{k-1}, x_k]$, $k = 1, \dots, n$, where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, and consider the following partition of F to the sets

$$F_k := \{(x, y) : \; y \in [0, f(x)], \; x \in [x_{k-1}, x_k]\},$$

$k = 1, \dots, n$. Since f is a continuous, its values vary little on $[x_{k-1}, x_k]$, if $\Delta x_k = x_k - x_{k-1}$ is small. Consequently, we should expect that the area of F_k should be close to the area of the rectangle with sides Δx_k and $f(\xi_k)$ which equals $f(\xi_k)\Delta x_k$, where ξ_k are points from the intervals $[x_{k-1}, x_k]$. Thus, one can expect that

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \to S(F), \quad \text{as} \quad \max_k |\Delta x_k| \to 0. \tag{21}$$

Limit of the type (21) really exists, and will be studied in the next sections.

16.2 Definition of the Integral

Definition 16.1.
- Let $[a, b]$ be an interval and $n \in \mathbb{N}$. A set of points $x_0, x_1, \dots, x_n$ such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ is called a **partition of the interval** $[a, b]$ and is denoted by λ.

- The number $|\lambda| = \max\{\Delta x_k : \ 1 \leq k \leq n\}$, where $\Delta x_k = x_k - x_{k-1}$, is called the **mesh of a partition** λ.

Let $f : [a,b] \to \mathbb{R}$ be a function, $\lambda = \{x_0, x_1, \dots, x_n\}$ be a partition of the interval $[a,b]$ and $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$. The sum

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{22}$$

is called the **Riemann sum**.

Definition 16.2. A function f is said to be **integrable on** $[a,b]$, if there exists a limit J of Riemann sums (22) as $|\lambda| \to 0$ and this limit does not depend on the choice of partitions λ and points ξ_k. More precisely, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each partition $\lambda = \{x_0, x_1, \dots, x_n\}$ with $|\lambda| < \delta$ and points $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$,

$$\left| J - \sum_{k=1}^{n} f(\xi_k)\Delta x_k \right| < \varepsilon.$$

The number J is called the **Riemann integral of** f **over** $[a,b]$ and is denoted by $\int_a^b f(x)dx$.

Shortly, we will write

$$\int_a^b f(x)dx = \lim_{|\lambda| \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k. \tag{23}$$

If $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, then we will write $f \in R([a,b])$.

Exercise 16.1. Show that a constant function $f(x) = c$, $x \in [a,b]$, is integrable on $[a,b]$ and compute $\int_a^b c dx$.

Exercise 16.2. Show that the Dirichlet function $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$, is not integrable on any interval $[a,b]$, $a < b$.

Exercise 16.3. Let $f, g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Show that $f + g$ is also integrable on $[a,b]$.

Theorem 16.1. *If a function $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, then f is bounded on $[a,b]$.*

Exercise 16.4. Prove Theorem 16.1.

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$.

Definition 16.3.
- The **upper Darboux sum** of f with respect to a partition λ is the sum

$$U(f, \lambda) = \sum_{k=1}^{n} M_k \Delta x_k,$$

where $M_k := \sup_{x \in [x_{k-1}, x_k]} f(x)$.

- The **lower Darboux sum** of f with respect to a partition λ is the sum

$$L(f, \lambda) = \sum_{k=1}^{n} m_k \Delta x_k,$$

where $m_k := \inf_{x \in [x_{k-1}, x_k]} f(x)$.

Theorem 16.2 (Integrability criterion)**.** *A function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists $\lambda = \lambda([a, b])$ such that*

$$U(f, \lambda) - L(f, \lambda) < \varepsilon.$$

Exercise 16.5. Let $f \in R([a, b])$. Show that
a) $|f| \in R([a, b])$; b) $\sin f \in R([a, b])$; c) $f^2 \in R([a, b])$; d) $\max\{0, f\} \in R([a, b])$.

Exercise 16.6. Let $f, g \in R([a, b])$. Show that $fg \in R([a, b])$.

16.3 Classes of Integrable Functions

16.3.1 Integrability of Monotone Functions

Theorem 16.3. *Let $f : [a, b] \to \mathbb{R}$ be a monotone function on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. We assume that f is increasing on $[a, b]$ and $f(a) < f(b)$. To prove the theorem, we are going to use the integrability criterion (see Theorem 16.2). For any $\varepsilon > 0$ we take a partition λ of the interval $[a, b]$ such that $|\lambda| < \frac{\varepsilon}{f(b)-f(a)}$. For such a partition we have

$$\begin{aligned} U(f, \lambda) - L(f, \lambda) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\Delta x_k \\ &\le |\lambda| \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = |\lambda|(f(x_n) - f(x_0)) = |\lambda|(f(b) - f(a)) < \varepsilon. \end{aligned}$$

□

Exercise 16.7. For any bounded function $f : [a, b] \to \mathbb{R}$ we set $g(x) = \sup_{u\in[a,x]} f(u)$ and $h(x) = \inf_{u\in[a,x]} f(u)$, $x \in [a, b]$. Show that $g, h \in R([a, b])$.

16.3.2 Integrability of Continuous Functions

Theorem 16.4. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. We will use the integrability criterion again, to prove the theorem. By the Cantor theorem (see Theorem 9.4), f is uniformly continuous on $[a, b]$. Thus, for a number $\frac{\varepsilon}{b-a} > 0$ there exists $\delta > 0$ such that for each $x', x'' \in [a, b]$, $|x' - x''| < \delta$ it follows $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$. Next, we choose a partition λ of $[a, b]$ with $|\lambda| < \delta$. Thus, by the 2nd Weierstrass theorem (see Theorem 9.2), for each $k = 1, \ldots, n$

$$M_k - m_k = \sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x) = f(x^*) - f(x_*) < \frac{\varepsilon}{b - a},$$

where x^* and x_* are points where f takes its maximum and minimum value on $[x_{k-1}, x_k]$, respectively. Consequently,

$$U(f, \lambda) - L(f, \lambda) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\varepsilon}{b - a} \sum_{k=1}^{n} \Delta x_k = \varepsilon.$$

□

16.4 Properties of Riemann Integral

Theorem 16.5 (Linearity and addidivity)**.** *(i) Let $f \in R([a,b])$ and $c \in \mathbb{R}$. Then $cf \in R([a,b])$ and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

(ii) Let $f, g \in R([a,b])$. Then $f + g \in R([a,b])$ and

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(iii) Let $f \in R([a,b])$ and $c \in (a,b)$. Then $f \in R([a,c])$ and $f \in R([c,b])$. Moreover,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Exercise 16.8. Prove (i) and (ii) of Theorem 16.5.

Exercise 16.9. Let $c \in (a,b)$. Show that $f \in R([a,b])$, if $f \in R([a,c])$ and $f \in R([c,b])$.

Theorem 16.6. *Let $f, g \in R([a,b])$ and $f(x) \le g(x)$, $x \in [a,b]$. Then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.*

Proof. The statement immediately follows from the definition of the integral. □

Exercise 16.10. Prove Theorem 16.6.

Corollary 16.1. *Let $f \in R([a,b])$ and $m := \inf_{x\in[a,b]} f(x)$, $M := \sup_{x\in[a,b]} f(x)$. Then*

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a). \tag{24}$$

Proof. We first note that m and M exists, since f is bounded (see Theorem 16.1). Inequality (24) follows from the inequality $m \le f(x) \le M$, $x \in [a,b]$, and Theorem 16.6. □

Corollary 16.2. *Let $f \in R([a,b])$. Then $|f| \in R([a,b])$ and*

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Exercise 16.11. Prove Corollary 16.2.

Theorem 16.7 (Mean value theorem for integrals)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then there exists $\theta \in [a,b]$ such that $\int_a^b f(x)dx = f(\theta)(b-a)$.*

Proof. By Corollary 16.1,

$$m \le L := \frac{1}{b-a}\int_a^b f(x)dx \le M.$$

Since f is continuous, we can apply the 2nd Weierstrass theorem (see Theorem 9.2) to f. Thus, there exist $x_*, x^* \in [a,b]$ such that $m = f(x_*)$ and $M = f(x^*)$. Consequently, $f(x_*) \le L \le f(x^*)$. By the intermediate value theorem (see Theorem 9.3), there exists θ between x^* and x_* such that $f(\theta) = L$. □

Exercise 16.12. Let $f : [a,b] \to \mathbb{R}$ be a non-negative continuous function on $[a,b]$ such that $f(x_0) > 0$ for some $x_0 \in [a,b]$. Show that $\int_a^b f(x)dx > 0$.

Exercise 16.13. Let $f \in C([a,b])$, $g \in R([a,b])$ and $g(x) \geq 0$, $x \in [a,b]$. Show that there exists $\theta \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(\theta) \int_a^b g(x)dx$.

Exercise 16.14. For functions $f, g \in R([a,b])$ compute the limit

$$\lim_{|\lambda| \to 0} \sum_{k=1}^{n} f(\xi_k) \int_{x_{k-1}}^{x_k} g(x)dx.$$

Exercise 16.15. For a function $f \in R([0,1])$ prove the equality

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{1} f(x)dx = \int_0^1 f(x)dx.$$

Exercise 16.16 (Cauchy inequality)**.** For $f, g \in R([a,b])$ prove the following inequality

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$

17 Lecture 17 – Fundamental Theorem of Calculus and Application of Riemann Integral

17.1 Fundamental Theorem of Calculus

We set $\int_a^a f(x)dx := 0$ and $\int_b^a f(x)dx := -\int_a^b f(x)dx$ for $a < b$.

Theorem 17.1. *Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then the function $\varphi(x) := \int_a^x f(u)du$, $x \in [a,b]$, is continuous on $[a,b]$.*

Proof. For every $x', x'' \in [a,b]$ we have

$$|\varphi(x') - \varphi(x'')| = \left| \int_a^{x'} f(x)dx - \int_a^{x''} f(x)dx \right| = \left| \int_{x'}^{x''} f(x)dx \right| \leq \int_{x'}^{x''} |f(x)|dx \leq \sup_{x \in [a,b]} |f(x)||x' - x''|,$$

by Theorem 16.5 (iii) and corollaries 16.1, 16.2. Consequently, φ is uniformly continuous on $[a,b]$. $\square$

Theorem 17.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then the function $\varphi(x) := \int_a^x f(u)du$, $x \in [a,b]$, is differentiable on $[a,b]$ and $\varphi'(x) = f(x)$, $x \in [a,b]$, that is, φ is an antiderivative of f on $[a,b]$.*

Proof. Let $x_0 \in [a,b]$ and $h \neq 0$. By the mean value theorem (see Theorem 16.7), there exists θ_h between x_0 and $x_0 + h$ such that

$$\frac{\varphi(x_0 + h) - \varphi(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} f(x)dx = f(\theta_h).$$

Since $\theta_h \to x_0$, $h \to 0$, and f is continuous, we obtain

$$\lim_{h \to 0} \frac{\varphi(x_0 + h) - \varphi(x_0)}{h} = \lim_{h \to 0} f(\theta_h) = f(x_0).$$

$\square$

Theorem 17.3 (Fundamental Theorem of Calculus)**.** *We assume that $f : [a,b] \to \mathbb{R}$ satisfies the following properties:*

1) *f is integrable on $[a,b]$;*

2) *f has an antiderivative F on $[a,b]$.*

Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

We will also denote $F(x)\Big|_a^b := F(b) - F(a)$.

Proof. We first prove the theorem in the case $f \in C([a,b])$. The function $\varphi(x) := \int_a^x f(u)du$, $x \in [a,b]$, is an antiderivative of f on $[a,b]$, by Theorem 17.2. Thus, using Remark 15.1, there exists $C \in \mathbb{R}$ such that $\varphi(x) = F(x) + C$, $x \in [a,b]$. In particular, $\varphi(a) = F(a) + C = 0$. Thus, $C = -F(a)$. Consequently, $\int_a^b f(x)dx = \varphi(b) = F(b) + C = F(b) - F(a)$.

Next, we give the second proof of the theorem in the general case. Let $\lambda = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. We first note that

$$F(b) - F(a) = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1})) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})).$$

We apply the Lagrange theorem (see Theorem 11.4) to the function F on $[x_{k-1}, x_k]$ for each $k = 1, \dots, n$. So, there exists $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$, such that

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} F'(\xi_k) \Delta x_k = \sum_{k=1}^{n} f(\xi_k) \Delta x_k.$$

Making $|\lambda| \to 0$, we have

$$F(b) - F(a) = \sum_{k=1}^{n} f(\xi_k) \Delta x_k \to \int_a^b f(x) dx,$$

since f is integrable on $[a, b]$. □

Exercise 17.1. Compute the following integrals:
a) $\int_{-1}^{8} \sqrt[3]{x} dx$; b) $\int_0^{\pi} \sin x dx$; c) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$; d) $\int_0^2 |1 - x| dx$; e) $\int_{-1}^{1} \frac{dx}{x^2 - 2x \cos \alpha + 1}$ for $\alpha \in (0, \pi)$.

Example 17.1 (Leibniz's rule)**.** Let a function $f : \mathbb{R} \to \mathbb{R}$ have an antiderivative on $\mathbb{R}$ and be integrable on each finite interval. Let functions $a, b\mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Then

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(u) du = f(b(x)) b'(x) - f(a(x)) a'(x), \quad x \in \mathbb{R}.$$

Indeed, let F be an antiderivative of f on $\mathbb{R}$. By the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(u) du = F(b(x)) - F(a(x)), \quad x \in \mathbb{R}. \tag{25}$$

Moreover, the right hand side of (25) is differentiable and

$$\frac{d}{dx} (F(b(x)) - F(a(x))) = F'(b(x)) b'(x) - F'(a(x)) a'(x) = f(b(x)) b'(x) - f(a(x)) a'(x), \quad x \in \mathbb{R},$$

by the chain rule.

Exercise 17.2. Compute the following derivatives:
a) $\frac{d}{dx} \int_a^b \sin x^2 dx$; b) $\frac{d}{da} \int_a^b \sin x^2 dx$; c) $\frac{d}{dx} \int_0^{x^2} \sqrt{1 + t^2} dt$; d) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{1+t^4}$.

Exercise 17.3. Compute the following limits:
a) $\lim\limits_{x \to 0} \frac{\int_0^x \cos t^2 dt}{x}$; b) $\lim\limits_{x \to +\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}$; c) $\lim\limits_{x \to +\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt}$.

17.2 Some Corollaries

Theorem 17.4 (Substitution rule). *We assume that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $u : [\alpha,\beta] \to [a,b]$ is continuously differentiable on $[\alpha,\beta]$. Then the following equality*

$$\int_\alpha^\beta f(u(t))u'(t)dt = \int_\alpha^\beta f(u(t))du(t) = \int_{u(\alpha)}^{u(\beta)} f(x)dx$$

holds.

Proof. Since the function f is continuous on $[u(\alpha), u(\beta)]$, it has an antiderivative F on $[u(\alpha), u(\beta)]$, by Theorem 17.2. Using the fundamental theorem of calculus,

$$\int_{u(\alpha)}^{u(\beta)} f(x)dx = F(u(\beta)) - F(u(\alpha)).$$

Moreover, the function $F(u)$ is an antiderivative of $f(u)u'$ on $[\alpha,\beta]$. Thus, by the fundamental theorem of calculus,

$$\int_\alpha^\beta f(u(t))u'(t)dt = F(u(\beta)) - F(u(\alpha)).$$

This proves the theorem. □

Exercise 17.4. Using the substitution rule, compute the following integrals:
a) $\int_0^{\sqrt{\pi}} x \sin x^2 dx$; b) $\int_0^1 e^{2x-1}dx$; c) $\int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$; d) $\int_0^{\ln 2} \sqrt{e^x - 1}dx$; e) $\int_0^{\frac{\pi}{6}} \frac{dx}{\cos x}$.

Theorem 17.5 (Integration by parts). *Let $u, v : [a,b] \to \mathbb{R}$ be continuously differentiable functions on $[a,b]$. Then*

$$\int_a^b u(x)dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)du(x),$$

i.e.

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx.$$

Proof. Since the function uv is an antiderivative of $uv' + u'v$ on $[a,b]$,

$$\int_a^b (u(x)v'(x) + u'(x)v(x))dx = u(b)v(b) - u(a)v(a),$$

by the fundamental theorem of calculus. Using Theorem 16.5 (ii), we obtain the integration by parts formula. □

Exercise 17.5. Using the integration by parts formula, compute the following integrals:
a) $\int_0^{\ln 2} xe^{-x}dx$; b) $\int_0^\pi x \sin x dx$; c) $\int_0^{2\pi} x^2 \cos x dx$; d) $\int_{\frac{1}{e}}^e |\ln x| dx$; e) $\int_0^1 \arccos x dx$.

17.3 Application of the Integral

17.3.1 Area of the Region under the Graph of Function

Theorem 17.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and $f(x) \geq 0$, $x \in [a, b]$. Then the area of the region*

$$F = \{(x, y) : \; 0 \leq y \leq f(x), \; a \leq x \leq b\}$$

under the graph of f is equal to

$$S(F) = \int_a^b f(x)dx.$$

Proof. We first note that f is integrable on $[a, b]$ because it is continuous (see Theorem 16.4). Thus, the formula for the area follows from the discussion in Section 16.1 and definition of the integral (see (23)). □

Example 17.2. The area of the region under the graph of the function $f(x) = x^2$, $x \in [0, 1]$, is equal

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Example 17.3. Compute the area of the region G enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$. In order to compute the area of G, it is enough to compute the area of

$$F = \left\{(x, y) : \; 0 \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}, \;\; 0 \leq x \leq a\right\}.$$

By Theorem 17.6,

$$S(G) = 4S(F) = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}dx = \left| \begin{array}{c} x = a \sin t \\ dx = a \cos t dt \end{array} \right| = 4ab \int_0^{\frac{\pi}{2}} \cos^2 t dt$$
$$= 4ab \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = 2abt\Big|_0^{\frac{\pi}{2}} + ab \sin 2t\Big|_0^{\frac{\pi}{2}} = \pi ab.$$

Exercise 17.6. Compute the area of regions bounded by the graphs of the following functions:
a) $2x = y^2$ and $2y = x^2$; b) $y = x^2$ and $x + y = 2$; c) $y = 2^x$, $y = 2$ and $x = 0$;
d) $y = \frac{a^3}{a^2+x^2}$ and $y = 0$, where $a > 0$.

17.3.2 Length of a Curve

Definition 17.1. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$. The set of points

$$\Gamma := \{(x, y) \in \mathbb{R}^2 : \; x = \varphi(t), \; y = \psi(t), \; t \in [a, b]\} \tag{26}$$

is called a **continuous (plane) curve**.

We first give a definition of the length of the continuous curve Γ. Let $\lambda = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. We consider the polygonal curve Γ_λ with vertices $(\varphi(t_k), \psi(t_k))$, $k = 0, \ldots, n$. Its length equals

$$l(\Gamma_\lambda) = \sum_{k=1}^n \sqrt{(\varphi(t_k) - \varphi(t_{k-1}))^2 + (\psi(t_k) - \psi(t_{k-1}))^2}.$$

Definition 17.2. The curve Γ is said to be a **rectifiable curve**, if there exists a finite limit

$$\lim_{|\lambda|\to 0} l(\Gamma_\lambda) =: l(\Gamma),$$

that is, if there exists a real number $l(\Gamma)$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall \lambda \;\; |\lambda| < \delta : \quad |l(\Gamma_\lambda) - l(\Gamma)| < \varepsilon.$$

The limit $l(\Gamma)$ is called the **length of rectifiable curve** Γ.

Theorem 17.7. *Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be continuously differentiable on $[a,b]$. Then Γ, defined by (26), is a rectifiable curve and its length equals*

$$l(\Gamma) = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}dt.$$

Proof. Using the Lagrange theorem (see Theorem 11.4), we have

$$l(\Gamma_\lambda) = \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\eta_k))^2}\Delta t_k = \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2}\Delta t_k + r_\lambda,$$

where $\xi_k, \eta_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$, and

$$r_\lambda := \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\eta_k))^2}\Delta t_k - \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2}\Delta t_k.$$

Since the function $f(t) = \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}$, $t \in [a,b]$, is continuous on $[a,b]$, it is integrable on $[a,b]$, by Theorem 16.4. Thus,

$$\lim_{|\lambda|\to 0} \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2}\Delta t_k = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}dt.$$

Moreover, using the inequality

$$|\sqrt{u^2 + v^2} - \sqrt{u^2 + w^2}| \leq |v - w|, \quad u, v, w \in \mathbb{R},$$

(see Exercise 12.5 b)), we have

$$|r_\lambda| \leq \sum_{k=1}^n |\psi'(\xi_k) - \psi'(\eta_k)|\Delta t_k \leq \sum_{k=1}^n (M_k - m_k)\Delta t_k,$$

where $M_k := \sup\limits_{t\in[t_{k-1},t_k]} \psi'(t)$ and $m_k := \inf\limits_{t\in[t_{k-1},t_k]} \psi'(t)$, $k = 1, \ldots, n$. Using theorems 16.2 and 16.4, we obtain

$$|r_\lambda| \leq \sum_{k=1}^n (M_k - m_k)\Delta t_k \to 0, \quad |\lambda| \to 0.$$

$\square$

Remark 17.1. If a curve Γ is given by the graph of a continuously differentiable function $f : [a, b] \to \mathbb{R}$, that is,

$$\Gamma = \{(x, y) : \ y = f(x), \quad x \in [a, b]\},$$

then its length equals

$$l(\Gamma) = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Example 17.4. We compute the length of the circle $x^2 + y^2 = r^2$, $r > 0$, that is, the length of the curve

$$\Gamma = \left\{(x, y) : \ x^2 + y^2 = r^2\right\} = \{(x, y) : \ x = r \cos t, \ y = r \sin t, \ t \in [0, 2\pi)\}.$$

By Theorem 17.7,

$$l(\Gamma) = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt = \int_0^{2\pi} r dt = 2\pi r.$$

Exercise 17.7. Compute the length of continuous curves defined by the following functions:
a) $y = x^{\frac{3}{2}}$, $x \in [0, 4]$; b) $y = e^x$, $0 \le x \le b$; c) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.

17.3.3 Volume of Solid of Revolution

Definition 17.3. Let $f : [a, b] \to \mathbb{R}$ be a positive continuous function. A **solid of revolution** G is a set of points in $\mathbb{R}^3$ obtained by rotating of the region under the graph of f around the x-axis, that is,

$$G = \left\{(x, y, z) : \ y^2 + z^2 \le f^2(x), \ x \in [a, b]\right\}.$$

Theorem 17.8. *Let $f : [a, b] \to \mathbb{R}$ be a positive continuous function. Then the volume of solid of revolution G is equal to*

$$V(G) = \pi \int_a^b f^2(x) dx.$$

Idea of Proof. We consider a partition $\lambda = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$ and split G into smaller sets

$$G_k = \left\{(x, y, z) : \ y^2 + z^2 \le f^2(x), \ x \in [x_{k-1}, x_k]\right\}, \quad k = 1, \ldots, n.$$

Then the volume of G_k is approximately equal the volume of the cylinder

$$\left\{(x, y, z) : \ y^2 + z^2 \le f^2(\xi_k), \ x \in [x_{k-1}, x_k]\right\},$$

where $\xi_k \in [x_{k-1}, x_k]$. Thus,

$$V(G) = \sum_{k=1}^{n} V(G_k) \approx \sum_{k=1}^{n} \pi f^2(\xi_k) \Delta x_k.$$

Passing to the limit as $|\lambda| \to 0$, we obtain

$$V(G) = \pi \int_a^b f^2(x) dx.$$

$\square$

Example 17.5. The volume of the cone

$$G = \{(x, y, z) : \; y^2 + z^2 \leq x^2, \quad x \in [0, 1]\}.$$

equals

$$V(G) = \pi \int_0^1 x^2 dx = \pi \frac{x^3}{3}\Big|_0^1 = \frac{\pi}{3},$$

since G can be obtained by rotating of the region under the graph of the function $f(x) = x$, $x \in [0, 1]$, around the x-axis.

Exercise 17.8. Compute the volume of the paraboloid of revolution

$$G = \{(x, y, z) : \; y^2 + z^2 \leq x, \quad x \in [0, 1]\}.$$

(*Hint:* G can be obtained by rotating of the region under the graph of the function $f(x) = \sqrt{x}$, $x \in [0, 1]$, around the x-axis)

18 Lecture 18 – Improper Integrals

18.1 Integrals over Unbounded Intervals

18.1.1 Definition and Elementary Properties

In this section, we assume that a function $f : [a, +\infty) \to \mathbb{R}$ is integrable on $[a, z]$ for all $z > a$ and set

$$\varphi(z) := \int_a^z f(x)dx, \quad z \geq a.$$

Definition 18.1. The finite limit

$$\lim_{z\to+\infty} \varphi(z) = \lim_{z\to+\infty} \int_a^z f(x)dx \tag{27}$$

is called the **improper integral of f over** $[a, +\infty)$ and is denoted by

$$\int_a^{+\infty} f(x)dx. \tag{28}$$

In this case, we will say that improper integral (28) **converges**. If limit (27) does not exist or is infinite, then improper integral (28) is said to be **divergent**.

Remark 18.1. If integral (28) converges, then for each $b > a$ the improper integral

$$\int_b^{+\infty} f(x)dx \tag{29}$$

also converges. If for some $b > a$ improper integral (29) converges, then improper integral (28) also converges. These both statements follow from the equality

$$\int_a^z f(x)dx = \int_a^b f(x)dx + \int_b^z f(x)dx, \quad a < b \leq z,$$

and the definition of the improper integral.

Example 18.1. The improper integral $\int_0^{+\infty} e^{-x}dx$ converges and equals 1. Indeed,

$$\int_0^{+\infty} e^{-x}dx = \lim_{z\to+\infty} \int_0^z e^{-x}dx = \lim_{z\to+\infty} \left(-e^{-x}\Big|_0^z\right) = \lim_{z\to+\infty} (1 - e^{-z}) = 1.$$

Example 18.2. The equality $\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$ is true, since

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{z\to+\infty} \int_0^z \frac{dx}{1+x^2} = \lim_{z\to+\infty} \left(\arctan x\Big|_0^z\right) = \lim_{z\to+\infty} (\arctan z - 0) = \frac{\pi}{2}.$$

Example 18.3. Let $p > 0$. The improper integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$. Indeed, for each $z \geq 1$

$$\varphi(z) = \int_1^z \frac{dx}{x^p} = \begin{cases} \ln z - \ln 1, & p = 1, \\ \frac{z^{-p+1}}{-p+1} - \frac{1}{-p+1}, & p \neq 1. \end{cases}$$

Thus,

$$\varphi(z) \to \begin{cases} +\infty, & p \leq 1, \\ \frac{1}{p-1}, & p > 1, \end{cases} \quad z \to +\infty.$$

Example 18.4. The integral $\int_0^{+\infty} \cos x dx$ diverges, since for the function $\varphi(z) = \int_0^z \cos x dx = \sin z$, $z \geq 0$, there is no limit as $z \to +\infty$.

Theorem 18.1 (Elementary properties of improper integrals)**.** *The following properties of improper integrals are true.*

1) *Let integrals $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ converge. Then the integrals $\int_a^{+\infty} cf(x)dx$, $c \in \mathbb{R}$, $\int_a^{+\infty}(f(x)+g(x))dx$ converge and*

$$\int_a^{+\infty} cf(x)dx = c\int_a^{+\infty} f(x)dx,$$
$$\int_a^{+\infty} (f(x)+g(x))dx = \int_a^{+\infty} f(x)dx + \int_a^{+\infty} g(x)dx.$$

2) *Let f have an antiderivative F on $[a,+\infty)$. If the limit*

$$F(+\infty) := \lim_{z\to+\infty} F(z) \tag{30}$$

exists, then $\int_a^{+\infty} f(x)dx = F(+\infty) - F(a)$. If limit (30) does not exists or is infinite, then the integral $\int_a^{+\infty} f(x)dx$ diverges.

3) *(Integration by parts) We assume that functions u, v are continuously differentiable on $[a,+\infty)$. If the integral $\int_a^{+\infty} u(x)v'(x)dx = \int_a^{+\infty} u(x)dv(x)$ converges and the limit $u(+\infty)v(+\infty) := \lim_{z\to+\infty} u(z)v(z)$ exists, then the integral $\int_a^{+\infty} u'(x)v(x)dx = \int_a^{+\infty} v(x)du(x)$ converges and*

$$\int_a^{+\infty} v(x)du(x) = u(x)v(x)\Big|_a^{+\infty} - \int_a^{+\infty} u(x)dv(x).$$

Example 18.5. The integral $\int_0^{+\infty} xe^{-x}dx$ converges and equals 1, since

$$\int_0^{+\infty} xe^{-x}dx = -\int_0^{+\infty} xde^{-x} = -xe^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}dx = 1,$$

according to examples 7.3 and 18.1.

18.1.2 Convergence of Improper Integrals of Non-Negative Functions

Theorem 18.2. *The improper integral $\int_a^{+\infty} f(x)dx$ of a non-negative function f converges iff there exists $C \in \mathbb{R}$ such that $\varphi(z) = \int_a^z f(x)dx \leq C$ for all $z \geq a$.*

Proof. Since f is non-negative function, the function φ non-decreases. Consequently, the upper boundedness of φ is equivalent to the existence of the limit $\lim_{z\to+\infty} \varphi(z)$, by Theorem 7.9 (i). $\square$

Theorem 18.3. *Let $f : [a,+\infty) \to \mathbb{R}$ and $g : [a,+\infty) \to \mathbb{R}$ satisfy $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then the convergence of the improper integral $\int_a^{+\infty} g(x)dx$ implies the convergence of $\int_a^{+\infty} f(x)dx$.*

Proof. The statement follows from Theorem 18.2 and the estimate

$$\varphi(z) = \int_a^z f(x)dx \le \int_a^z g(x)dx \le \lim_{z\to+\infty} \int_a^z g(x)dx = \int_a^{+\infty} g(x)dx =: C, \quad z \ge a.$$

Here the inequality for integrals follows from Theorem 16.6. □

Example 18.6. The integral $\int_0^{+\infty} \frac{\cos^2 x}{1+x^2}dx$ converges because we can apply Theorem 18.3 with $a = 0$, $f(x) = \frac{\cos^2 x}{1+x^2}$, $x \ge 0$, and $g(x) = \frac{1}{1+x^2}$, $x \ge 0$. For the convergence of the integral $\int_0^{+\infty} g(x)dx = \int_0^{+\infty} \frac{dx}{1+x^2}$ see Example 18.2.

Exercise 18.1. Show that the following improper integrals converge:
a) $\int_1^{+\infty} e^{-x^2}dx$; b) $\int_1^{+\infty} e^{-x}\ln x dx$; c) $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx$.

Corollary 18.1. *We assume that for some numbers $0 < C < +\infty$ and $p > 0$ $f(x) \sim \frac{C}{x^p}$, $x \to +\infty$, i.e. $\lim_{x\to+\infty} x^p f(x) = C$. Then the integral $\int_a^{+\infty} f(x)dx$ converges for $p > 1$ and diverges for $p \le 1$.*

Proof. Let $p > 1$. By Theorem 7.1 (iii), there exists $D \ge a$ such that $x^p f(x) \le 2C$ for all $x \ge D$. Thus, $f(x) \le \frac{2C}{x^p}$, $x \ge D$. Now applying Theorem 18.3 with $a = D$, $g(x) = \frac{2C}{x^p}$, $x \ge D$, and using Example 18.3, we obtain that $\int_D^{+\infty} f(x)dx$ converges. Hence, $\int_a^{+\infty} f(x)dx$ also converges, by Remark 18.1.

Let $p \le 1$. Similarly, there exists $D \ge a$ such that $f(x) \ge \frac{C}{2x^p}$ for all $x \ge D$. Since the integral $\int_D^{+\infty} \frac{Cdx}{2x^p}$ diverges (see Example 18.3), the integral $\int_D^{+\infty} f(x)dx$ also diverges. □

18.1.3 Absolute and conditional convergence

Definition 18.2. An improper integral

$$\int_a^{+\infty} f(x)dx \tag{31}$$

is said to be **absolutely convergent**, if the integral

$$\int_a^{+\infty} |f(x)|dx \tag{32}$$

converges. If integral (31) converges but integral (32) diverges, then (31) is called **conditionally convergent**.

Theorem 18.4. *If an improper integral absolutely converges, then it converges.*

Proof. Let integral (32) converge. We consider the following functions

$$f_-(x) := \frac{|f(x)| - f(x)}{2}, \quad x \ge a,$$
$$f_+(x) := \frac{|f(x)| + f(x)}{2}, \quad x \ge a,$$

and note that $0 \le f_-(x) \le |f(x)|$, $0 \le f_+(x) \le |f(x)|$ and $f(x) = f_+(x) - f_-(x)$ for all $x \ge a$.

By Theorem 18.3, the integrals $\int_a^{+\infty} f_-(x)dx$ and $\int_a^{+\infty} f_+(x)dx$ converge. Thus, using Theorem 18.1 1), we have that the improper integral $\int_a^{+\infty} f(x)dx = \int_a^{+\infty}(f_+(x) - f_-(x))dx$ also converges. □

Example 18.7. The integral $\int_1^{+\infty} \frac{\sin x}{x} dx$ is conditionally convergent. Indeed, according to the integration by parts formula, we have

$$\int_1^{+\infty} \frac{\sin x}{x} dx = \lim_{z \to +\infty} \int_1^z \frac{\sin x}{x} dx = -\lim_{z \to +\infty} \int_1^z \frac{1}{x} d\cos x = -\lim_{z \to +\infty} \left[\frac{\cos x}{x} \Big|_1^z - \int_1^z \cos x d\frac{1}{x} \right]$$
$$= -\lim_{z \to +\infty} \left[\frac{\cos z}{z} - \cos 1 + \int_1^z \frac{\cos x}{x^2} dx \right] = \cos 1 - \int_1^{+\infty} \frac{\cos x}{x^2} dx.$$

The integral $\int_1^{+\infty} \frac{\cos x}{x^2} dx$ absolutely converges because $\frac{|\cos x|}{x^2} \leq \frac{1}{x^2}$, $x \geq 1$, and the integral $\int_1^{+\infty} \frac{dx}{x^2}$ converges (see Example 18.3). Thus, $\int_1^{+\infty} \frac{\cos x}{x^2} dx$ converges, by Theorem 18.4. This implies the convergence of the integral $\int_1^{+\infty} \frac{\sin x}{x} dx$.

Next, we show that $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ diverges. We estimate

$$\int_\pi^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=2}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| dx = \frac{2}{\pi} \sum_{k=2}^n \frac{1}{k}.$$

In the next lecture, we will show that $\sum_{k=2}^n \frac{1}{k} \to +\infty$, $n \to \infty$. Thus, $\int_\pi^{+\infty} \frac{|\sin x|}{x} dx$ diverges and, consequently, $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ also diverges.

Theorem 18.5 (Dirichlet's test)**.** *Let functions f and g satisfy the following properties:*

1) *there exists $C \in \mathbb{R}$ such that $\left| \int_a^z f(x) dx \right| \leq C$ for all $z \geq a$;*

2) *the function g is monotone on $[a, +\infty)$;*

3) *$g(x) \to 0$, $x \to +\infty$.*

Then the integral $\int_a^{+\infty} f(x) g(x) dx$ converges.

Example 18.8. The integral $\int_1^{+\infty} \frac{\sin x}{x} dx$ converges, since the functions $f(x) = \sin x$, $x \geq 1$, and $g(x) = \frac{1}{x}$, $x \geq 1$, satisfy conditions 1)-3) of Theorem 18.5 with $C = 2$.

Example 18.9. The integral $\int_1^{+\infty} \sin x^3 dx$ converges, since the functions $f(x) = x^2 \sin x^3$, $x \geq 1$, and $g(x) = \frac{1}{x^2}$, $x \geq 1$, satisfy conditions 1)-3) of Theorem 18.5 with $C = \frac{2}{3}$.

Theorem 18.6 (Abel's test)**.** *Let functions f and g satisfy the following properties:*

1) *the integral $\int_a^{+\infty} f(x) dx$ converges;*

2) *the function g is monotone on $[a, +\infty)$;*

3) *the function g is bounded on $[a, +\infty)$.*

Then the integral $\int_a^{+\infty} f(x) g(x) dx$ converges.

Exercise 18.2. Prove the convergence of the following integrals:
a) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}} dx$; b) $\int_1^{+\infty} \cos x^2 dx$; c) $\int_0^{+\infty} \sin x^2 dx$; d) $\int_1^{+\infty} \frac{\sin 2x \cdot \sin x}{x} dx$.

Remark 18.2. The definition and properties of the improper integral $\int_{-\infty}^a f(x) dx$ are similar to ones of $\int_a^{+\infty} f(x) dx$. The integral $\int_{-\infty}^{+\infty} f(x) dx$ is defined as $\int_{-\infty}^a f(x) dx + \int_a^{+\infty} f(x) dx$ for any $a \in \mathbb{R}$.

18.2 Improper Integrals of Unbounded Functions

In this section, we will consider a function $f[a,b) \to \mathbb{R}$ such that for all $c \in (a,b)$ it is integrable on $[a,c]$ and unbounded on (c,b). The case of a function $f(a,b] \to \mathbb{R}$, which is unbounded near a can be considered similarly. We set

$$\varphi(z) = \int_a^z f(x)dx, \quad z \in (a,b).$$

Definition 18.3. The finite limit

$$\lim_{z\to b-} \varphi(z)dz = \lim_{z\to b-} \int_a^z f(x)dx \tag{33}$$

is called the **improper integral** of f over $[a,b)$ and is denoted by

$$\int_a^b f(x)dx. \tag{34}$$

In this case, we will say that the improper integral (34) **converges**. If limit (33) does not exist or is infinite, then the improper integral (34) is said to be **divergent**.

Exercise 18.3. The improper integral $\int_0^1 \frac{dx}{\sqrt{1-x}}$ converges, since

$$\int_0^z \frac{dx}{\sqrt{1-x}} = -\int_0^z \frac{d(1-x)}{\sqrt{1-x}} = -\int_0^z (1-x)^{-\frac{1}{2}} d(1-x) = -2(1-x)^{\frac{1}{2}}\Big|_0^z = -2(1-z)^{\frac{1}{2}}+2 \to 2, \quad z \to 1-.$$

Example 18.10. The improper integral $\int_0^1 \ln x dx$ converges, since

$$\int_z^1 \ln x dx = (x\ln x - x)\Big|_z^1 = -1 - z\ln z + z \to -1, \quad z \to 0+,$$

by Exeample 13.1 b). For the computation of an antiderivative of $\ln x$ see Example 15.7.

Exercise 18.4. Prove that the improper integral $\int_a^b \frac{dx}{(b-x)^p}$, $p > 0$, converges for $p < 1$ and diverges for $p \geq 1$.

The following properties of improper integrals of unbounded functions can be proved similarly as properties of improper integrals over unbounded intervals.

1. Let $f(x) \geq 0$, $x \in [a,b)$. The improper integral $\int_a^b f(x)dx$ converges iff there exists $C \in \mathbb{R}$ such that $\int_a^z f(x)dx \leq C$ for all $z \in [a,b)$.

2. Let $0 \leq f(x) \leq g(x)$, $x \in [a,b)$. If the improper integral $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.

3. If for some $p > 0$ and $0 < C < +\infty$ $f(x) \sim \frac{C}{(b-x)^p}$, $x \to b-$, that is, $\lim\limits_{x\to b-} (b-x)^p f(x) = C$, then the integral $\int_a^b f(x)dx$ converges for $p < 1$ and diverges for $p \geq 1$.

4. If a function f has an antiderivative F on $[a,b)$ and there exists a limit $F(b-) := \lim\limits_{x\to b-} F(x)$, then $\int_a^b f(x)dx = F(b-) - F(a)$.

Exercise 18.5. Prove the convergence of the integral $\int_0^1 \frac{dx}{\sqrt[3]{1-x^2}}$.

19 Lecture 19 – Series

19.1 Definition and Elementary Properties of Series

Let $(a_n)_{n\geq 1}$ be a sequence of the real numbers. For each $n \in \mathbb{N}$ we set

$$s_n := a_1 + a_2 + \ldots + a_n.$$

Definition 19.1. The sequence $(s_n)_{n\geq 1}$ is called a **series** and is denoted by

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n. \tag{35}$$

Elements of the sequence $(s_n)_{n\geq 1}$ are called the **partial sums of series** (35). If the sequence $(s_n)_{n\geq 1}$ converges to a real number s, then series (35) is said to be **convergent**, and the number s is called the **sum of series** (35) and is denoted by

$$s = \sum_{n=1}^{\infty} a_n.$$

If the sequence $(s_n)_{n\geq 1}$ has no a finite limit, then series (35) is said to be **divergent**.

Theorem 19.1. *If a series $\sum\limits_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$, $n \to \infty$.*

Proof. Indeed, since $a_n = s_n - s_{n-1}$ for all $n \geq 2$, we have $a_n = s_n - s_{n-1} \to s - s = 0$, $n \to \infty$. $\square$

Exercise 19.1. Prove that the convergence of a series $\sum\limits_{n=1}^{\infty} a_n$ implies that $a_n + a_{n+1} + \ldots + a_{2n} \to 0$, $n \to \infty$.

Example 19.1. The series

$$1 + 1 + \ldots + 1 + \ldots$$

and

$$1 - 1 + 1 - 1 + \ldots + (-1)^{n+1} + \ldots$$

diverge, since their terms $a_n = 1$, $n \geq 1$, for the first series and $a_n = (-1)^{n+1}$, $n \geq 1$, for the second one do not converge to 0.

Example 19.2 (Geometric series)**.** For $q \in \mathbb{R}$ the series

$$1 + q + q^2 + \ldots + q^n + \ldots = \sum_{n=1}^{\infty} q^{n-1} = \sum_{n=0}^{\infty} q^n \tag{36}$$

is called the **geometric series**. Its partial sums $s_n = 1 + q + q^2 + \ldots + q^{n-1}$ are equal to $\frac{1-q^n}{1-q}$ for all $n \geq 1$. Thus, series (36) converges and

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

for $|q| < 1$. If $|q| \geq 1$, then the geometric series diverges.

Example 19.3 (Harmonic series)**.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

diverges. In order to prove this, we assume that the series converges and its sum equal s. Then $s_{2n} - s_n \to s - s = 0$, $n \to \infty$. But for each $n \geq 1$

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n} \geq n\frac{1}{2n} = \frac{1}{2}$$

that contradicts the convergence of $(s_{2n} - s_n)_{n\geq 1}$ to 0.

Exercise 19.2. Show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots = 1$.

Theorem 19.2. *Let series* $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ *converge and* $c \in \mathbb{R}$*. Then the series* $\sum\limits_{n=1}^{\infty} ca_n$, $\sum\limits_{n=1}^{\infty} (a_n + b_n)$ *also converge and* $\sum\limits_{n=1}^{\infty} ca_n = c\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$.

Proof. The proof of the statement immediately follows from Definition 19.1 and Theorem 3.8. Indeed,

$$\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} \sum_{k=1}^{n} ca_n = c \lim_{n\to\infty} \sum_{k=1}^{n} a_n = c\sum_{n=1}^{\infty} a_n$$

and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \lim_{n\to\infty} \sum_{k=1}^{n}(a_k + b_k) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

□

Theorem 19.3 (Cauchy criterion)**.** *A series* $\sum\limits_{n=1}^{\infty} a_n$ *converges iff*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall p \in \mathbb{N}: \quad |a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon.$$

Proof. We remark that $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n)_{n\geq 1}$ converges. Thus, using Theorem 5.3, we have that the convergence of $(s_n)_{n\geq 1}$ is equivalent to

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall p \in \mathbb{N}: \quad |s_{n+p} - s_n| < \varepsilon.$$

Hence, the statement follows from the equality $s_{n+p} - s_n = a_{n+1} + a_{n+2} + \ldots + a_{n+p}$. □

19.2 Series with Positive Terms

Theorem 19.4. *Let terms of a series* $\sum\limits_{n=1}^{\infty} a_n$ *are non-negative, that is,* $a_n \geq 0$ *for all* $n \geq 1$*. The series* $\sum\limits_{n=1}^{\infty} a_n$ *converges iff the sequence of its partial sums* $(s_n)_{n\geq 1}$ *is bounded.*

Proof. We note that the sequence $(s_n)_{n\geq 1}$ increases. Hence, the statement follows from theorems 4.1 and 3.5. □

Theorem 19.5 (Integral criterion for convergence)**.** *Let $f : [1, +\infty) \to \mathbb{R}$ be a non-negative decreasing function and $f(n) = a_n$ for all $n \geq 1$. Then the convergence of the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ is equivalent to the convergence of the improper integral $\int_1^{+\infty} f(x)dx$.*

Proof. Using the monotonicity of the function f and Corollary 16.1, we can estimate for each $n \geq 2$

$$a_n = f(n) \leq \int_{n-1}^{n} f(x)dx \leq f(n-1) = a_{n-1}.$$

So, if the improper integral $\int_1^{+\infty} f(x)dx$ converges, then for every $n \geq 1$

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k = a_1 + \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx = a_1 + \int_1^n f(x)dx \leq a_1 + \int_1^{+\infty} f(x)dx.$$

Hence, the sequence $(s_n)_{n\geq 1}$ is bounded and, consequently, the series $\sum_{n=1}^{\infty} a_n$ converges, by Theorem 19.4.

Next, if the series $\sum_{n=1}^{\infty} a_n$ converges, then for each $z > 1$

$$\varphi(z) = \int_1^z f(x)dx \leq \int_1^n f(x)dx = \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx \leq \sum_{k=2}^{n} a_{k-1} = \sum_{k=1}^{n-1} a_k \leq \sum_{k=1}^{\infty} a_k =: C,$$

where $n := \lfloor z \rfloor + 1$. Thus, the integral $\int_1^{+\infty} f(x)dx$ converges, by Theorem 18.2. □

Example 19.4. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$, converges for $p > 1$ and diverges for $p \leq 1$. This follows from Theorem 19.5 and the fact that the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$ (see Example 18.3).

Exercise 19.3. Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 19.6 (Comparison criterion)**.** *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series.*

(i) If $0 \leq a_n \leq b_n$, $n \geq 1$, then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.

(ii) Let $a_n > 0$, $b_n > 0$, $n \geq 1$, and there exists a limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C, \quad 0 \leq C \leq +\infty.$$

If $C < +\infty$, then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$. If $C > 0$, then the divergence of $\sum_{n=1}^{\infty} b_n$ implies the divergence of $\sum_{n=1}^{\infty} a_n$. Consequently, the convergences of both series are equivalent in the case $0 < C < +\infty$.

(iii) If $a_n > 0$, $b_n > 0$ and $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq 1$, then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.

Proof. We prove only (i). We estimate for each $n \geq 1$

$$0 \leq s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{k=1}^{+\infty} b_k.$$

Thus, the sequence $(s_n)_{n\geq 1}$ is bounded that implies the convergence of the series $\sum_{n=1}^{\infty} a_n$, according to Theorem 19.4. □

Exercise 19.4. Prove Theorem 19.6 (ii), (iii). (*Hint:* To prove (iii), note that $\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} \leq \dots \frac{a_1}{b_1}$)

Remark 19.1. We will write, $a_n \sim b_n$, $n \to \infty$, if $\frac{a_n}{b_n} \to 1$, $n \to \infty$. So, Theorem 19.6 (ii) implies that the convergence of $\sum_{n=1}^{\infty} a_n$ is equivalent to the convergence of $\sum_{n=1}^{\infty} b_n$, if $a_n \sim b_n$, $n \to \infty$.

Example 19.5. The series $\sum_{n=1}^{\infty} n \sin \frac{1}{n^3}$ converges, since $n \sin \frac{1}{n^3} \sim \frac{n}{n^3} = \frac{1}{n^2}$, $n \to \infty$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (see Example 19.4).

Exercise 19.5. Prove the convergence of the following series:
a) $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$; d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)^2$; f) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$; g) $\sum_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$; h) $\sum_{n=2}^{\infty} \left(\ln \frac{n}{n-1} - \frac{1}{n}\right)$.

20 Lecture 20 – Series with Arbitrary Terms

20.1 Root and Ratio Tests for Series with Positive Terms

Theorem 20.1 (Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$, $n \geq 1$, and let there exist a limit $r := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$.*

Proof. Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. We take $q \in (r, 1)$. Then there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} < q = \frac{q^{n+1}}{q^n}$ for all $n \geq N$. Thus, using Theorem 19.6 (iii) and the convergence of the geometric series for $|q| < 1$ (see Example 19.2), we have that the series $a_N + a_{N+1} + \ldots = \sum_{n=N}^{\infty} a_n$ converges and, hence, $\sum_{n=1}^{\infty} a_n$ also converges.

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$, then there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq N$. Consequently, $a_n < a_{n+1}$ for all $n \geq N$. So, we obtain that $0 < a_N < a_{N+1} < a_{N+2} < \ldots$. This implies that $a_n \not\to 0$, $n \to \infty$. Hence, the series $\sum_{n=1}^{\infty} a_n$ diverges, according to Theorem 19.1. $\square$

Example 20.1. The series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all $x > 0$. Indeed,

$$r = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1.$$

Exercise 20.1. Prove that the following series converge:
a) $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$; b) $\sum_{n=1}^{\infty} \frac{7^n (n!)^2}{n^{2n}}$.

Theorem 20.2 (Root Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$, $n \geq 1$, and let $r := \varlimsup_{n\to\infty} \sqrt[n]{a_n}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$.*

Proof. Let $\varlimsup_{n\to\infty} \sqrt[n]{a_n} = r < 1$ and let q be a number from $(r, 1)$. Then there exists $N \in \mathbb{N}$ such that $\sqrt[n]{a_n} < q$ for all $n \geq N$. So, $a_n < q^n$ for all $n \geq N$. By Theorem 19.6 (i), the series $\sum_{n=N}^{\infty} a_n$ converges due to the convergence of the geometric series $\sum_{n=1}^{\infty} q^n$ for $|q| < 1$.

If $\varlimsup_{n\to\infty} \sqrt[n]{a_n} = r > 1$, then there exists a subsequence $(\sqrt[n_k]{a_{n_k}})_{k\geq 1}$ such that $\sqrt[n_k]{a_{n_k}} \to r$, $k \to \infty$, since the upper limit is also a subsequential limit (see Theorem 5.1). Hence, there exists $K \in \mathbb{N}$ such that $\sqrt[n_k]{a_{n_k}} > 1$ for all $k \geq K$. Consequently, $a_{n_k} > 1$ for all $k \geq K$. This implies that $a_n \not\to 0$, $n \to \infty$, since the sequence $(a_n)_{n\geq 1}$ has an subsequence which does not converge to 0. $\square$

Example 20.2. The series $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges, since $r = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^3}{2} = \frac{1}{2} < 1$.

Exercise 20.2. Prove that the following series converge:
a) $\sum_{n=1}^{\infty} \frac{3^n}{(\ln n)^n}$; b) $\sum_{n=1}^{\infty} \frac{n^{n^2} 2^n}{(n+1)^{n^2}}$.

20.2 Series with Arbitrary Terms

20.2.1 Absolute and Conditional Convergence

Definition 20.1. A series

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \tag{37}$$

is said to be **absolutely convergent**, if the series

$$|a_1| + |a_2| + \ldots + |a_n| + \ldots = \sum_{n=1}^{\infty} |a_n| \tag{38}$$

converges. If series (38) diverges but (37) converges, then series (37) is called **conditionally convergent**.

Theorem 20.3. *If a series $\sum\limits_{n=1}^{\infty} a_n$ absolutely converges, then it converges and*

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Proof. We note that terms of the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|) \tag{39}$$

satisfy the property $0 \leq a_n + |a_n| \leq 2|a_n|$, $n \geq 1$. Thus, series (39) converges due to the convergence of the series $\sum_{n=1}^{\infty} 2|a_n|$ and Theorem 19.6 (i). Summing series (39) with the series $\sum\limits_{n=1}^{\infty} (-|a_n|)$, which also converges, we have that the series $\sum\limits_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum\limits_{n=1}^{\infty} a_n$ converges, by Theorem 19.2. $\square$

We set $a_n^+ := \max\{a_n, 0\}$ and $a_n^- := -\min\{a_n, 0\}$, $n \geq 1$. Then $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$ for all $n \geq 1$.

Theorem 20.4. *A series $\sum\limits_{n=1}^{\infty} a_n$ absolutely converges iff the series $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ converge. Moreover,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-, \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.$$

Exercise 20.3. Prove Theorem 20.4. (*Hint:* Use the equalities $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$)

Corollary 20.1. *Let a series $\sum\limits_{n=1}^{\infty} a_n$ conditionally converge. Then the series $\sum\limits_{n=1}^{\infty} a_n^+$ and $\sum\limits_{n=1}^{\infty} a_n^-$ diverge.*

Proof. We assume that $\sum\limits_{n=1}^{\infty} a_n^+$ converges. Using Theorem 19.2, we obtain that the series $\sum\limits_{n=1}^{\infty} a_n^+ - \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (a_n^+ - a_n) = \sum\limits_{n=1}^{\infty} a_n^-$ also converges. But then, by Theorem 20.4, the series $\sum\limits_{n=1}^{\infty} a_n$ absolutely converges that contradicts the assumption of the corollary. $\square$

Exercise 20.4. Show that the following series absolutely converge:
a) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$; b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$.

20.2.2 Dirichlet's and Abel's Tests

Theorem 20.5 (Dirichlet's test). *Let sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the following properties:*

1) $(a_n)_{n\geq 1}$ is a monotone sequence;

2) $a_n \to 0$, $n \to \infty$;

3) there exists $C > 0$ such that $\left|\sum\limits_{k=1}^{n} b_n\right| \leq C$ for all $n \geq 1$.

Then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

Proof. For proof of the theorem see Theorem 3.42 [2]. □

Example 20.3. The series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
conditionally converges. Indeed, taking $a_n := \frac{1}{n}$ and $b_n := (-1)^{n+1}$, $n \geq 1$, we can see that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the conditions of Dirichlet's test (condition 3) is satisfied with $C = 1$). Thus, the series $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. But the series $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (see Example 19.3).

Example 20.4. The series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n}$ converges. To prove this, we take $a_n := \frac{1}{n}$, $b_n := \sin n$, $n \geq 1$. The sequence $(a_n)_{n\geq 1}$ is monotone and converges to 0. Next, we compute for $n \geq 1$
$$\sum_{k=1}^{n} \sin k = \frac{1}{\sin\frac{1}{2}} \sum_{k=1}^{n} \sin k \cdot \sin\frac{1}{2} = \frac{1}{2\sin\frac{1}{2}} \sum_{k=1}^{n} \left(\cos\left(k - \frac{1}{2}\right) - \cos\left(k + \frac{1}{2}\right)\right)$$
$$= \frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)\right).$$
Hence,
$$\left|\sum_{k=1}^{n} \sin k\right| = \left|\frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)\right)\right| \leq \frac{1}{\sin\frac{1}{2}}, \quad n \geq 1,$$
and, consequently, condition 3) of Dirichlet's test is satisfied. Hence, the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n}$ converges.

Exercise 20.5. Show that the series $\sum\limits_{n=1}^{\infty} \frac{|\sin n|}{n}$ diverges. (*Hint:* Use the equality $|\sin a| \geq \sin^2 a = \frac{1-\cos 2a}{2}$ and then show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverges and $\sum\limits_{n=1}^{\infty} \frac{\cos 2n}{2n}$ converges).

Exercise 20.6. Prove the convergence of the following sequences:
a) $\sum\limits_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{1}{\sqrt{n}}$; b) $\sum\limits_{n=1}^{\infty}\frac{\sin 3n}{\sqrt{n}}$; c) $\sum\limits_{n=1}^{\infty}\frac{\cos n}{n}$.

Corollary 20.2 (Leibniz's test)**.** *Let a sequence $(a_n)_{n\geq 1}$ satisfy the following properties:*

1) $0 \leq a_{n+1} \leq a_n$ *for* $n \geq 1$*;*

2) $a_n \to 0$, $n \to \infty$.

Then the series

$$a_1 - a_2 + a_3 - a_4 + \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges.

Proof. The corollary follows from Dirichlet's test taking $b_n := (-1)^{n+1}$, $n \geq 1$. □

Example 20.5. The series $\sum\limits_{n=1}^{\infty}(-1)^n \ln\frac{n+1}{n}$ converges due to Leibniz's test, since the sequence $(a_n)_{n\geq 1} = \left(\ln\frac{n+1}{n}\right)_{n\geq 1}$ decreases to 0. Indeed, $a_n = \ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n}\right) > \ln\left(1+\frac{1}{n+1}\right) = a_{n+1} > 0$ because $1+\frac{1}{n} > 1+\frac{1}{n+1}$ and ln is an increasing function.

Theorem 20.6 (Abel's test)**.** *Let sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the following properties:*

1) $(a_n)_{n\geq 1}$ *is monotone;*

2) $(a_n)_{n\geq 1}$ *is bounded;*

3) the series $\sum\limits_{n=1}^{\infty} b_n$ *converges.*

Then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ *converges.*

Proof. In order to prove Abel's test, we are going to use Dirichlet's test. Since the sequence $(a_n)_{n\geq 1}$ is monotone and bounded, it has a limit $a \in \mathbb{R}$, by Theorem 4.1. Applying Dirichlet's test to the sequences $(a_n - a)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we get that the series $\sum\limits_{n=1}^{\infty}(a_n - a)b_n$ convergence. Thus, the series $\sum\limits_{n=1}^{\infty}(a_n - a)b_n + a\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} a_n b_n$ is convergent due to the convergence of $\sum\limits_{n=1}^{\infty} b_n$ and Theorem 19.2. □

Exercise 20.7. Prove the convergence of the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\arctan n}{\sqrt{n}}$.

20.2.3 Permutation of Terms of a Series

Definition 20.2. A bijection $\sigma : \mathbb{N} \to \mathbb{N}$ is called a **permutation**.

In this section, we will study series obtained from permutation of their terms, i.e.

$$a_{\sigma(1)} + a_{\sigma(2)} + \ldots + a_{\sigma(n)} + \ldots = \sum_{n=1}^{\infty} a_{\sigma(n)}. \tag{40}$$

According to Example 20.3, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Moreover, one can show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

But it turns out that a rearrangement of the series gives other finite sum, e.g.

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots = \frac{3}{2} \ln 2.$$

So, we see that there exist series whose sums depend on order of their terms.

Theorem 20.7. *Let $\sum\limits_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then for every permutation σ the permuted series (40) converges to the same sum, i.e.*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n.$$

Proof. For proof of the theorem see Theorem 3.55 [2]. □

Theorem 20.8 (Riemann rearrangement theorem)**.** *Let $\sum\limits_{n=1}^{\infty} a_n$ be conditionally convergent and $s \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then there exists a permutation σ such that*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = s.$$

Proof. For proof of the theorem in more general setting see Theorem 3.54 [2]. □

21 Lecture 21 – Complex Numbers

21.1 Definition and Basic Properties

We recall that the equation $az^2 + bz + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, has solutions if and only if $D := b^2 - 4ac \geq 0$ which can be computed by the formula $z_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$. Thus, e.g. the equation

$$z^2 - 2z + 2 = 0 \tag{41}$$

has no solutions, since $D = 4 - 4 \cdot 1 \cdot 2 = -4 < 0$. However, we can formally take $z_1 := \frac{2+\sqrt{-4}}{2}$ and $z_2 := \frac{2-\sqrt{-4}}{2}$. If $\sqrt{-4}$ was a number such that $(\sqrt{-4})^2 = -4$, then a simple computation would show that z_1 and z_2 are solutions to (41). We are going to give this idea the rigorous meaning, namely, we extend the set of real numbers and later show that any polynomial equation has solutions in that class of numbers.

We consider a new symbol i and postulate that $i = \sqrt{-1}$, that is, $i^2 = -1$.

Definition 21.1. A number $z = x + yi$, where $x, y \in \mathbb{R}$ and $i^2 = -1$, is called a **complex number**. The number x is called the **real part** of z and is denoted by $x = \operatorname{Re} z$. The number y is called the **imaginary part** of z and is denoted by $y = \operatorname{Im} z$.

The set of all complex numbers is denoted by $\mathbb{C}$, i.e $\mathbb{C} = \{z = x + yi : \ x, y \in \mathbb{R}\}$.

Remark 21.1. If $\operatorname{Im} z = 0$, that is, $z = x + 0i$, then we will identify z with the real number x and write $z \in \mathbb{R}$.

Next, we introduce operations on complex numbers.

Addition and subtraction of complex numbers: For $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ from $\mathbb{C}$ we define

$$z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2)i. \tag{42}$$

Example 21.1. a) $(1-2i)+(2+4i) = (1+2)+(-2+4)i = 3+2i$; b) $i+(2-2i) = (0+2)+(1-2)i = 2-i$.

Exercise 21.1. Prove that $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

Multiplication and division of complex numbers: For $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ from $\mathbb{C}$ we define

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2)i, \tag{43}$$

$$z_1 / z_2 = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} i, \quad z_2 \neq 0. \tag{44}$$

Remark 21.2. The multiplication rule is motivated by the multiplication rule of polynomials and the equality $i^2 = -1$. Indeed, multiplying $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ as two polynomials, we have

$$z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2 = (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2)i.$$

Remark 21.3. The division of two complex numbers is motivated by the following observation: $(x_2 + y_2 i)(x_2 - y_2 i) = x_2^2 + y_2^2$. Thus, for $z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i) \cdot (x_2 - y_2 i)}{(x_2 + y_2 i) \cdot (x_2 - y_2 i)} = \frac{(x_1 x_2 + y_1 y_2) + (y_1 x_2 - x_1 y_2)i}{x_2^2 + y_2^2}.$$

Example 21.2. a) $(2-i)(1+3i) = 2+6i-i-3i^2 = 2+5i-3\cdot(-1) = 5+5i$;
b) $\frac{1-i}{1-2i} = \frac{(1-i)(1+2i)}{(1-2i)(1+2i)} = \frac{1+2-i+2i}{1^2+2^2} = \frac{3+i}{5} = \frac{3}{5}+\frac{1}{5}i$; c) $\frac{1}{i} = \frac{1\cdot(-i)}{i\cdot(-i)} = -i$.

Exercise 21.2. Express the following complex numbers in the form $x+yi$ for $x,y\in\mathbb{R}$:
a) $(-2+3i)(1+i)$; b) $(\sqrt{2}-i)^2$; c) $\frac{3-i}{2+2i}$; d) $\frac{i}{(1-i)^2}$.

Exercise 21.3. Show that for $z_k = x_k + y_k i \in \mathbb{C}$, $k=1,2,3$,
a) $z_1\cdot z_2 = z_2\cdot z_1$; b) $(z_1\cdot z_2)\cdot z_3 = z_1\cdot(z_2\cdot z_3)$; c) $z_1\cdot(z_2+z_3) = z_1\cdot z_2 + z_1\cdot z_3$;
d) $z_1\cdot z_2\in\mathbb{R}$ if $z_1,z_2\in\mathbb{R}$.

21.2 Complex Conjugate and Absolute Value of Complex Numbers

Definition 21.2. The number $\overline{z} := x-yi$ is sad to be the **conjugate** of a complex number $z = x+yi\in\mathbb{C}$.

Theorem 21.1. *Let $z_1, z_2, z\in\mathbb{C}$. Then the following equalities hold:*

a) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$;

b) $\overline{z_1\cdot z_2} = \overline{z_1}\cdot\overline{z_2}$;

c) $z+\overline{z} = 2\operatorname{Re} z$ *and* $z-\overline{z} = 2i\operatorname{Im} z$;

d) $z\cdot\overline{z} = \operatorname{Re}^2 z + \operatorname{Im}^2 z = |z|^2$ *(for the definition of $|z|$ see Definition 21.3 below);*

e) $\frac{1}{z} = \frac{\overline{z}}{\operatorname{Re}^2 z+\operatorname{Im}^2 z} = \frac{\overline{z}}{|z|^2}$, $z\neq 0$;

f) $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{z_1}/\overline{z_2}$, $z_2\neq 0$;

g) $\overline{\overline{z}} = z$.

Proof. Equalities a), b), c), d), f) and g) immediately follow from the definition of complex conjugate and (42), (43), (44). Equality e) follows from d). Indeed, multiplying the nominator and denominator of $\frac{1}{z}$ by $\overline{z}$ and using d), we have

$$\frac{1}{z} = \frac{\overline{z}}{z\cdot\overline{z}} = \frac{\overline{z}}{\operatorname{Re}^2 z+\operatorname{Im}^2 z} = \frac{\overline{z}}{|z|^2}.$$

□

Exercise 21.4. Prove equalities a)-d), f) and g).

Definition 21.3. The number $|z| = \sqrt{x^2+y^2} = \sqrt{\operatorname{Re}^2 z+\operatorname{Im}^2 z}$ is called the **absolute value** of a complex number $z = x+yi\in\mathbb{C}$.

Theorem 21.2. *Let $z_1, z_2, z\in\mathbb{C}$. Then the absolute value satisfies the following properties:*

a) $|z| = \sqrt{z\cdot\overline{z}}$;

b) $|z|>0$ *unless* $z=0$;

c) $|z| = |\overline{z}|$;

d) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$;

e) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$;

f) $|\operatorname{Re} z| \leq |z|$, $|\operatorname{Im} z| \leq |z|$;

g) $|z_1 + z_2| \leq |z_1| + |z_2|$.

Proof. Properties a), b), c) and f) easily follow from Definition 21.3. To show d), we compute for $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$$\begin{aligned} |z_1 \cdot z_2|^2 &= \operatorname{Re}^2(z_1 \cdot z_2) + \operatorname{Im}^2(z_1 \cdot z_2) \overset{(43)}{=} (x_1x_2 - y_1y_2)^2 + (y_1x_2 + x_1y_2)^2 \\ &= x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + y_1^2x_2^2 + 2x_1x_2y_1y_2 + x_1^2y_2^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_1^2y_2^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2 \cdot |z_2|^2. \end{aligned}$$

Thus, d) holds. Next, we prove e). So, for $z_2 \neq 0$ we have

$$\left|\frac{z_1}{z_2}\right| \overset{\text{Thm 21.1 e)}}{=} \left|\frac{z_1 \cdot \overline{z_2}}{|z_2|^2}\right| \overset{\text{d)}}{=} \frac{|z_1| \cdot |\overline{z_2}|}{|z_2|^2} \overset{\text{c)}}{=} \frac{|z_1|}{|z_2|}.$$

Now, we check triangle inequality g):

$$\begin{aligned} |z_1 + z_2|^2 &\overset{\text{a)}}{=} (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \overset{\text{Thm 21.1 a)}}{=} (z_1 + z_2) \cdot (\overline{z_1} + \overline{z_2}) = z_1 \cdot \overline{z_1} + z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1} + z_2 \cdot \overline{z_2} \\ &\overset{\text{a) \& Thm 21.1 g)}}{=} |z_1|^2 + z_1 \cdot \overline{z_2} + \overline{\overline{\overline{z_2} \cdot z_1}} + |z_2|^2 \overset{\text{Thm 21.1 b), g)}}{=} |z_1|^2 + z_1 \cdot \overline{z_2} + \overline{\overline{z_2} \cdot z_1} + |z_2|^2 \\ &\overset{\text{Thm 21.1 c)}}{=} |z_1|^2 + 2\operatorname{Re}(z_1 \cdot \overline{z_2}) + |z_2|^2 \overset{\text{f)}}{\leq} |z_1|^2 + 2|z_1 \cdot \overline{z_2}| + |z_2|^2 \\ &\overset{\text{c), d)}}{=} |z_1|^2 + 2|z_1| \cdot |z_2| + |z_2|^2 = (|z_1| + |z_2|)^2. \end{aligned}$$

□

Exercise 21.5. Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$.

Exercise 21.6. Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$ such that either $|z| = 1$ or $|w| = 1$. Prove that

$$\left|\frac{z - w}{1 - \overline{z}w}\right| = 1.$$

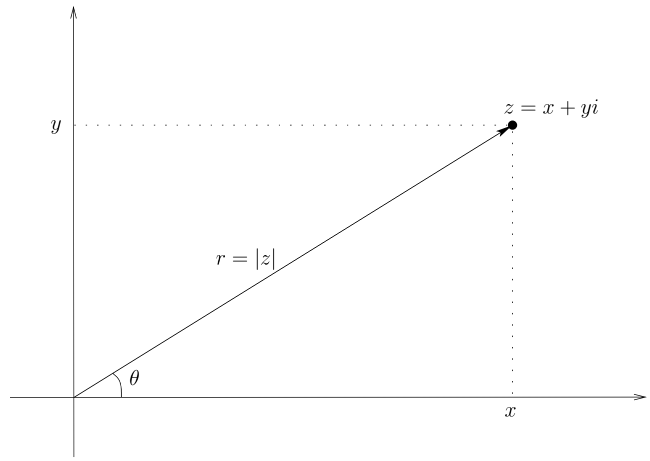
Exercise 21.7. Let z be a complex number with $|z| < \frac{1}{2}$. Show that

$$|(1 + i)z^2 + iz| < \frac{3}{4}.$$

Exercise 21.8. Solve the following equations:
a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

21.3 Complex Plane and Polar form of complex numbers

In this section, we will identify complex numbers with points of a plane which we will call the **complex plane**. So, we will identify a number $z = x + iy \in \mathbb{C}$ with the point (x, y) of $\mathbb{R}^2$. The point (x, y) is called the **rectangular coordinates** of z. We will also identify z with its polar coordinates (r, θ), where r is the length of the vector $\overline{(0,0),(x,y)}$ and equals the absolute volume of z, and θ is the angle between the positive real axis and the vector $\overline{(0,0),(x,y)}$. The angle θ is called the **argument** of z and is denoted by $\theta = \arg z$. We remark that for $z \neq 0$ the argument θ is uniquely determined up to integer multiples of 2π.



By the definition of sin and cos, it is easy to see that

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

Consequently, we can write the number $z = x + yi$ in the form

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z|$ and $\theta = \arg z$. This form is called the **polar form** of the complex number z.

Example 21.3. Let us write the number $z = 1 + i$ in the polar form. For this we compute $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. The argument θ can be found from the equalities $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$. Thus, $\theta = \frac{\pi}{4}$. Hence, $z = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

Exercise 21.9. Write the following complex numbers in the polar form:
a) i; b) $1 - i$; c) $-1 + \sqrt{3}i$; d) $-2 - 2i$.

It turns out, that the polar form of complex numbers is convenient for the multiplication and division.

Theorem 21.3. *Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers, written in the polar form. Then*

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \tag{45}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)), \quad z_2 \neq 0. \tag{46}$$

Proof. The equalities immediately follows from (43), (44) and the formulas

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2,$$
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2.$$

□

Remark 21.4. Setting

$$e^{i\theta} = \cos\theta + i\sin\theta, \text{ [1]}$$

equalities (45) and (46) can be rewritten as follows

$$z_1 \cdot z_2 = \left(r_1 e^{i\theta_1}\right) \cdot \left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1 + \theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad z_2 \neq 0.$$

[1] This formula is called Euler's formula and can be obtain from the Taylor expansion of functions of complex argument.

Exercise 21.10. Simplify the expression $\frac{\cos\theta+i\sin\theta}{\cos\varphi-i\sin\varphi}$.

Exercise 21.11. Compute $\frac{(1-\sqrt{3}i)(\cos\theta+i\sin\theta)}{2(1-i)(\cos\theta-i\sin\theta)}$.

Corollary 21.1 (De Moivre's formula)**.** *Let $z = r(\cos\theta + i\sin\theta) \neq 0$ be a complex number. Then for each $n \in \mathbb{Z}$*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof. The corollary immediately follows from Theorem 21.3. □

Exercise 21.12. Compute: a) $(1+i)^{25}$; b) $(\sqrt{3}-3i)^{15}$; c) $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}$; d) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$.

21.4 Roots of Complex Numbers

Let $n \in \mathbb{N}$ be fixed.

Definition 21.4. A complex number w is called an n**-th root** of $z \in \mathbb{C}$ if $w^n = z$.

Theorem 21.4. *Let $z = r(\cos\theta + i\sin\theta) \neq 0$ be a complex number. Then z has n different n-th roots given by the formula*

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

where $\sqrt[n]{r}$ is the usual n-th root of the positive real number r.

Proof. Let $w = \rho(\cos\varphi + i\sin\varphi)$ be a complex number written in the polar form such that $w^n = z$. Then

$$w^n = \rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta),$$

by Corollary 21.1. Thus, $\rho^n = r$ and $n\varphi = \theta + 2\pi k$, $k \in \mathbb{Z}$. This implies that $\rho = \sqrt[n]{r}$ and $\varphi = \frac{\theta+2\pi k}{n}$, $k \in \mathbb{Z}$. So, we obtain that the numbers

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k \in \mathbb{Z},$$

are n-th roots of z. By the periodicity of sin and cos, one can see that there are only n different w_k, $k = 0, \ldots, n-1$. □

Example 21.4. Let us compute 4-th root of $z = -1$. First we write -1 in the polar form: $-1 = 1(\cos\pi + i\sin\pi)$. Then $w_k = \cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}$, $k = 0, 1, 2, 3$, are 4-th roots of -1, by Theorem 21.4. Thus, $w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $w_1 = \cos\frac{\pi+2\pi}{4} + i\sin\frac{\pi+2\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $w_2 = \cos\frac{\pi+4\pi}{4} + i\sin\frac{\pi+4\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, $w_3 = \cos\frac{\pi+6\pi}{4} + i\sin\frac{\pi+6\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Remark 21.5. The n-th roots of $z \neq 0$ form a regular n-gon in the complex plane with center 0. The vertices of this n-gon lie on the circle with center 0 and the radius $\sqrt[n]{|z|}$.

Exercise 21.13. Solve the following equations:
a) $z^5 - 2 = 0$; b) $z^4 + i = 0$; c) $z^3 - 4i = 0$.

Exercise 21.14. Compute a) 6-th roots of $\frac{1-i}{\sqrt{3}+i}$; b) 8-th roots of $\frac{1+i}{\sqrt{3}-i}$.

22 Lecture 22 – Fundamental Theorem of Algebra and Definition of Vector Space

22.1 Fundamental Theorem of Algebra

For more details see [3, Chapter 4].

Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n$ be a complex numbers. We set

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \quad z \in \mathbb{C}.$$

The function f is called a **polynomial function**. The numbers $a_0, a_1, \ldots, a_n$ are coefficients of the polynomial f. If $a_n \neq 0$, then the number n is called the **degree** of f and is denoted by $\deg f := n$.

In this section, we are going to prove that the equation $f(z) = 0$ as at most n solutions. The next theorem is called the fundamental theorem of algebra and we formulate it without prof. The prof will be given at the course of complex analysis using Liouville's theorem.

Theorem 22.1 (Fundamental theorem of algebra). *For every $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$, the equation*

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

has at least one solution in $\mathbb{C}$.

Theorem 22.2. *Let f be a polynomial function of degree $n \in \mathbb{N}$. Then*

1) for any $w \in \mathbb{C}$ we have that $f(w) = 0$ iff there exists a polynomial g of degree $n-1$ such that

$$f(z) = (z - w) g(z), \quad z \in \mathbb{C};$$

2) there exist at most n distinct complex solutions of the polynomial equation $f(z) = 0$;

3) there exist $w_1, \ldots, w_n \in \mathbb{C}$ (not necessary distinct) such that

$$f(z) = a_n (z - w_1) \ldots (z - w_n), \quad z \in \mathbb{C},$$

where a_n denotes the coefficient about z^n.

Proof. To prove 1), we first recall that for each $a, b \in \mathbb{R}$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + \ldots + a b^{n-2} + b^{n-1}) = (a - b) \sum_{k=1}^{n} a^{n-k} b^{k-1}.$$

Let $f(w) = 0$. Then

$$\begin{aligned}
f(z) = f(z) - f(w) &= a_n (z^n - w^n) + a_{n-1} (z^{n-1} - w^{n-1}) + \ldots + a_1 (z - w) \\
&= a_n (z - w) \sum_{k=1}^{n} z^{n-k} w^{k-1} + a_{n-1} (z - w) \sum_{k=1}^{n-1} z^{n-k-1} w^{k-1} + \ldots + a_1 (z - w) \\
&= (z - w) \left(a_n \sum_{k=1}^{n} z^{n-k} w^{k-1} + a_{n-1} \sum_{k=1}^{n-1} z^{n-k-1} w^{k-1} + \ldots + a_1 \right) = (z - w) g(z),
\end{aligned}$$

where $g(z) = a_n \sum_{k=1}^{n} z^{n-k} w^{k-1} + a_{n-1} \sum_{k=1}^{n-1} z^{n-k-1} w^{k-1} + \ldots + a_1$, $z \in \mathbb{C}$, is a polynomial of degree $n-1$.

Now, we prove 3). Let w_1 be a complex number such that $f(w_1) = 0$, which exists according to Theorem 22.1. Applying the part 1) of Theorem 22.2, we have that there exists a polynomial g_1 such that $f(z) = (z - w_1)g_1(z)$, $z \in \mathbb{C}$. Next, using Theorem 22.1 again, we obtain that there exists w_2 such that $f(w_2) = 0$. By Theorem 22.2 1), there exists a polynomial g_2 such that $g_1(z) = (z - w_2)g_2(z)$, $z \in \mathbb{C}$. Consequently, $f(z) = (z - w_1)(z - w_2)g_2(z)$, $z \in \mathbb{C}$. Applying theorems 22.1 and 22.2 1) n times, we get that

$$f(z) = (z - w_1) \ldots (z - w_n) g_n, \quad z \in \mathbb{C}, \tag{47}$$

for some $w_1, w_2, \ldots, w_n \in \mathbb{C}$ and a polynomial g_n of degree 0 which is a constant function. Since the right hand side of (47) is a polynomial with the coefficient g_n about z_n, we can conclude that $g_n = a_n$.

The part 2) of the theorem easily follows from 3).

□

Exercise 22.1. For a complex number α show that the coefficients of the polynomial

$$p(z) = (z - \alpha)(z - \overline{\alpha})$$

are real numbers.

Exercise 22.2. Let $p(z)$ be a polynomial with real coefficients and let α be a complex number. Prove that $p(\alpha) = 0$ if and only if $p(\overline{\alpha}) = 0$.

Exercise 22.3. Prove that any polynomial $p(z)$ with real coefficients can be decomposed into a product of polynomials of the form $az^2 + bz + c$, where $a, b, c \in \mathbb{R}$.

22.2 Definition and some Examples of Vector Spaces

For more details see [3, Chapter 5].

Let $\mathbb{F}$ denote the set of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We will call $\mathbb{F}$ a **field**. We also consider a set V, whose elements are called **vectors** and will be denoted by $\mathbf{v}, \mathbf{u}, \mathbf{w} \ldots$ etc. We define on V two operations:

- **vector addition** $+ : V \times V \to V$, that maps two elements $\mathbf{u}, \mathbf{v}$ of V to $\mathbf{u} + \mathbf{v} \in V$;
- **scalar multiplication** $\cdot : \mathbb{F} \times V \to V$, that maps $a \in \mathbb{F}$ and $\mathbf{u} \in V$ to $a \cdot \mathbf{u} = a\mathbf{u} \in V$.

Definition 22.1. A **vector space** over $\mathbb{F}$ is a set V together with operations of vector addition and scalar multiplication which satisfy the following properties:

1) **commutativity**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$;
2) **associativity**: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and $(ab)\mathbf{v} = a(b\mathbf{v})$ for all $a, b \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$;
3) **additive identity**: there exists a vector $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;
4) **additive inverse**: for every $\mathbf{v} \in V$ there exists a vector $\mathbf{w} \in V$ (denoted by $-\mathbf{v}$) such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$;
5) **multiplicative identity**: $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$;

6) **distributivity**: $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$ and $(a+b)\mathbf{v} = a\mathbf{v}+b\mathbf{v}$ for all $a,b \in \mathbb{F}$, $\mathbf{u},\mathbf{v} \in V$.

A vector space over $\mathbb{R}$ will be called a **real vector space** and a vector space over $\mathbb{C}$ is similarly called a **complex vector space**.

Example 22.1. The set $V = \mathbb{F}$ with the usual operations of addition and multiplication is trivially a vector space over $\mathbb{F}$.

Example 22.2. The set

$$\mathbb{F}^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : \ x_k \in \mathbb{F}, \ k = 1, \dots, n\}$$

with operations

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

and

$$a\mathbf{x} = (ax_1, ax_2, \dots, ax_n),$$

for all $a \in \mathbb{F}$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{F}^n$, is a vector space. It is easily to see that the additive identity is $\mathbf{0} = (0, 0, \dots, 0)$ and the additive inverse of $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is $-\mathbf{x} = (-x_1, -x_2, \dots, -x_n)$.

Example 22.3. Similarly, the set

$$\mathbb{F}^\infty = \{\mathbf{x} = (x_1, x_2, \dots) : \ x_k \in \mathbb{F}, \ k \in \mathbb{N}\}$$

with operations

$$\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, \dots)$$

and

$$a\mathbf{x} = (ax_1, ax_2, \dots),$$

for all $a \in \mathbb{F}$, $\mathbf{x} = (x_1, x_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots) \in \mathbb{F}^\infty$, is also a vector space, where $\mathbf{0} = (0, 0, \dots)$ is additive identity and $-\mathbf{x} = (-x_1, -x_2, \dots)$ is the additive inverse of $\mathbf{x} = (x_1, x_2, \dots)$.

Example 22.4. The set of all polynomials of degree at most n

$$\mathbb{F}^n[z] = \{\mathbf{p}(z) = a_n z^n + \ldots + a_1 z + a_0, \ z \in \mathbb{F} : \ a_k \in \mathbb{F}, \ k = 1, \dots, n\}$$

with the addition

$$(\mathbf{p}+\mathbf{q})(z) = (a_n+b_n)z^n + \ldots + (a_1+b_1)z + a_0 + b_0, \quad z \in \mathbb{F},$$

and scalar multiplication

$$(a\mathbf{p})(z) = aa_n z^n + \ldots + aa_1 z + aa_0, \quad z \in \mathbb{F},$$

for all $a \in \mathbb{F}$ and $\mathbf{p}(z) = a_n z^n + \ldots + a_1 z + a_0$, $\mathbf{q}(z) = b_n z^n + \ldots + b_1 z + b_0$, $z \in \mathbb{F}$, from $\mathbb{F}^n[z]$ is also a vector space.

Exercise 22.4. The vector space $\mathbb{F}[z]$ of all polynomials of any degree can be defined similarly as $\mathbb{F}^n[z]$ and is also a vector space.

Example 22.5. The set of (real-valued) continuous functions on an interval $[a, b]$ with the usual addition of functions and multiplication by a constant is a real vector space.

Exercise 22.5. The set of complex numbers $\mathbb{C} = \{x + iy : \ x, y \in \mathbb{R}\}$ can be considered as a real vector space with the usual addition of complex numbers and the multiplication by the real number.

Example 22.6. The set $V = \{\mathbf{0}\}$, where $\mathbf{0}$ is any element, with addition $\mathbf{0} + \mathbf{0} := \mathbf{0}$ and scalar multiplication $a \cdot \mathbf{0} := \mathbf{0}$ is a vector space. (Here $\mathbf{0}$ also plays a role of the additive identity).

Exercise 22.6. Show that the sets from the previous examples are vector spaces under corresponding addition and scalar multiplication.

Exercise 22.7. For each of the following sets, either show that the set is a vector space over $\mathbb{F}$ or explain why it is not a vector space.

a) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{R}$.

b) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{C}$.

c) The set $\{f \in \mathrm{C}[0, 1] : f(0) = 2\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.

d) The set $\{f \in \mathrm{C}[0, 1] : f(0) = f(1) = 0\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.

e) The set $\{(x, y, z) \in \mathbb{R}^3 : x - 2y + z = 0\}$ under the usual operations of addition and multiplication on $\mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$.

f) The set $\{(x, y, z) \in \mathbb{C}^3 : 2x + z + i = 0\}$ under the usual operations of addition and multiplication on $\mathbb{C}^3$, $\mathbb{F} = \mathbb{C}$.

22.3 Elementary Properties of Vector Spaces

In this section, we prove some important and simple properties of vector spaces. Let V denote a vector space over $\mathbb{F}$.

Proposition 22.1. *Any vector space has a unique additive identity.*

Proof. Let us assume that there exist two additive identities $\mathbf{0}$ and $\mathbf{0}'$. Then

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}',$$

where the first identity holds since $\mathbf{0}'$ is an identity and the second equality holds since $\mathbf{0}$ is an identity. □

Proposition 22.2. *Every* $\mathbf{v} \in V$ *has a unique inverse.*

Proof. We assume that $\mathbf{w}$ and $\mathbf{w}'$ are additive inverses of $\mathbf{v}$ so that $\mathbf{v} + \mathbf{w} = \mathbf{0}$ and $\mathbf{v} + \mathbf{w}' = \mathbf{0}$. Then

$$\mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{w} + (\mathbf{v} + \mathbf{w}') = (\mathbf{w} + \mathbf{v}) + \mathbf{w}' = \mathbf{0} + \mathbf{w}' = \mathbf{w}'.$$

□

Since the additive inverse of $\mathbf{v}$ is unique, we will denote it by $-\mathbf{v}$. We also define $\mathbf{w}-\mathbf{v} := \mathbf{w}+(-\mathbf{v})$.

Proposition 22.3. *For every* $\mathbf{v} \in V$ $0 \cdot \mathbf{v} = \mathbf{0}$.

Proof. For $\mathbf{v} \in V$ we have that

$$0 \cdot \mathbf{v} = (0+0) \cdot \mathbf{v} = 0 \cdot \mathbf{v} + 0 \cdot \mathbf{v}.$$

Adding the additive inverse of $0\mathbf{v}$ to both sides, we obtain

$$\mathbf{0} = 0\mathbf{v} - 0\mathbf{v} = (0\mathbf{v} + 0\mathbf{v}) - 0\mathbf{v} = 0\mathbf{v}.$$

□

Proposition 22.4. *For every* $a \in \mathbb{F}$ $a \cdot \mathbf{0} = \mathbf{0}$.

Exercise 22.8. Prove Proposition 22.4.

Proposition 22.5. *For every* $\mathbf{v} \in V$ $(-1) \cdot \mathbf{v} = -\mathbf{v}$.

We recall that the vector $-\mathbf{v}$ denotes the additive inverse of $\mathbf{v}$.

Proof. For $\mathbf{v} \in V$, we have

$$\mathbf{v} + (-1) \cdot \mathbf{v} = 1 \cdot \mathbf{v} + (-1) \cdot \mathbf{v} = (1 + (-1))\mathbf{v} = 0 \cdot \mathbf{v} = \mathbf{0},$$

by Proposition 22.3. □

23 Lecture 23 – Vector Subspaces and Span

23.1 Vector Subspaces

Throughout this section, V denotes a vector space over $\mathbb{F}$.

Definition 23.1. Let V be a vector space over $\mathbb{F}$, and let $U \subset V$ be a subset of V. Then U is called a **subspace** of V if U is a vector space over $\mathbb{F}$

To check that a subset $U \subset V$ is a subspace, it is suffices to check only a few of the conditions of a vector space.

Lemma 23.1. *Let $U \subset V$ be a subset of a vector space V over $\mathbb{F}$. Then U is a subspace of V iff the following conditions holds:*

(1) ***closure under addition:*** $\mathbf{u}, \mathbf{v} \in U$ *implies* $\mathbf{u} + \mathbf{v} \in U$*;*

(2) ***closure under scalar multiplication:*** $a \in \mathbb{F}$*,* $\mathbf{u} \in U$ *implies that* $a\mathbf{u} \in U$*.*

Exercise 23.1. Prove Lemma 23.1.

Example 23.1. In every vector space V, the subset $U = \{\mathbf{0}\}$ is a vector subspace of V.

Exercise 23.2. Show that the set $\{(x_1, 0) : \ x_1 \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^2$.

Exercise 23.3. Show that the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + 2x_2 - x_3 + 1 = 0\}$ is not a vector subspace of $\mathbb{R}^3$.

Exercise 23.4. Let U_1 and U_2 be a vector subspaces of V. Prove that the intersection $U_1 \cap U_2$ is also a vector subspace. Is the union $U_1 \cup U_2$ a vector subspace?

23.2 Sums and Direct Sums of Vector Subspaces

Let U_1, U_2 be a vector subspaces of V.

Definition 23.2. Let U_1, U_2 be a vector subspaces of V. The set

$$U_1 + U_2 = \{\mathbf{u}_1 + \mathbf{u}_2 : \ \mathbf{u}_1 \in U_1, \ \mathbf{u}_2 \in U_2\}$$

is said to be a **sum of vector subspaces** U_1 and U_2.

Exercise 23.5. Check that a direct sum of two vector subspaces is a vector space.

Example 23.2. Let

$$\begin{aligned} U_1 &= \{(x, 0, 0) \in \mathbb{F}^3 : \ x \in \mathbb{F}\} \\ U_2 &= \{(0, y, 0) \in \mathbb{F}^3 : \ y \in \mathbb{F}\} \\ U_3 &= \{(y, y, 0) \in \mathbb{F}^3 : \ y \in \mathbb{F}\}. \end{aligned}$$

Then

$$U_1 + U_2 = U_1 + U_3 = \{(x, y, 0) \in \mathbb{F}^3 : \ x, y \in \mathbb{F}\}.$$

We remark that $\mathbf{u} \in U = U_1 + U_2$ if and only if there exist vectors $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 23.3. If every vector $\mathbf{u} \in U = U_1 + U_2$ can be uniquely written as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ for $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$. Then we call the vector space U the **direct sum** of U_1, U_2 and denote by

$$U = U_1 \oplus U_2.$$

Example 23.3. Let

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{R}^3 : \ x, y \in \mathbb{R}\}, \\ U_2 &= \{(0, 0, z) \in \mathbb{R}^3 : \ z \in \mathbb{R}\}, \\ U_3 &= \{(0, y, z) \in \mathbb{R}^3 : \ y, z \in \mathbb{R}\}. \end{aligned}$$

Then $\mathbb{R}^3 = U_1 \oplus U_2$. But $\mathbb{R}^3 = U_1 + U_3$ and $\mathbb{R}^3 \neq U_1 \oplus U_3$ (the vector $(0, 0, 0)$ can be written as $(0, 0, 0) + (0, 0, 0)$ and $(0, -1, 0) + (0, 1, 0)$).

Proposition 23.1. *Let U_1 and U_2 be a vector subspaces of V. Then $V = U_1 \oplus U_2$ iff the following conditions hold:*

(1) $V = U_1 + U_2$;

(2) *If $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$ with $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$, then $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$.*

Proof. We assume that $V = U_1 \oplus U_2$. Then Condition (1) follows from the definition. Since $\mathbf{0}$ can be uniquely written as $\mathbf{0} + \mathbf{0}$, we have that Condition (2) is also true.

Next, let conditions (1) and (2) hold. By Condition (1), for every vector $\mathbf{u} \in V$ there exist $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. We assume that $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{v}_1 \in U_1$ and $\mathbf{v}_2 \in U_2$. Subtracting the two equations, we obtain

$$0 = (\mathbf{u}_1 - \mathbf{v}_1) + (\mathbf{u}_2 - \mathbf{v}_2),$$

where $\mathbf{u}_1 - \mathbf{v}_1 \in U_1$ and $\mathbf{u}_2 - \mathbf{v}_2 \in U_2$. By Condition (2), we have that $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{v}_2$. This implies that $V = U_1 \oplus U_2$. □

Proposition 23.2. *Let U_1 and U_2 be a vector subspaces of V. Then $V = U_1 \oplus U_2$ iff the following conditions hold:*

(1) $V = U_1 + U_2$;

(2) $U_1 \cap U_2 = \{\mathbf{0}\}$.

Proof. We assume that $V = U_1 \oplus U_2$. Then Condition (1) follows from the definition. Next, we suppose that $\mathbf{u} \in U_1 \cap U_2$. Then by Exercise 23.4, $-\mathbf{u}$ also belongs to $U_1 \cap U_2$ because $U_1 \cap U_2$ is a vector space. Thus, $\mathbf{0} = \mathbf{u} + (-\mathbf{u})$, where $\mathbf{u} \in U_1 \cap U_2 \subset U_1$ and $-\mathbf{u} \in U_1 \cap U_2 \subset U_2$. By Proposition 23.1, $\mathbf{u} = \mathbf{0}$.

Next, we assume that conditions (1) and (2) hold. In order to prove that $V = U_1 \oplus U_2$, we show that $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$, implies $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$. Since $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1 = -\mathbf{u}_2$. So, $\mathbf{u}_1 = -\mathbf{u}_2 \in U_2$ because U_2 is a vector space. Thus, $u_1 \in U_1 \cap U_2$ and, consequently, $\mathbf{u}_1 = -\mathbf{u}_2 = \mathbf{0}$, according to Condition (2). USsing Proposition 23.1, we obtain that V is the direct sum of U_1 and U_2. □

Exercise 23.6. Prove or give a counterexample to the following claim:

1) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are vector subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.

2) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are vector subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.

Exercise 23.7. Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let

$$U = \{az^2 + bz^5 : \ a, b \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}[z]$ such that $F[z] = U \oplus W$.

23.3 Linear Span

In order to give a definition of one of the main notion of the linear algebra: basis of a vector space, we need to introduce a notion of a linear span of vectors.

Definition 23.4. A vector $\mathbf{v} \in V$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, if there exists scalars $a_1, a_2, \ldots, a_n$ from $\mathbb{F}$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n.$$

Definition 23.5. The set

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} := \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n : \ a_1, a_2, \ldots, a_n \in \mathbb{F}\}$$

is called a **linear span** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The following lemma follows from the definitions of a vector spaces and linear span.

Proposition 23.3. *Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Then*

(i) *the vector $\mathbf{v}_i$ belongs to* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$*;*

(ii) $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a subspace of V;*

(iii) *If U is a subspace of V such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in U$, then* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset U$.

Proposition 23.3 implies that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the smallest vector space of V which contains the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Definition 23.6. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$, then we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ **span** V and we call V **finite-dimensional**. If a vector space is not finite dimensional, then we call it **infinite-dimensional**.

Example 23.4. The vectors $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, ..., $\mathbf{e}_n = (0, \ldots, 0, 1)$ span $\mathbb{F}^n$. According to the previous definition the space $\mathbb{F}^n$ is finite-dimensional.

Example 23.5. Let $\mathbf{p}_k(z) = z^k$, for $k = 0, \ldots, n$. Then the set $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ span $\mathbb{F}_n[z]$. It is easy to see that the space $\mathbb{F}[z]$ of all polynomials is infinite-dimensional.

Exercise 23.8. Consider the complex vector space $V = \mathbb{C}^3$ and the list $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of vectors in V, where $\mathbf{v}_1 = (i, 0, 0)$, $\mathbf{v}_2 = (i, 1, 0)$ and $\mathbf{v}_3 = (i, i, -1)$.
a) Prove that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = V$.

24 Lecture 24 – Basis

24.1 Linear Independence

In this section, we are going to define the notion of linear independence of a list of vectors.

Definition 24.1. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called **linearly independent** if the only solution for $a_1, a_2, \ldots, a_n \in \mathbb{F}$ to the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0$$

is $a_1 = a_2 = \ldots = a_n = 0$. Otherwise, the vectors $v_1, v_2, \ldots, v_n$ are said to be **linearly dependent**.

Example 24.1. The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ from Example 23.4 are linearly independent, since

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_n\mathbf{e}_n = (a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$$

provided $a_1 = a_2 = \ldots = a_n = 0$.

Example 24.2. The vectors $\mathbf{v}_1 = (1, 1, 3)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$ are linearly dependent because

$$\mathbf{v}_1 - \mathbf{v}_2 - 3\mathbf{v}_3 = (0, 0, 0).$$

Example 24.3. The vectors $(1, z, z^2, \ldots, z^n)$ in $\mathbb{F}_n[z]$ are linearly independent.

Exercise 24.1. Show that the vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 3)$, and $\mathbf{v}_3 = (2, -1, 1)$ are linearly independent in $\mathbb{R}^3$. Write $\mathbf{v} = (1, -2, 5)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Exercise 24.2. Consider the complex vector space $V = \mathbb{C}^3$ and the vectors $\mathbf{v}_1 = (i, 0, 0)$, $\mathbf{v}_2 = (i, 1, 0)$, $\mathbf{v}_3 = (i, i, -1)$.
a) Prove that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = V$.
b) Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ a basis of $\mathbb{C}^3$?

Exercise 24.3. Determine the value of $\lambda \in \mathbb{R}$ for which each vectors $(\lambda, -1, -1)$, $(-1, \lambda, -1)$, $(-1, -1, \lambda)$ are linearly dependent in $\mathbb{R}^3$.

Theorem 24.1. *Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent iff each vector $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be unequally written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent. If $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be written as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = a_1'\mathbf{v}_1 + a_2'\mathbf{v}_2 + \ldots + a_n'\mathbf{v}_n,$$

then $\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1 - a_1')\mathbf{v}_1 + (a_2 - a_2')\mathbf{v}_2 + \ldots + (a_n - a_n')\mathbf{v}_n$, which implies that $a_1 = a_1'$, $a_2 = a_2'$, ..., $a_n = a_n'$. The sufficiency can be proved trivially, taking $\mathbf{v} = \mathbf{0}$. □

Theorem 24.2. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$. Then there exists $j \in \{2, \ldots, n\}$ such that*

1) $\mathbf{v}_j \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$;

2) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Proof. Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent, there exist $a_1, a_2, \dots, a_n \in \mathbb{F}$ such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_n\mathbf{v}_n = 0$. Since $\mathbf{v}_1 \neq \mathbf{0}$, not all of $a_2, \dots, a_n$ are 0. Let $j \in \{2, \dots, n\}$ be the largest index such that $a_j \neq 0$. Then we have

$$\mathbf{v}_j = -\frac{a_1}{a_j}\mathbf{v}_1 - \frac{a_2}{a_j}\mathbf{v}_2 - \dots - \frac{a_{j-1}}{a_j}\mathbf{v}_{j-1}. \tag{48}$$

This implies 1).

Let $\mathbf{v}$ be an arbitrary vector from $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. It means that there exist $b_1, b_2, \dots, b_n \in \mathbb{F}$ such that

$$v = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n.$$

According to (48), $\mathbf{v}$ can be rewritten as linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n$. This proves 2). □

Theorem 24.3. *Let V be a finite-dimensional vector space, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be linearly independent and span V, and let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be vectors that span V. Then $n \leq m$.*

Proof. For the proof of the theorem see the proof of Theorem 5.2.9 [3]. □

Exercise 24.4. Let V be a vector space over $\mathbb{F}$, and suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$ are linearly independent. Let $\mathbf{w}$ be a vector from V such that the vectors $\mathbf{v}_1 + \mathbf{w}, \mathbf{v}_2 + \mathbf{w}, \dots, \mathbf{v}_n + \mathbf{w}$ are linearly dependent. Prove that $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

24.2 Bases

Definition 24.2. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of a finite-dimensional vector space V if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent and span V, i.e. $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

Remark 24.1. We remark that each vector $\mathbf{v} \in V$ can be uniquely written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ iff $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of V.

Example 24.4. The set of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of $\mathbb{F}^n$.

Exercise 24.5. Prove that the set of vectors $(1,1,0)$, $(1,0,0)$, $(0,0,1)$ is a basis of $\mathbb{F}^3$.

Example 24.5. The set $1, z^2, \dots, z^n$ is a basis of $\mathbb{F}_n[z]$.

Theorem 24.4 (Basis reduction theorem)**.** *If $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then either the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis of V or some $\mathbf{v}_k$ can be removed to obtain a basis of V.*

Proof. Suppose $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. We start with the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and sequentially run through all vectors $\mathbf{v}_k$ for $k = 1, 2, \dots, m$ to determine whether to keep or remove them from S:

Step 1. If $\mathbf{v}_1 = \mathbf{0}$, then remove $\mathbf{v}_1$ from S. Otherwise, leave S unchanged.

Step k. If $\mathbf{v}_k \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$, then remove $\mathbf{v}_k$ from S. Otherwise, leave S unchanged.

The final set S still spans V since, at each step, a vector was only discarded if it was already in the span of the previous vectors. The process also ensures that no vector is in the span of the previous vectors. Hence, by Theorem 24.2, the final list S is linearly independent. It follows that S is a basis of V. □

Example 24.6. The set of vectors $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, $\mathbf{v}_4 = (0, -1, 2)$ are linearly dependent, since

$$0\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

But the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $\mathbb{R}^3$. Indeed, each element $\mathbf{v} = (x, y, z)$ can be uniquely written as follows

$$(x, y, z) = (x - z)\mathbf{v}_1 + (x + y - 2z)\mathbf{v}_2 + z\mathbf{v}_3.$$

Corollary 24.1. *Every finite-dimensional vector space has a basis.*

Proof. The statement immediately follows from Theorem 24.4. □

Theorem 24.5 (Basis Extension Theorem)**.** *Every linearly independent set of vectors in a finite-dimensional vector space V can be extended to a basis of V.*

Proof. Let V be finite-dimensional and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent. Since V is finite-dimensional, there exists a set of vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ that spans V. We are going to adjoin some of the $\mathbf{w}_k$ to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in order to create a basis of V.

Step 1. If $\mathbf{w}_1 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then let $S := \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Otherwise, $S := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, w_1\}$.

Step k. If $\mathbf{w}_k \in \operatorname{span} S$, then leave S unchanged. Otherwise, adjoin $\mathbf{w}_k$ to S.

After each step, the set S is still linearly independent, since we only adjoined $\mathbf{w}_k$ if $\mathbf{w}_k$ was not in the span of the previous vectors. After m steps, $\mathbf{w}_k \in \operatorname{span} S$ for all $k = 1, 2, \ldots, m$. Since the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ spans V, S also spans V. Consequently, S is a basis of V. □

24.3 Dimension

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be two bases of a finite-dimensional vector space V, that is, they both are linearly independent and span V. Then by Theorem 24.3, it follows that $n = m$.

Definition 24.3. We call the length of any basis of V the **dimension** of V and denote by $\dim V$.

Example 24.7. According to Example 24.4, the dimension of $\mathbb{F}^n$ equals n.

Example 24.8. By Example 24.5, the dimension of $\mathbb{F}_n[z]$ equals $n + 1$.

Exercise 24.6. Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{F}_n[z]$ satisfy $\mathbf{p}_j(2) = 0$ for all $j = 0, 1, \ldots, n$. Prove that $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

Remark 24.2. We note that $\dim \mathbb{C}^n = n$ as a complex vector space, whereas $\dim \mathbb{C}^n = 2n$ as a real vector space. This comes from the fact that we can view $\mathbb{C}$ itself as a real vector space of dimension 2 with basis $\{1, i\}$.

Theorem 24.6. *Let V be a finite-dimensional vector space with $\dim V = n$. Then*

(i) If $U \subset V$ is a subspace of V, then $\dim U \leq \dim V$.

(ii) If $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

(iii) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent in V, then they form a basis of V.

Proof. To prove statement (i), first we note that U is necessarily finite-dimensional (otherwise we could find a list of linearly independent vectors longer than $\dim V$). Therefore, by Corollary 24.1, U has a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ which are linearly independent in both U and V. By Theorem 24.5, we can extend $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ to a basis of V, which is of length n since $\dim V = n$. This implies that $m \le n$.

In order to prove statement (ii), we suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V. Then, by the basis reduction theorem (see Theorem 24.4), this set can be reduced to a basis. However, every basis of V has length n. Hence, no vector needs to be removed from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

To prove statement (iii), we assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. By the basis extension theorem (see Theorem 24.5), this set can be extended to a basis. However, every basis has length n. Hence, no vector needs to be added to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V. $\square$

Theorem 24.7. *Let $U \subset V$ be a subspace of a finite-dimensional vector space V. Then there exists a subspace $W \subset V$ such that $V = U \oplus W$.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis of U. By Theorem 24.6 (i), we know that $m \le \dim V$. Hence, by the basis extension theorem (see Theorem 24.5), the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ can be extended to a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of V. Let $W := \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$.

We now show that $V = U \oplus W$. Since the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis of V, each element $\mathbf{v}$ of V can be uniquely written as follows

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_m\mathbf{u}_m + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \ldots + b_k\mathbf{w}_k = \mathbf{u} + \mathbf{w}$$

for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k \in \mathbb{F}$, where $\mathbf{u} := a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_m\mathbf{u}_m$ and $\mathbf{w} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \ldots + b_k\mathbf{w}_k$. Since $\mathbf{u} \in U$ and $\mathbf{w} \in W$, V is the direct sum of U and W, according to Definition 23.3. $\square$

Exercise 24.7. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim V = n$ for some $n \in \mathbb{N}$. Prove that there exist n one-dimensional subspaces $U_1, U_2, \ldots, U_n$ of V such that

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n.$$

Theorem 24.8. *If $U, W \subset V$ are subspaces of a finite-dimensional vector space, then*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. For the proof of the theorem see the proof of Theorem 5.4.6 [3]. $\square$

Exercise 24.8. Let V be a finite-dimensional vector space over $\mathbb{F}$, and let U be a vector subspace of V for which $\dim U = \dim V$. Prove that $U = V$.

25 Lecture 25 – Linear Maps

26 Lecture 26 – Matrices

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.

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[3] I. Lankham, B. Nachtergaele, and A. Schilling. *Linear Algebra As An Introduction To Abstract Mathematics*. WSPC, 2016.

A Selected exercises

1. Show that

   a) $A \cup \emptyset = A$, $A \cup A = A$, $A \cup B = B \cup A$, $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;

   b) $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup B = B \cap A$, $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;

   c) $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$, $A \setminus B = A \cap B^c$;

   d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   e) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

2. Let $A_n = \{1, \dots, n\}$ for each $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \mathbb{N}, \quad \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \{1\}.$$

3. Prove that
   a) $\sqrt{6} \notin \mathbb{Q}$; b)$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$; c) for each $n \in \mathbb{N}$ either $\sqrt{n} \in \mathbb{N}$ or $\sqrt{n} \notin \mathbb{Q}$.

4. Using mathematical induction prove that:
   a) $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n;
   b) $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for each $n \in \mathbb{N}$;
   c) $11^n - 4^n$ is divisible by 7 for each $n \in \mathbb{N}$;
   d) $5^n - 4n - 1$, $n \in \mathbb{N}$, are divisible by 16;
   e) $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

5. Prove that there does not exist a rational number x solving the equation $x^2 = 2$.

6. Prove that the following sets are bounded:

   a) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$;

   b) $\left\{\frac{(-1)^n n+1}{n-(-1)^n} : n \in \mathbb{N}\right\}$.

7. For each $a < b$ prove that $\inf[a,b] = \inf(a,b] = a$ and $\sup[a,b] = \sup[a,b) = b$.

8. Show that
   a) $2^n \geq n+1$, $n \in \mathbb{N}$; b) $3^n \geq 2n+1$, $n \in \mathbb{N}$; c) $2^n > (\sqrt{2}-1)^2 n^2$, $n \in \mathbb{N}$.

9. Let $x_1, \dots, x_n$ be a positive real numbers. Prove that

$$(1 + x_1) \cdot \ldots \cdot (1 + x_n) \geq 1 + x_1 + \ldots + x_n.$$

10. Prove the boundedness of the following sequences:

    a) $\left(\frac{2^n}{n!}\right)_{n \geq 1}$; b) $\left(a_n = \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \sqrt{2}}}}}_{n \text{ square roots}}\right)_{n \geq 1}$.

11. Prove the following statements:

    a) $a_n \to a,\ n \to \infty \ \Leftrightarrow\ a_n - a \to 0,\ n \to \infty \ \Leftrightarrow\ |a_n - a| \to 0,\ n \to \infty$;

    b) $a_n \to 0,\ n \to \infty \ \Leftrightarrow\ |a_n| \to 0,\ n \to \infty$;

    c) $a_n \to a,\ n \to \infty \ \Leftrightarrow\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}:\ \{a_N, a_{N+1}, \dots\} \subset (x - \varepsilon, x + \varepsilon)$;

    d) $a_n \to 0,\ n \to \infty \ \Leftrightarrow\ \sup\{|a_k| :\ k \geq n\} \to 0,\ n \to \infty$;

    e) $a_n \to a,\ n \to \infty \ \Rightarrow\ |a_n| \to |a|,\ n \to \infty$.

12. Prove that for a sequence $(a_n)_{n\geq 1}$ with $a_n \neq 0$ the equality $\lim\limits_{n\to\infty} |a_n| = +\infty$ is equivalent to $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

13. Compute the following limits:
    a) $\lim\limits_{n\to\infty} \frac{n^3 - 2n^2 \cos n + n}{\sqrt{n} - 3n^3 + 1}$; b) $\lim\limits_{n\to\infty} (\sqrt{n^2 + n} - \sqrt{n})$; c) $\lim\limits_{n\to\infty} \sqrt[n]{n^2 2^n + 3^n}$; d) $\lim\limits_{n\to\infty} \frac{\sin^2 n}{\sqrt{n}}$; e) $\lim\limits_{n\to\infty} \frac{n^2 + \sin n}{n^2 + n \cos n}$; f) $\lim\limits_{n\to\infty} \frac{2^n + n^3}{3^n + 1}$; g) $\sqrt[n+1]{n}$.

14. Let $(a_n)_{n\geq 1}$ be a bounded sequence and $b_n \to 0$, $n \geq \infty$. Prove that $a_n b_n \to 0$, $n \to \infty$.

15. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1, a_2, \dots, a_n\}}{n} \to 0$, $n \to \infty$.

16. Let $(a_n)_{n\geq 1}$ be a bounded sequence and $b_n \to +\infty$, $n \geq \infty$. Prove that $a_n + b_n \to +\infty$, $n \to \infty$.

17. Using the monotonicity compute the following limits:

    a) $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$; b) $\lim\limits_{n\to\infty} \frac{n!}{2^{n^2}}$; c) $\lim\limits_{n\to\infty} \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \sqrt{2}}}}}_{n \text{ square roots}}$; d) $\lim\limits_{n\to\infty} \frac{n}{2^{\sqrt{n}}} = 0$.

18. Identify the set of subsequential limits of the following sequences:
    a) $\left(\sin \frac{2\pi n}{3}\right)_{n\geq 1}$; b) $(\sin 3\pi n)_{n\geq 1}$; c) $(a_n)_{n\geq 1}$,
    where $a_n = \begin{cases} (-1)^{\frac{n+1}{2}} + n, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} + \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$

19. Prove that $a_n \to a$, $n \to \infty \Leftrightarrow \varliminf\limits_{n\to\infty} a_n = \varlimsup\limits_{n\to\infty} a_n = a$.

20. For a sequence $(a_n)_{n\geq 1}$ compute $\varliminf\limits_{n\to\infty} a_n$ and $\varlimsup\limits_{n\to\infty} a_n$, if for all $n \geq 1$
    a) $a_n = 1 - \frac{1}{n}$; b) $a_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$; c) $a_n = \frac{n-1}{n+1} \cos \frac{2n\pi}{3}$; d) $a_n = 1 + n \sin \frac{n\pi}{2}$;
    e) $a_n = \left(1 + \frac{1}{n}\right)^n \cdot (-1)^n + \sin \frac{n\pi}{4}$; f) $a_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$.

21. Check whether the following sequences are Cauchy sequences.
    a) $\left(\frac{1}{2^n}\right)_{n\geq 1}$; b) $((-1)^n)_{n\geq 1}$; c) $\left(a_n = \frac{\sin 1}{2^1} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}\right)_{n\geq 1}$.

22. Show that $(a_n)_{n\geq 1}$ is a Cauchy sequence iff $\sup\limits_{m\geq N, n\geq N} |a_m - a_n| \to 0$, $N \to \infty$.

23. Find the domain and the range of the following functions:
    a) $f(x) = \frac{1}{(x+1)^2}$; b) $f(x) = \sqrt{1 - x^2}$; c) $f(x) = \ln(1 + x)$.

24. Let $f: X \to Y$ and $A_1 \subset X$, $A_2 \subset X$. Check that
 a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; b) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$; c) $(f(A_1) \setminus f(A_2)) \subset f(A_1 \setminus A_2)$;
 d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$; e) $A_1 \subset f^{-1}(f(A_1))$; f) $(f(X) \setminus f(A_1)) \subset f(X \setminus A_1)$.

25. Let $f: X \to Y$ and $B_1 \subset Y$, $B_2 \subset Y$. Show that
 a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$; b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
 c) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$; d) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$;
 e) $f(f^{-1}(B_1)) = B_1 \cap f(X)$; f) $f^{-1}(B_1^c) = (f^{-1}(B_1))^c$.

26. Prove that the set of all limit points of $\mathbb{Q}$ equals $\mathbb{R} \cup \{-\infty, +\infty\}$.

27. Prove that $\frac{1}{f(x)} \to 0$, $x \to a$, if $f(x) \to +\infty$, $x \to a$.

28. Prove that the limit of the function $f(x) = \cos \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exist at the point $a = 0$.

29. Using $\varepsilon - \delta$ definition, show that
 a) $\lim\limits_{x \to 4} \sqrt{x} = 2$; b) $\lim\limits_{x \to +\infty} \frac{\ln x}{x} = 0$.

30. Compute the following limits:
 a) $\lim\limits_{x \to 0} \frac{\tan x}{x}$; b) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2}$; c) $\lim\limits_{x \to 1} \frac{x^2 - x}{x^2 - 3x + 2}$; d) $\lim\limits_{x \to +\infty} \frac{x^3 - x \sin x + x}{1 - 3x^3 + \ln x}$; e) $\lim\limits_{x \to +\infty} \frac{x^2 + \cos x + 1}{\sqrt{x^4 + 1} + x + 3}$;
 f) $\lim\limits_{x \to 0} \left(\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right)$; g) $\lim\limits_{x \to 0} \frac{x^2 + x}{\sqrt[3]{1 + \sin x} - 1}$; h) $\lim\limits_{x \to +\infty} \left(x(\sqrt{x^2 + 2x + 2} - x - 1) \right)$;
 i) $\lim\limits_{x \to +\infty} (\sqrt{ax + 1} - \sqrt{x})$, for some $a > 0$.

31. Compute the following limits:
 a) $\lim\limits_{x \to 0-} \frac{x}{\sqrt{1 - \cos^2 x}}$; b) $\lim\limits_{x \to 0+} \frac{x}{\sqrt{1 - \cos^2 x}}$; c) $\lim\limits_{x \to \frac{\pi}{2}-} \frac{x - \frac{\pi}{2}}{\sqrt{1 - \sin x}}$; d) $\lim\limits_{x \to \frac{\pi}{2}+} \frac{x - \frac{\pi}{2}}{\sqrt{1 - \sin x}}$;
 e) $\lim\limits_{x \to 0+} e^{-\frac{1}{x}}$; f) $\lim\limits_{x \to 0+} \frac{e^{-\frac{1}{x}}}{x}$.

32. Let f be an increasing function on an interval $[a, b]$.
 a) For each $c \in (a, b)$ show that the one-sided limits $f(a+)$, $f(c-)$, $f(c+)$, $f(b-)$ exist.
 b) Check the inequalities

$$f(a) \leq f(a+) \leq f(c-) \leq f(c) \leq f(c+) \leq f(b-) \leq f(b),$$

 for all $c \in (a, b)$.
 c) Prove that $\lim\limits_{x \to c+} f(x-) = f(c+)$ and $\lim\limits_{x \to c-} f(x+) = f(c-)$ for all $c \in (a, b)$.

33. Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. For which a, b the function f is continuous on $\mathbb{R}$?

34. Compute the following limits:
 a) $\lim\limits_{x \to 0} (\tan x - e^x)$; b) $\lim\limits_{x \to 2} \frac{x^2 - 3^x + 1}{x - \sin \pi x}$; c) $\lim\limits_{x \to 3} \frac{x \cos x + 1}{x^3 + 1}$.

35. Prove that the function $f(x) = \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

36. Show that the Dirichlet function $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at any point of $\mathbb{R}$.

37. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\ln(1+x)+\arcsin x^2}{\arccos x+\cos x}$; b) $\lim\limits_{x\to 1} \frac{\arctan x}{1+\arctan x^2}$; c) $\lim\limits_{x\to 0} \frac{\arcsin x}{x}$; d) $\lim\limits_{x\to 0} \frac{x}{\sin x+\arcsin x}$; e) $\lim\limits_{x\to 0} \frac{\arctan x}{x}$;
f) $\lim\limits_{x\to 0} \frac{\arccos x-\frac{\pi}{2}}{x}$; g) $\lim\limits_{x\to 0} \frac{\sin(\arctan x)}{\tan x}$.

38. Compute the following limits:
a) $\lim\limits_{x\to 0}(\cos x)^x$; b) $\lim\limits_{x\to+\infty} x(\ln(1+x)-\ln x)$; c) $\lim\limits_{x\to 0}\left(\frac{1+\sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$; d) $\lim\limits_{x\to 0}\frac{1-\cos x}{1-\cos 2x}$; e) $\lim\limits_{x\to 0}\frac{\ln(1+x)+e^x-\cos x}{e^{x^2}-1+\sin x}$;
f) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$; g) $\lim\limits_{x\to 0}\frac{\arcsin(x-1)}{x^m-1}$ for $m\in\mathbb{N}$; h) $\lim\limits_{x\to 0}\frac{1-(\cos mx)^m}{x^2}$ for $m\in\mathbb{N}$;
i) $\lim\limits_{x\to 0}\frac{1-(\cos mx)^{\frac{1}{m}}}{x^2}$ for $m\in\mathbb{N}$; k) $\lim\limits_{x\to 0}\frac{\sqrt[7]{\cos x}-1}{\sqrt[3]{1+x^2}-1}$; l) $\lim\limits_{x\to 0}\frac{e^{\sin 2x}-e^{\tan x}}{x}$.

39. Prove that the function $P(x)=x^3+7x^2-1$, $x\in\mathbb{R}$, has at least one root, that is, there exists $x_0\in\mathbb{R}$ such that $P(x_0)=0$.

40. Let $g:[0,1]\to[0,1]$ be a continuous function on $[0,1]$. Show that there exists $x_0\in[0,1]$ such that $g(x_0)=x_0$.

41. Let $f,g:[0,1]\to[0,1]$ be continuous and f be a surjection. Prove that there exists $x_0\in[0,1]$ such that $f(x_0)=g(x_0)$.

42. Using the definition of derivative, check that $(x|x|)'=2|x|$, $x\in\mathbb{R}$.

43. Show that the following functions are not differentiable at 0.
a) $f(x)=|x|$, $x\in\mathbb{R}$; b) $f(x)=\sqrt[3]{x}$, $x\in\mathbb{R}$; c) $f(x)=x\sin\frac{1}{x}$, $x\in\mathbb{R}\setminus\{0\}$, and $f(0)=0$.

44. For the function $f(x)=|x^2-x|$, $x\in\mathbb{R}$, compute $f'(x)$ for each $x\in\mathbb{R}\setminus\{0,1\}$. Compute left and right derivatives at points 0 and 1.

45. Let

$$f(x)=\begin{cases} x^2, & x\le 1,\\ ax+b, & x>1.\end{cases}$$

For which $a,b\in\mathbb{R}$ the function f:
a) is continuous on $\mathbb{R}$; b) is differentiable on $\mathbb{R}$? Compute also f'.

46. Check whether the following functions are differentiable at 0. Justify your answer.
a) $f(x)=\begin{cases}\frac{\cos x-1}{x}, & x\ne 0,\\ 0, & x=0;\end{cases}$ b)$f(x)=\sqrt[5]{x^2}$, $x\in\mathbb{R}$; c) $f(x)=|\sin x|$, $x\in\mathbb{R}$.

47. Prove that f is continuous at a point a if $f'_-(a)$ and $f'_+(a)$ exist.

48. Compute derivatives of the following functions:
a) $f(x)=x^2\sin x$; b) $f(x)=e^{-\frac{x^2}{2}}\cos x$; c) $f(x)=\frac{x}{1+x^2}$; d) $f(x)=\frac{e^x+e^{-x}}{e^x-e^{-x}}$;
e) $f(x)=2^{\tan(x^2-1)}$; f) $f(x)=\sin(\cos^2(\tan^3 x))$; g) $f(x)=\sqrt[3]{\frac{1+x^3}{1-x^3}}$;
h) $y=e^{-x^2\sin x}$; i) $y=\frac{\sin^2 x}{\sin x^2}$; j) $y=e^x\left(1+\cot\frac{x}{2}\right)$; k) $f(x)=e^{ax}\cdot\frac{a\sin bx-b\cos bx}{\sqrt{a^2+b^2}}$,
where a,b are some constants.

49. Let $f(x)=\frac{1}{x^3}e^{-\frac{1}{x^2}}$ for $x\ne 0$ and $f(0)=0$. Prove that $f'(0)=0$.

50. Compute derivatives of the following functions:
    a) $f(x) = \frac{1}{4}\ln\frac{x^2-1}{x^2+1}$; b) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$; c) $f(x) = \ln\tan\frac{x}{2}$;
    d) $f(x) = \arcsin\frac{1-x}{\sqrt{2}}$; e) $f(x) = \arctan\frac{1+x}{1-x}$; f) $f(x) = x^x$; g) $f(x) = \sqrt[x]{x}$.

51. Let a function $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in (a,b)$. Show that f is uniformly continuous on (a,b).

52. Prove that
    a) $x - \frac{x^3}{3!} \leq \sin x \leq x$ for all $x \geq 0$;
    b) $1 - \frac{x^2}{2} \leq \cos x \leq 1$ for all $x \geq 0$;
    c) $\frac{x}{1+x} \leq \ln(1+x) \leq x$ for all $x > -1$.

53. For each $\alpha > 1$, prove that $(1+x)^\alpha \geq 1 + \alpha x$ for all $x > -1$.

54. Identify the intervals on which the following functions are monotone.
    a) $f(x) = 3x - x^3$, $x \in \mathbb{R}$ b)$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{x^2}{2^x}$, $x \in \mathbb{R}$;
    d) $f(x) = x + \sqrt{|1-x^2|}$, $x \in \mathbb{R}$; e) $f(x) = \frac{1}{x^3} - \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$.

55. Identify $a \in \mathbb{R}$ for which the function $f(x) = x + a\sin x$, $x \in \mathbb{R}$, is increasing on $\mathbb{R}$.

56. Using L'Hospital's Rule, show that
    a) $\lim\limits_{x\to0}\frac{1-\cos x}{x^2} = 1$; b) $\lim\limits_{x\to0}\frac{\ln(1+x)}{\sin x} = 1$; c) $\lim\limits_{x\to e}\frac{(\ln x)^\alpha - \left(\frac{x}{e}\right)^\beta}{x-e} = \frac{\alpha-\beta}{e}$,
    where α, β are some real numbers; d) $\lim\limits_{x\to1}\frac{\left(\frac{4}{\pi}\arctan x\right)^\alpha - 1}{\ln x} = \frac{2\alpha}{\pi}$, $\alpha \in \mathbb{R}$;
    e) $\lim\limits_{x\to0+}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = e^{-\frac{1}{2}}$; f) $\lim\limits_{x\to+\infty}\frac{x}{2^x} = 0$; g) $\lim\limits_{x\to+\infty}\frac{\ln x}{x^\varepsilon} = 0$ for all $\varepsilon > 0$;
    h) $\lim\limits_{x\to+0}x^\varepsilon\ln x = 0$ for all $\varepsilon > 0$; i) $\lim\limits_{x\to+0}(\ln(1+x))^x = 1$.

57. Compute the following limits:
    a) $\lim\limits_{x\to0}\frac{\ln(1+x)-x}{x^2}$; b) $\lim\limits_{x\to0}\frac{e^x - e^{\sin x}}{x - \sin x}$; c) $\lim\limits_{x\to+\infty}\left(x\left(\frac{\pi}{2} - \arctan x\right)\right)$; d) $\lim\limits_{x\to+\infty}\frac{\ln(x+1)-\ln(x-1)}{\sqrt{x^2+1}-\sqrt{x^2-1}}$;
    e) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x}\right)^x$; f) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x^2}\right)^x$; g) $\lim\limits_{x\to0}\frac{(1+x)^{\frac{1}{x}} - e}{x}$; h) $\lim\limits_{x\to+\infty}\frac{x^{\ln x}}{(\ln x)^x}$.

58. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0.$$

59. Write Taylor's expansion of the function e^{2x-x^2}, $x \in \mathbb{R}$ at the point $x_0 = 0$ up to the term with x^5.

60. Use Taylor's formula to compute the limits
    a) $\lim\limits_{x\to0}\frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$; b) $\lim\limits_{x\to0}\frac{e^x\sin x - x(1+x)}{x^3}$; c) $\lim\limits_{x\to0}\frac{e^x - 1 - x}{x^2}$; d) $\lim\limits_{x\to0}\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}}$;
    e) $\lim\limits_{x\to0}\frac{\ln(1+x+x^2)+\ln(1-x-x^2)}{x\sin x}$; f) $\lim\limits_{x\to0}\frac{\cos(xe^x) - \cos(xe^{-x})}{x^3}$.

61. Find points of local extrema of the following functions:
a) $f(x) = x^4(1-x)^3$, $x \in \mathbb{R}$; b) $f(x) = x^2 e^x$, $x \in \mathbb{R}$; c) $f(x) = x + \frac{1}{x}$, $x > 0$;
d) $f(x) = \frac{x^2}{2} - \frac{1}{4} + \frac{9}{4(2x^2+1)}$, $x \in \mathbb{R}$; e) $f(x) = |x|e^{-x^2}$, $x \in \mathbb{R}$; f) $f(x) = x^x$, $x > 0$;
g) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$, $x \in \mathbb{R}$.

62. Identify intervals on which the following functions are convex or concave:
a) $f(x) = e^x$, $x \in \mathbb{R}$; b) $f(x) = \ln x$, $x > 0$; c) $f(x) = \sin x$, $x \in \mathbb{R}$; d) $f(x) = \arctan x$, $x \in \mathbb{R}$;
e) $f(x) = x^\alpha$, $x > 0$, $\alpha \in \mathbb{R}$.

63. Compute the following indefinite integrals:
a) $\int \cos 6x dx$, $x \in \mathbb{R}$; b) $\int x \sin x dx$, $x \in \mathbb{R}$; c) $\int \sin^2 x dx$, $x \in \mathbb{R}$;
d) $\int \sin 2x \sin 3x dx$, $x \in \mathbb{R}$; e) $\int \sin^3 x dx$, $x \in \mathbb{R}$; f) $\int \frac{dx}{\sin x \cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$;
g) $\int x \cos x^2 dx$, $x \in \mathbb{R}$; h) $\int \frac{dx}{1-x}$ on $(-\infty, 1)$ and $(1, +\infty)$; i) $\int \frac{dx}{x \ln x}$, $x > 0$;
j) $\int \frac{(2x+1)dx}{\sqrt[3]{1+x+x^2}}$, $x \in \mathbb{R}$; k) $\int \frac{\sqrt{\tan x}}{\cos^2 x} dx$, $x \in \left(0, \frac{\pi}{2}\right)$; l) $\int \frac{dx}{\cos x}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;
m) $\int \cos^2 x \sin^3 x dx$, $x \in \mathbb{R}$; n) $\int \frac{dx}{x^2+x+1}$, $x \in \mathbb{R}$; o) $\int x^2 \sin x dx$, $x \in \mathbb{R}$;
p) $\int (\ln x)^2 dx$, $x > 0$; q) $\int e^{2x} \cos x dx$, $x \in \mathbb{R}$. r) $\int \frac{1}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$; s) $\int \frac{e^{\frac{1}{x}} dx}{x^2}$, $x > 0$;
t) $\int \sqrt{1-3x} dx$, $x < \frac{1}{3}$; u) $\int \frac{dx}{1-x^2}$ on $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$;
v) $\int \ln(x^2 + x + 1) dx$, $x \in \mathbb{R}$.

64. Let $f : [0, 1] \to \mathbb{R}$ be integrable on $[0, 1]$. Prove the equality

$$\lim_{n \to \infty} \int_{\frac{1}{n}}^{1} f(x) dx = \int_0^1 f(x) dx.$$

65. Compute the following integrals:
a) $\int_{-1}^{8} \sqrt[3]{x} dx$; b) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$; c) $\int_0^{\frac{\pi}{2}} \sin 2x dx$; d) $\int_0^1 e^{2x-1} dx$; e) $\int_0^2 |1-x| dx$; f) $\int_0^{\ln 2} x e^{-x} dx$;
g) $\int_0^{\sqrt{\pi}} x \sin x^2 dx$; h) $\int_0^{2\pi} x^2 \cos x dx$; i) $\int_{-1}^{1} \frac{x dx}{\sqrt{5-4x}}$; j) $\int_0^{\ln 2} \sqrt{e^x - 1} dx$;
k) $\int_{-1}^{1} \frac{dx}{x^2 - 2x\cos\alpha + 1}$ for $\alpha \in (0, \pi)$; l) $\int_{\frac{1}{e}}^{e} |\ln x| dx$; m) $\int_0^1 \arccos x dx$.

66. Compute the following derivatives:
a) $\frac{d}{dx} \int_a^b \sin x^2 dx$; b) $\frac{d}{da} \int_a^b \sin x^2 dx$; c) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t^2} dt$; d) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{1+t^4}$.

67. Compute the following limits:
a) $\lim\limits_{x \to 0} \frac{\int_0^x \cos t^2 dt}{x}$; b) $\lim\limits_{x \to +\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}$.

68. Compute the area of regions bounded by the graphs of the following functions:
a) $2x = y^2$ and $2y = x^2$; b) $y = x^2$ and $x + y = 2$; c) $y = 2^x$, $y = 2$ and $x = 0$;
d) $y = \frac{a^3}{a^2+x^2}$ and $y = 0$, where $a > 0$.

69. Compute the length of the circle $x^2 + y^2 = r^2$, $r > 0$.

70. Compute the length of continuous curves defined by the following functions:
a) $y = x^{\frac{3}{2}}$, $x \in [0, 4]$; b) $y = e^x$, $0 \le x \le b$; c) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.

71. Compute the following improper integrals:
a) $\int_0^{+\infty} xe^{-x}dx$; b) $\int_0^{+\infty} \frac{dx}{x^2+x+1}$; c) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; d) $\int_0^{+\infty} x^2e^{-x}dx$.

72. Identify all $p \in \mathbb{R}$ for which the improper integral $\int_1^{+\infty} x^pe^{-x}dx$ converges. Justify your answer.

73. Show that the following improper integrals converge:
a) $\int_1^{+\infty} e^{-x^2}dx$; b) $\int_1^{+\infty} \frac{x-2}{x^3+x+1}dx$; c) $\int_0^{+\infty} \frac{\sin x}{1+x^2}dx$; d) $\int_1^{+\infty} e^{-x}\ln x dx$; e) $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx$
f) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}}dx$ $\int_1^{+\infty} \cos x^2 dx$; g) $\int_1^{+\infty} \frac{\sin 2x \cdot \sin x}{x}dx$; h) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; i) $\int_0^1 \ln x dx$.

74. Prove that the convergence of a series $\sum\limits_{n=1}^{\infty} a_n$ implies that $a_n + a_{n+1} + \ldots + a_{2n} \to 0$, $n \to \infty$.

75. Identify all $p > 0$ for which the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges. Justify your answer.

76. Prove the convergence of the following series:
a) $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n^3-n^2+1}$; b) $\sum\limits_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum\limits_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right)$; d) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum\limits_{n=1}^{\infty} \left(\sqrt{n^2+1}-n\right)^2$;
f) $\sum\limits_{n=1}^{\infty} \frac{n^2}{3^n}$; g) $\sum\limits_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$; h) $\sum\limits_{n=2}^{\infty} \left(\ln\frac{n}{n-1}-\frac{1}{n}\right)$.

77. Investigate the convergence of the following series:
a) $\sum\limits_{n=1}^{\infty} \frac{3^n n!}{n^n}$; b) $\sum\limits_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$; c) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; d) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum\limits_{n=1}^{\infty} \frac{3^n(n!)^2}{(2n)!}$;
f) $\sum\limits_{n=1}^{\infty} \frac{7^n(n!)^2}{n^{2n}}$; g) $\sum\limits_{n=1}^{\infty} \frac{3^n}{(\ln n)^n}$; h) $\sum\limits_{n=1}^{\infty} \frac{n^{n^2}2^n}{(n+1)^{n^2}}$.

78. Prove the convergence of the following sequences:
a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$; b) $\sum\limits_{n=1}^{\infty} (-1)^{\frac{n(n+1)}{2}}\frac{1}{\sqrt{n}}$; c) $\sum\limits_{n=1}^{\infty} \frac{\sin 3n}{\sqrt{n}}$; d) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n}$.

79. Investigate the absolute and conditional convergence of the following series:
a) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$; b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$; c) $\sum\limits_{n=1}^{\infty} (-1)^n \sin^{\frac{1}{3}}\frac{1}{n}$; d) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{\sqrt{n}}$.

80. Show that for each $x \in \mathbb{R}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

81. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:
a) $(-2+3i)(1+i)$; b) $(\sqrt{2}-i)^2$; c) $(2+3i)^2(1+2i)$;
d) $\frac{2+3i}{2-i}$; e) $\frac{3-i}{2+2i}$; f) $\frac{i}{(1-i)^2}$; g) $\frac{1}{i} - \frac{1}{(1+i)^2}$.

82. Compute the real and imaginary parts of $\frac{1}{z^2}$, where $z = x + iy$, $x, y \in \mathbb{R}$.

83. Solve the following equations:
a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

84. Write the following complex numbers in the polar form:
a) i; b) $1 - i$; c) $-1 + \sqrt{3}i$; d) $-2 - 2i$.

85. Compute $\frac{(1-\sqrt{3}i)(\cos\theta+i\sin\theta)}{2(1-i)(\cos\theta-i\sin\theta)}$.

86. Compute a) $(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)^{21}$; b) $(\sqrt{3}-3i)^{15}$; c) $(1+i)^{25}$; d) $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}$; e) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$.

87. Solve the following equations:
a) $z^2+z+3=0$; b) $z^3-i=0$; c) $z^5-2=0$; d) $z^4+i=0$; e) $z^3-4i=0$.

88. Let $z,w\in\mathbb{C}$. Prove the parallelogram law $|z-w|^2+|z+w|^2=2(|z|^2+|w|^2)$.

89. For a complex number α show that the coefficients of the polynomial
$$p(z)=(z-\alpha)(z-\overline{\alpha})$$
are real numbers.

90. Let $p(z)$ be a polynomial with real coefficients and let α be a complex number. Prove that $p(\alpha)=0$ if and only if $p(\overline{\alpha})=0$.

91. For each of the following sets, either show that the set is a vector space over $\mathbb{F}$ or explain why it is not a vector space.

    a) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F}=\mathbb{R}$.
    b) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F}=\mathbb{C}$.
    c) The set $\{f\in \mathrm{C}[0,1]: f(0)=2\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F}=\mathbb{R}$.
    d) The set $\{f\in \mathrm{C}[0,1]: f(0)=f(1)=0\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F}=\mathbb{R}$.
    e) The set $\{(x,y,z)\in\mathbb{R}^3: x-2y+z=0\}$ under the usual operations of addition and multiplication on $\mathbb{R}^3$, $\mathbb{F}=\mathbb{R}$.
    f) The set $\{(x,y,z)\in\mathbb{C}^3: 2x+z+i=0\}$ under the usual operations of addition and multiplication on $\mathbb{C}^3$, $\mathbb{F}=\mathbb{C}$.

92. Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let
$$U=\{az^2+bz^5:\ a,b\in\mathbb{F}\}.$$
Find a subspace W of $\mathbb{F}[z]$ such that $F[z]=U\oplus W$.

93. Consider the complex vector space $V=\mathbb{C}^3$ and the list $\{v_1,v_2,v_3\}$ of vectors in V, where $v_1=(i,0,0)$, $v_2=(i,1,0)$ and $v_3=(i,i,-1)$.
a) Prove that $\mathrm{span}\{v_1,v_2,v_3\}=V$.
b) Prove or disprove that $\{v_1,v_2,v_3\}$ is a basis of V.

94. Let V be a vector space over $\mathbb{F}$, and suppose that $v_1,v_2,\dots,v_n\in V$ are linearly independent. Let w be a vector from V such that the vectors $v_1+w,v_2+w,\dots,v_n+w$ are linearly dependent. Prove that $w\in\mathrm{span}\{v_1,v_2,\dots,v_n\}$.

95. Let $p_0,p_1,\dots,p_n\in\mathbb{F}_n[z]$ satisfy $p_j(2)=0$ for all $j=0,1,\dots,n$. Prove that $p_0,p_1,\dots,p_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

96. Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
    a) Show that T is linear; b) show that T is surjective; c) find $\dim(\ker T)$.

97. Show that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ is surjective if

$$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = 5x_2, \ x_3 = 7x_4\}.$$

98. Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V. Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

99. Let U, V and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

$$\dim(\ker(TS)) \leq \dim(\ker T) + \dim(\ker S).$$

B Homework

B.1 Problem sheet 1

1. **[1+1 points]** List elements of the following sets:
a) $\left\{n \in \mathbb{N} : \ (n-4)^2 < 5^2\right\}$; b) $\left\{n \in \mathbb{N} : \ n^3 > 4n\right\}$.

2. **[2+2+2 points]** Check the following relations:
a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; b) $(A \cup B)^c = A^c \cap B^c$; c) $\left(\bigcap\limits_{t \in T} A_t\right)^c = \bigcup\limits_{t \in T} A_t^c$

3. **[2+2+2 points]** Prove that
a) $\sqrt{6} \notin \mathbb{Q}$; b)$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$; c) for each $n \in \mathbb{N}$ either $\sqrt{n} \in \mathbb{N}$ or $\sqrt{n} \notin \mathbb{Q}$.

4. **[3+3 points]** Using mathematical induction prove that:
a) $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for each $n \in \mathbb{N}$;
b) $11^n - 4^n$ is divisible by 7 for each $n \in \mathbb{N}$.

5. **[2+2+3 points]** Prove that a) $\sup A = -\inf(-A)$, where A is a subset of $\mathbb{R}$ bounded from above and $-A := \{-a : \ a \in A\}$;
b) Let A and B be subsets of $\mathbb{R}$ bounded from above. Show that $\sup(A \cup B) = \max\{\sup A, \sup B\}$;
c) Let $A = \{0, \alpha_1 \alpha_2 \ldots \alpha_n \ldots : \ \forall n \in \mathbb{N} \ \ \alpha_n \in \{1, 2, 3, 4, 5, 6, 7, 8\}\}$. Find $\inf A$ and $\sup A$.

B.2 Problem sheet 2

1. **[1+1+1 points]** Using the definition of the limit show that
   a) $\lim\limits_{n\to\infty} \frac{n-1}{n} = 1$; b) $\lim\limits_{n\to\infty} n^2 = +\infty$; c) $\lim\limits_{n\to\infty} (-1)^n$ doe not exist.

2. **[3 points]** Assume that $a_n \to a$, $n \to \infty$, and $b_n \to b$, $n \to \infty$. Show that $\max\{a_n, b_n\} \to \max\{a, b\}$, $n \to \infty$.

3. **[2+2+2 points]** Compute the following limits:
   a) $\lim\limits_{n\to\infty} \frac{n^3-2n^2\cos n+n}{\sqrt{n}-3n^3+1}$; b) $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n})$; c) $\lim\limits_{n\to\infty} \sqrt[n]{n^2 2^n + 3^n}$.

4. **[3 points]** Let $(a_n)_{n\geq 1}$ be a bounded sequence and $b_n \to 0$, $n \geq \infty$. Prove that $a_n b_n \to 0$, $n \to \infty$.

5. **[3+3 points]** Using the monotonicity compute the following limits:
   a) $\lim\limits_{n\to\infty} \frac{n!}{2^{n^2}}$; b) $\lim\limits_{n\to\infty} \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n \text{ square roots}}$.

6. **[2 points]** Show that $\lim\limits_{n\to\infty} n \ln\left(1+\frac{1}{n}\right) = 1$.

7. **[2 points]** Identify the set of subsequential limits of the sequence $\left(\sin\frac{2\pi n}{3}\right)_{n\geq 1}$.

B.3 Problem sheet 3

1. **[1+1+1 points]** For a sequence $(a_n)_{n\geq 1}$ compute $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$, if for all $n \geq 1$
   a) $a_n = 1 + \frac{1}{n}$; b) $a_n = 1 + n \sin \frac{n\pi}{2}$; c) $a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$.
2. **[3 points]** Show that $a := \varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k$, for the case $a \in \mathbb{R}$.
   (The equality also holds in the case $a \in \{-\infty, +\infty\}$)
3. **[2 points]** Check that the sequence $\left(a_n = \frac{\sin 1}{2^1} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}\right)_{n\geq 1}$ is a Cauchy sequence.
4. **[2+2+2 points]** Find the domain and the range of the following functions:
   a) $f(x) = \frac{1}{(x+1)^2}$; b) $f(x) = \sqrt{1-x^2}$; c) $f(x) = \ln(1+x)$.
5. **[2+2 points]** Find the formulas for the following implicitly defined functions. What are their domains?
   a) $y = f(x)$ is the solution to the equation $x^3 y + 2y = 5$;
   b) $y = f(x)$ is the largest solution to the equation $y^2 = 3x^2 - 2xy$.

B.4 Problem sheet 4

1. **[1+1+1 points]** Let $f : X \to Y$. Check that
   a) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$ for $A_1 \subset X$, $A_2 \subset X$;
   b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ for $B_1 \subset Y$, $B_2 \subset Y$;
   c) $f(f^{-1}(B)) = B \cap f(X)$ for $B \subset Y$.

2. **[2 points]** Show that the set of all limit points of the set $A = \{r \in [0,1] : \ r \text{ is rational}\}$ coincides with the interval $[0,1]$. *(Hint: Use Theorem 2.3)*

3. **[2 point]** Prove that the limit of the function $f(x) = \cos\frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exist at the point $a = 0$.

4. **[2+2 point]** Using $\varepsilon - \delta$ definition, show that
   a) $\lim\limits_{x\to 4} \sqrt{x} = 2$; b) $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$.

5. **[2+2+2 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$; b) $\lim\limits_{x\to+\infty} \frac{x^3 - x\sin x + x}{1-3x^3+\ln x}$; c) $\lim\limits_{x\to 1} \frac{x^2-x}{x^2-3x+2}$.

6. **[2 points]** Let a be a limit point of $A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$ satisfy the following properties: 1) f is bounded on A; 2) $g(x) \to 0$, $x \to a$. Show that $\lim\limits_{x\to a}(f(x) \cdot g(x)) = 0$.
   (Hint: Use Squeeze theorem for functions)

7. **[2+2+2 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0-} \frac{x}{\sqrt{1-\cos^2 x}}$; b) $\lim\limits_{x\to 0+} \frac{x}{\sqrt{1-\cos^2 x}}$; c) $\lim\limits_{x\to 0+} e^{-\frac{1}{x}}$.

B.5 Problem sheet 5

1. **[1+1 points]** Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. a) For which a, b the function f is monotone on $\mathbb{R}$? b) For which a, b the function f is continuous on $\mathbb{R}$?

2. **[1+1 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0}(\tan x - e^x)$; b) $\lim\limits_{x\to 2} \frac{x^2-3^x+1}{x-\sin \pi x}$.

3. **[2 points]** Let $f(x) = \lfloor x \rfloor \sin \pi x$, $x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ and sketch its graph. (*Hint:* If $x \in [k, k+1)$ for some $k \in \mathbb{Z}$, then $\lfloor x \rfloor = k$ and $f(x) = k \sin \pi x$. Find $f(k-)$ and $f(k+)$ at the points k.)

4. **[2 points]** Prove that the function $f(x) = \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

5. **[2x6 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0} \frac{\arcsin x}{x}$; b) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$; c) $\lim\limits_{x\to+\infty} x(\ln(1+x) - \ln x)$; d) $\lim\limits_{x\to 0} \left(\frac{1+\sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$;
   e) $\lim\limits_{x\to 0} \frac{\sqrt[7]{\cos x}-1}{\sqrt[3]{1+x^2}-1}$; f) $\lim\limits_{x\to 0} \frac{e^{\sin 2x}-e^{\tan x}}{x}$.

6. **[2 points]** Prove that the function $P(x) = x^3 + 7x^2 - 1$, $x \in \mathbb{R}$, has at least one root, that is, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = 0$.

7. **[3 points]** Let $f : [a, b] \to \mathbb{R}$ strictly increase on $[a, b]$ and for each $y_0 \in [f(a), f(b)]$ there exist $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Prove that f is continuous on $[a, b]$.

8. **[2 points]** Using the definition, show that the function $f(x) = \sqrt{x}$, $x \in [1, +\infty)$, is uniformly continuous on $[1, +\infty)$.

B.6 Problem sheet 6

1. **[1+1+1 points]** Express through $f'(a)$ the following limits:
   a) $\lim\limits_{h\to 0} \frac{f(a+2h)-f(a)}{h}$; b) $\lim\limits_{h\to 0} \frac{f(a+h)-f(a-h)}{h}$; c) $\lim\limits_{n\to\infty} n\left(f\left(\frac{n+1}{n}a\right)-f(a)\right)$.

2. **[2 points]** Using the definition of derivative, check that $(x|x|)' = 2|x|$, $x \in \mathbb{R}$.

3. **[3 points]** For the function $f(x) = |x^2 - x|$, $x \in \mathbb{R}$, compute $f'(x)$ for each $x \in \mathbb{R} \setminus \{0, 1\}$. Compute left and right derivatives at points 0 and 1.

4. **[1+2 points]** Let
$$f(x) = \begin{cases} x^2, & x \leq 1, \\ ax + b, & x > 1. \end{cases}$$
   For which $a, b \in \mathbb{R}$ the function f:
   a) is continuous on $\mathbb{R}$; b) is differentiable on $\mathbb{R}$? Compute also f'.

5. **[2x3 points]** Check whether the following functions are differentiable at 0. Justify your answer.
   a) $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x \neq 0, \\ 0, & x = 0; \end{cases}$ b)$f(x) = \sqrt[5]{x^2}$, $x \in \mathbb{R}$; c) $f(x) = |\sin x|$, $x \in \mathbb{R}$.

6. **[1x8 points]** Compute derivatives of the following functions:
   a) $f(x) = x^2 \sin x$; b) $f(x) = e^{-\frac{x^2}{2}} \cos x$; c) $f(x) = \frac{x}{1+x^2}$; d) $f(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$;
   e) $f(x) = 2^{\tan(x^2-1)}$; f) $f(x) = \sin(\cos^2(\tan^3 x))$; g) $f(x) = \sqrt[3]{\frac{1+x^3}{1-x^3}}$;
   h) $f(x) = e^{ax} \cdot \frac{a \sin bx - b \cos bx}{\sqrt{a^2+b^2}}$, where a, b are some constants.

B.7 Problem sheet 7

1. **[1x6 points]** Compute derivatives of the following functions:
   a) $f(x) = \frac{1}{4} \ln \frac{x^2-1}{x^2+1}$; b) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$; c) $f(x) = \ln \tan \frac{x}{2}$;
   d) $f(x) = \arcsin \frac{1-x}{\sqrt{2}}$; e) $f(x) = \arctan \frac{1+x}{1-x}$; f) $f(x) = \sqrt[x]{x}$.

2. **[2 points]** Let a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

3. **[2 points]** Prove the equality

$$3 \arccos x - \arccos(3x - 4x^3) = \pi, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

   (*Hint:* Compute derivatives of the left and right hand sides of the equality)

4. **[3 points]** Let functions $f, g : (a, b) \to (0, +\infty)$ be differentiable on (a, b) and for every $x \in (a, b)$ $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$. Prove that there exists $L > 0$ such that $f(x) = Lg(x)$ for all $x \in (a, b)$.
   (*Hint:* Consider the functions $\ln f$ and $\ln g$)

5. **[3 points]** (*Generalised Bernoulli inequality*) For each $\alpha > 1$, prove that $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x > -1$. Moreover, $(1 + x)^\alpha = 1 + \alpha x$ iff $x = 0$.

6. **[2x4 points]** Identify the intervals on which the following functions are monotone.
   a) $f(x) = 3x - x^3$, $x \in \mathbb{R}$ b)$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{x^2}{2^x}$, $x \in \mathbb{R}$;
   d) $f(x) = x + \sqrt{|1 - x^2|}$, $x \in \mathbb{R}$.

B.8 Problem sheet 8

1. **[1x4 points]** Using L'Hospital's Rule, show that

   a) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{\sin x} = 1$; b) $\lim\limits_{x\to e} \frac{(\ln x)^\alpha - \left(\frac{x}{e}\right)^\beta}{x-e} = \frac{\alpha-\beta}{e}$, where α, β are some real numbers;
   c) $\lim\limits_{x\to+\infty} \frac{\ln x}{x^\varepsilon} = 0$ for all $\varepsilon > 0$; d) $\lim\limits_{x\to+0} (\ln(1+x))^x = 1$.

2. **[3x3 points]** Using L'Hospital's Rule, compute the following limits:

   a) $\lim\limits_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$; b) $\lim\limits_{x\to 0+} \left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}}$; c) $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$.

3. **[2x3 points]** Compute the n-th derivative of the following functions:
   a) $f(x) = 2^{x-1}$, $x \in \mathbb{R}$; b) $f(x) = \sqrt{2x-1}$, $x > \frac{1}{2}$; c) $(x^2e^x)^{(n)}$, $x \in \mathbb{R}$.

   (*Hint:* Use the Leibniz Formula in c))

4. **[3 points]** Write Taylor's expansion of the function e^{2x-x^2}, $x \in \mathbb{R}$ at the point $x_0 = 0$ up to the term with x^5.

5. **[2x2 points]** Use Taylor's formula to compute the limits

   a) $\lim\limits_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$; b) $\lim\limits_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3}$.

B.9 Problem sheet 9

1. **[1x3 points]** Find points of local extrema of the following functions:
   a) $f(x) = x^2 e^x$, $x \in \mathbb{R}$; b) $f(x) = x + \frac{1}{x}$, $x > 0$; c) $f(x) = |x| e^{-x^2}$, $x \in \mathbb{R}$.

2. **[1x3 points]** Identify intervals on which the following functions are convex or concave:
   a) $f(x) = \frac{1}{x}$, $x > 0$; b) $f(x) = \arctan x$, $x \in \mathbb{R}$.

3. **[3 points]** *(Young's inequality)* Let $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
   Prove that $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for all $x, y \in (0, +\infty)$.
   (*Hint:* Consider the function $f(x) = -\ln x$, $x > 0$, and use its convexity on $(0, +\infty)$)

4. **[2x9 points]** Compute the following indefinite integrals:
   a) $\int \frac{dx}{x \ln x}$, $x > 0$; b) $\int \frac{(2x+1)dx}{\sqrt[3]{1+x+x^2}}$, $x \in \mathbb{R}$; c) $\int \frac{\sqrt{\tan x}}{\cos^2 x} dx$, $x \in \left(0, \frac{\pi}{2}\right)$;
   d) $\int \frac{dx}{\cos x}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; e) $\int \cos^2 x \sin^3 x dx$, $x \in \mathbb{R}$; f) $\int \frac{dx}{x^2+x+1}$, $x \in \mathbb{R}$;
   g) $\int x^2 \sin x dx$, $x \in \mathbb{R}$; h) $\int (\ln x)^2 dx$, $x > 0$; i) $\int e^{2x} \cos x dx$, $x \in \mathbb{R}$.

B.10 Problem sheet 10

1. **[3 points]** Using the definition of the integral, prove that the function $f(x) = x$, $x \in [0, 1]$, is integrable on $[0, 1]$ and compute $\int_0^1 x dx$.

2. **[2 points]** Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Show that f is integrable on $[a, b]$, if it is integrable on $[a, c]$ and $[c, b]$.

   (*Hint:* Use the integrability criterion (Theorem 16.2))

3. **[3 points]** Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and g be a non-negative integrable function on $[a, b]$. Show that there exists $\theta \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(\theta) \int_a^b g(x)dx$.

4. **[2 points]** Let $f : [0, 1] \to \mathbb{R}$ be integrable on $[0, 1]$. Prove the equality

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f(x)dx = \int_0^1 f(x)dx.$$

5. **[2x5 points]** Compute the following integrals:
   a) $\int_0^{\frac{\pi}{2}} \sin 2x dx$; b) $\int_0^2 |1 - x| dx$; c) $\int_0^{2\pi} x^2 \cos x dx$; d) $\int_{-1}^1 \frac{x dx}{\sqrt{5-4x}}$; e) $\int_0^{\ln 2} \sqrt{e^x - 1} dx$.

6. **[3 points]** Compute the area of the region bounded by the graphs of the following functions: $y = x^2$ and $x + y = 2$.

7. **[3 points]** Compute the length of the cycloid, the continuous curve defined by the following functions: $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.

B.11 Problem sheet 11

1. **[2x3 points]** Compute the following improper integrals:
   a) $\int_0^{+\infty} \frac{dx}{x^2+x+1}$; b) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; c) $\int_0^{+\infty} x^2 e^{-x} dx$.

2. **[3 points]** Identify all $p \in \mathbb{R}$ for which the improper integral $\int_1^{+\infty} x^p e^{-x} dx$ converges. Justify your answer.

3. **[2x4 points]** Show that the following improper integrals converge:
   a) $\int_1^{+\infty} e^{-x^2} dx$; b) $\int_1^{+\infty} \frac{x-2}{x^3+x+1} dx$; c) $\int_0^{+\infty} \frac{\sin x}{1+x^2} dx$; d) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}} dx$.

4. **[2 points]** Show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots = 1$.
   (*Hint:* Use the equality $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$)

5. **[3 points]** Identify all $p > 0$ for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges. Justify your answer.

6. **[2x3 points]** Prove the convergence of the following series:
   a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n^2+1}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)^2$.

B.12 Problem sheet 12

1. **[2x4 points]** Investigate the convergence of the following series:
   a) $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$; b) $\sum_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$; c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$.

2. **[3x2 points]** Investigate the absolute and conditional convergence of the following series:
   a) $\sum_{n=1}^{\infty} (-1)^n \sin^\alpha \frac{1}{n}$, where $\alpha > 0$; b) $\sum_{n=1}^{\infty} \frac{\cos n}{n}$.

3. **[2 points]** Show that for each $x \in \mathbb{R}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

   (*Hint:* Use Taylor's formula with Lagrangian remainder term (see Theorem 14.1 and Example 14.1) to show that the remainder term converges to 0)

4. **[1x3 points]** Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:
   a) $(2+3i)^2(1+2i)$; b) $\frac{2+3i}{2-i}$; c) $\frac{1}{i} - \frac{1}{(1+i)^2}$.

5. **[2 points]** Compute the real and imaginary parts of $\frac{1}{z^2}$, where $z = x + iy$, $x, y \in \mathbb{R}$.

6. **[2 points]** Compute $(-\frac{1}{2} + \frac{\sqrt{3}}{2})^{21}$.

7. **[2x2 points]** Solve the following equations:
   a) $z^2 + z + 3 = 0$; b) $z^3 - i = 0$.

8. **[2 points]** Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$.

B.13 Problem sheet 13

1. **[2 points]** For a complex number α show that the coefficients of the polynomial

$$p(z) = (z - \alpha)(z - \overline{\alpha})$$

are real numbers.

2. **[3 points]** Let $p(z)$ be a polynomial with real coefficients and let α be a complex number. Prove that $p(\alpha) = 0$ if and only if $p(\overline{\alpha}) = 0$.

3. **[3 points]** Prove that any polynomial $p(z)$ with real coefficients can be decomposed into a product of polynomials of the form $az^2 + bz + c$, where $a, b, c \in \mathbb{R}$.

(*Hint:* Use the fundamental theorem of algebra and exercises 1., 2.)

4. **[6x2 points]** For each of the following sets, either show that the set is a vector space over $\mathbb{F}$ or explain why it is not a vector space.

   a) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{R}$.
   b) The set $\mathbb{R}$ of real numbers under the usual operations of addition and multiplication, $\mathbb{F} = \mathbb{C}$.
   c) The set $\{f \in \mathrm{C}[0, 1] : f(0) = 2\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.
   d) The set $\{f \in \mathrm{C}[0, 1] : f(0) = f(1) = 0\}$ under the usual operations of addition and multiplication of functions, $\mathbb{F} = \mathbb{R}$.
   e) The set $\{(x, y, z) \in \mathbb{R}^3 : x - 2y + z = 0\}$ under the usual operations of addition and multiplication on $\mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$.
   f) The set $\{(x, y, z) \in \mathbb{C}^3 : 2x + z + i = 0\}$ under the usual operations of addition and multiplication on $\mathbb{C}^3$, $\mathbb{F} = \mathbb{C}$.

B.14 Problem sheet 14

1. **[1 point]** Let V be a vector space over $\mathbb{F}$. Then, given $a \in \mathbb{F}$ and $v \in V$ such that $av = 0$, prove that either $a = 0$ or $v = 0$.

2. **[2x3 points]** Prove or give a counterexample to the following claim:

   1) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.
   2) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.

3. **[2 points]** Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let

   $$U = \{az^2 + bz^5 : \ a, b \in \mathbb{F}\}.$$

   Find a subspace W of $\mathbb{F}[z]$ such that $F[z] = U \oplus W$.

4. **[2x2 points]** Consider the complex vector space $V = \mathbb{C}^3$ and the list $\{v_1, v_2, v_3\}$ of vectors in V, where $v_1 = (i, 0, 0)$, $v_2 = (i, 1, 0)$ and $v_3 = (i, i, -1)$.
   a) Prove that $\text{span}\{v_1, v_2, v_3\} = V$.
   b) Prove or disprove that $\{v_1, v_2, v_3\}$ is a basis of V.

5. **[2 points]** Let V be a vector space over $\mathbb{F}$, and suppose that $v_1, v_2, \ldots, v_n \in V$ are linearly independent. Let w be a vector from V such that the vectors $v_1 + w, v_2 + w, \ldots, v_n + w$ are linearly dependent. Prove that $w \in \text{span}\{v_1, v_2, \ldots, v_n\}$.

6. **[3 points]** Let $p_0, p_1, \ldots, p_n \in \mathbb{F}_n[z]$ satisfy $p_j(2) = 0$ for all $j = 0, 1, \ldots, n$. Prove that $p_0, p_1, \ldots, p_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

7. **[3x1 points]** Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
   a) Show that T is linear; b) show that T is surjective; c) find $\dim(\ker T)$.

8. **[3 points]** Show that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ is surjective if

   $$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = 5x_2, \ x_3 = 7x_4\}.$$

9. **[3 points]** Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V. Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

10. **[3 points]** Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional. Given $T \in \mathcal{L}(V, W)$, prove that there is a subspace U of V such that $U \cap \ker T = \{0\}$ and $\text{range}\, T = \{T(u) : \ u \in U\}$.

11. **[3 points]** Let U, V and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

    $$\dim(\ker(TS)) \leq \dim(\ker T) + \dim(\ker S).$$

C Exam

1. Show that for every $n \in \mathbb{N}$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1, a_2, \ldots, a_n\}}{n} \to 0$, $n \to \infty$.

3. Is the function
$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad x \in (-\pi, \pi),$$
continuous on $(-\pi, \pi)$? Is f differentiable on $(-\pi, \pi)$? Compute the derivative of f at each point where it exists.

4. Compute the limit $\lim\limits_{x\to 0} \left(1 + \arcsin^2 x\right)^{\frac{1}{\tan^2 x}}$.

5. Find points of local maximum and minimum of the function $f(x) = x^2(x-5)^3$, $x \in \mathbb{R}$.

6. Compute the length of continuous curve defined by the function $y = x^{\frac{3}{2}}$, $x \in [0, 4]$.

7. Compute the improper integral $\int_2^{+\infty} \frac{\ln x}{x^2}\, dx$.

8. Investigate the absolute and conditional convergence of the series $\sum\limits_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{\sqrt{n}}\right)$.

9. Compute $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{12}$.

10. Show that there does not exist any linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with
$$\ker T = \{(x_1, x_2, x_3, x_4, x_5) : \; x_1 = x_2, \; x_3 = x_4 = -x_5\}.$$

11. Write the Taylor formulas with the Peano remainder term and the Lagrangian remainder term (indicate also conditions on the function under which the Taylor formulas are true).

12. Give definitions of linearly independent set of vectors, linear span and basis.

D Exam Solutions

1. Show that for every $n \in \mathbb{N}$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{49}$$

Solution. To prove equality (49), we use the mathematical induction. For $n = 1$ we have $1^2 = \frac{1\cdot 2\cdot 3}{6}$. We assume that (49) is true for $n \in \mathbb{N}$ and check it for $n+1$. So,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$
$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6}.$$

In order to finish the proof, we have to check that $n(2n+1)+6(n+1) = (n+2)(2(n+1)+1)$. For this, we compute $n(2n+1)+6(n+1) = 2n^2+7n+6$ and $(n+2)(2(n+1)+1) = (n+2)(2n+3) = 2n^2 + 7n + 6$.

2. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n \to \infty$.

Solution. Let $\varepsilon > 0$ be fixed. Since $\frac{a_n}{n} \to 0$, $n \to \infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ $\left|\frac{a_n}{n} - 0\right| = \frac{|a_n|}{n} < \varepsilon$. We next choose $N_2 \in \mathbb{N}$ such that $\frac{|a_k|}{N_2} < \varepsilon$ for all $k = 1, \ldots, N_1$. Thus, taking $N := \max\{N_1, N_2\}$, we can estimate for every $n \geq N$ and $k = 1, \ldots, n$

$$\frac{|a_k|}{n} \leq \frac{|a_k|}{N_2} < \varepsilon, \quad \text{if} \quad k \leq N_1,$$

and

$$\frac{|a_k|}{n} \leq \frac{|a_k|}{k} < \varepsilon, \quad \text{if} \quad N_1 < k \leq n.$$

Hence, we have

$$\left|\frac{\max\{a_1,a_2,\ldots,a_n\}}{n}\right| \leq \frac{\max\{|a_1|,|a_2|,\ldots,|a_n|\}}{n} < \varepsilon.$$

This implies that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n \to \infty$.

3. Is the function

$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad x \in (-\pi, \pi),$$

continuous on $(-\pi, \pi)$? Is f differentiable on $(-\pi, \pi)$? Compute the derivative of f at each point where it exists.

Solution. Since sin and cos are continuous functions and $\sin x \neq 0$ for all $x \in (-\pi,\pi) \setminus \{0\}$, the function f is continuous at each point of $(-\pi,\pi) \setminus \{0\}$. To check the continuity of f at 0, we compute

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{x^2(1-\cos x)}{x^2 \sin x}$$
$$= \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} = 0 \cdot 1 \cdot \frac{1}{2} = 0 = f(0).$$

Hence, the function f is continuous at 0 and, consequently, it is continuous on $(-\pi, \pi)$.

Similarly, f is differentiable at each point of $(-\pi, \pi) \setminus \{0\}$ because sin and cos are differentiable and $\sin x \neq 0$ for all $x \in (-\pi, \pi) \setminus \{0\}$. Moreover, for every $x \in (-\pi, \pi) \setminus \{0\}$

$$\begin{aligned} f'(x) &= \left(\frac{1-\cos x}{\sin x}\right)' = \frac{(1-\cos x)'\sin x - (1-\cos x)(\sin x)'}{\sin^2 x} \\ &= \frac{\sin^2 x - (1-\cos x)\cos x}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x - \cos x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x}. \end{aligned}$$

We next compute

$$\begin{aligned} f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0} \frac{1-\cos x}{x \sin x} = \lim_{x\to 0} \frac{x(1-\cos x)}{x^2 \sin x} \\ &= \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}. \end{aligned}$$

Hence the function f is differentiable on $(-\pi, \pi)$ and

$$f'(x) = \begin{cases} \frac{1-\cos x}{\sin^2 x}, & x \neq 0, \\ \frac{1}{2}, & x = 0, \end{cases} \qquad x \in (-\pi, \pi),$$

4. Compute the limit $\lim\limits_{x\to 0} \left(1 + \arcsin^2 x\right)^{\frac{1}{\tan^2 x}}$.

Solution.

$$\begin{aligned} \lim_{x\to 0} \left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}} &= \lim_{x\to 0} e^{\ln\left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}}} = \lim_{x\to 0} e^{\frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x}} \\ &= e^{\left(\lim\limits_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x}\right)}, \end{aligned}$$

by the continuity of the exponential function. So, we compute

$$\begin{aligned} \lim_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\tan^2 x} &= \lim_{x\to 0} \frac{\arcsin^2 x \cdot \ln\left(1+\arcsin^2 x\right)}{\arcsin^2 x \cdot \tan^2 x} = \lim_{x\to 0} \frac{\ln\left(1+\arcsin^2 x\right)}{\arcsin^2 x} \cdot \lim_{x\to 0} \frac{\arcsin^2 x}{\tan^2 x} \\ &= 1 \cdot \lim_{x\to 0} \frac{x^2 \cdot \arcsin^2 x}{x^2 \cdot \tan^2 x} = \lim_{x\to 0} \frac{\arcsin^2 x}{x^2} \cdot \lim_{x\to 0} \frac{x^2}{\tan^2 x} \\ &= \left(\lim_{x\to 0} \frac{\arcsin x}{x}\right)^2 \cdot \left(\lim_{x\to 0} \frac{x}{\tan x}\right)^2 = 1. \end{aligned}$$

Hence, $\lim\limits_{x\to 0} \left(1+\arcsin^2 x\right)^{\frac{1}{\tan^2 x}} = e^1 = e$.

5. Find points of local maximum and minimum of the function $f(x) = x^2(x-5)^3$, $x \in \mathbb{R}$.

Solution. We first compute critical points of f:

$$\begin{aligned} f'(x) &= \left(x^2(x-5)^3\right)' = \left(x^2\right)'(x-5)^3 + x^2\left((x-5)^3\right)' = 2x(x-5)^3 + 3x^2(x-5)^2 \\ &= x(x-5)^2\left(2(x-5)+3x\right) = x(x-5)^2(5x-10) = 0. \end{aligned}$$

Hence, the points $x = 0$, $x = 2$, $x = 5$ are critical.

The point 0 is a point of strict local maximum because the derivative changes its sign from "+" to "–", passing through 0.

The point 2 is a point of strict local minimum because the derivative changes its sign from "–" to "+", passing through 2.

The point 5 is not a point of local extrema because the derivative stays positive, passing through 5.

6. Compute the length of continuous curve defined by the function $y = x^{\frac{3}{2}}$, $x \in [0, 4]$.

   Solution. The length of the curve Γ defined by the function $y = x^{\frac{3}{2}}$, $x \in [0, 4]$, can be computed by the formula

$$l(\Gamma) = \int_0^4 \sqrt{1 + \left(\left(x^{\frac{3}{2}}\right)'\right)^2} dx = \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{\frac{1}{2}}\right)^2} dx = \int_0^4 \sqrt{1 + \frac{9}{4} x dx} = \left| \begin{array}{l} y = 1 + \frac{9}{4}x, \\ x = \frac{4}{9}(y-1), \\ dx = \frac{4}{9} dy \end{array} \right|$$
$$= \frac{4}{9} \int_1^{10} y^{\frac{1}{2}} dy = \frac{4}{9} \cdot \left. \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_1^{10} = \frac{8}{27} \cdot y^{\frac{3}{2}} \Big|_1^{10} = \frac{8}{27}(10\sqrt{10} - 1).$$

7. Compute the improper integral $\int_2^{+\infty} \frac{\ln x}{x^2} dx$.

   Solution. First we change the variable and then use the integration by parts formula:

$$\int_2^{+\infty} \frac{\ln x}{x^2} dx = \left| \begin{array}{l} y = \ln x, \\ x = e^y, \\ dx = e^y dy \end{array} \right| = \int_{\ln 2}^{+\infty} \frac{y e^y}{e^{2y}} dy = \int_{\ln 2}^{+\infty} y e^{-y} dy = -\int_{\ln 2}^{+\infty} y de^{-y}$$
$$= -y e^{-y} \Big|_{\ln 2}^{+\infty} + \int_{\ln 2}^{+\infty} e^{-y} dy = \ln 2 \cdot e^{-\ln 2} - e^{-y} \Big|_{\ln 2}^{+\infty} = \frac{\ln 2}{2} + e^{-\ln 2} = \frac{1 + \ln 2}{2}.$$

   Another way of the computation without change of variable:

$$\int_2^{+\infty} \frac{\ln x}{x^2} dx = -\int_2^{+\infty} \ln x d\frac{1}{x} = -\ln x \frac{1}{x} \Big|_2^{+\infty} + \int_2^{+\infty} \frac{1}{x} d \ln x = \frac{\ln 2}{2} + \int_2^{+\infty} \frac{1}{x^2} dx$$
$$= \frac{\ln 2}{2} - \frac{1}{x} \Big|_2^{+\infty} = \frac{1 + \ln 2}{2}$$

8. Investigate the absolute and conditional convergence of the series $\sum_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{\sqrt{n}}\right)$.

   Solution. The series

$$\sum_{n=1}^{\infty} \left| (-1)^n \ln\left(1 + \frac{1}{\sqrt{n}}\right) \right| = \sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{\sqrt{n}}\right)$$

   diverges because $\ln\left(1 + \frac{1}{\sqrt{n}}\right) \sim \frac{1}{\sqrt{n}}$, $n \to \infty$, and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Since the sequence $\ln\left(1+\frac{1}{\sqrt{n}}\right)$, $n \geq 1$, is monotone and converges to 0, the series

$$\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{\sqrt{n}}\right)$$

converges, according to Leibniz's test. This implies the conditional convergence of the series.

9. Compute $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{12}$.

Solution. We first write the numbers $1+\sqrt{3}i$ and $1-i$ in the polar form. We compute the absolute volume r and the argument θ of $1+\sqrt{3}i$. So, $r = \sqrt{1^2+(\sqrt{3})^2} = 2$ and $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$. Thus, $\theta = \frac{\pi}{3}$. So, we obtain

$$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Similarly,

$$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right).$$

Hence

$$\begin{aligned}\left(\frac{1+\sqrt{3}i}{1-i}\right)^{12} &= \left(\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)}\right)^{12} = 2^6\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)^{12} \\ &= 64\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)^{12} = 64\left(\cos\frac{12\cdot 7\pi}{12}+i\sin\frac{12\cdot 7\pi}{12}\right) = -64.\end{aligned}$$

10. Show that there does not exist any linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with

$$\ker T = \{(x_1,x_2,x_3,x_4,x_5) : \ x_1 = x_2, \ x_3 = x_4 = -x_5\}. \tag{50}$$

Solution. We assume that there exists a map $T : \mathbb{R}^5 \to \mathbb{R}^2$ with the kernel given by (50). We first note that

$$\ker T = \{(x_1,x_2,x_3,x_4,x_5) : \ x_1 = x_2, \ x_3 = x_4 = -x_5\} = \{(a,a,b,b,-b) : \ a,b \in \mathbb{R}\}.$$

Thus, the vectors $v_1 = (1,1,0,0,0)$ and $v_2 = (0,0,1,1,-1)$ form a basis of $\ker T$, since they are linearly independent and span $\ker T$. Hence, $\dim(\ker T) = 2$. Since $5 = \dim(\mathbb{R}^5) = \dim(\ker T) + \dim(\operatorname{range} T) = 2 + \dim(\operatorname{range} T)$, we have that $\dim(\operatorname{range} T) = 3$. But that is impossible because $\operatorname{range} T \subset \mathbb{R}^2$ and $\dim(\mathbb{R}^2) = 2$.

E Retake

1. For which $n \in \mathbb{N}$ the following inequality holds?
$$3^n > 5n + 2.$$

2. Compute the following limit
$$\lim_{n\to\infty} \sqrt[n]{n^5 5^n + n 3^n}.$$

3. Show that a sequence $(a_n)_{n\geq 1}$ of real numbers is a Cauchy sequence if and only if $\sup\limits_{n\geq k, m\geq k} |a_n - a_m| \to 0,\ k \to +\infty$.

4. For which $a \in \mathbb{R}$ the following function f is differentiable on $\mathbb{R}$?
$$f(x) = \begin{cases} \frac{\sin x}{e^x - 1}, & x \neq 0, \\ a, & x = 0, \end{cases} \quad x \in \mathbb{R}.$$
Compute the derivative of f.

5. Prove that the function $f(x) = x^x$ is increasing on $\left(\frac{1}{e}, \infty\right)$. Is it convex on $\left(\frac{1}{e}, \infty\right)$?

6. Compute the area of the region bounded by the graphs of the following functions $2x = y^2$ and $2y = x^2$.

7. Compute the improper integral $\int_0^\infty |x-1| e^{-x} dx$.

8. Does the following series converges?
$$\sum_{n=1}^{\infty} \frac{1}{n\left(\ln^2 n + 1\right)}.$$

9. Write the following complex numbers in algebraic form: $\frac{i}{(1-i)^2}$, $(1 - \sqrt{3}i)^{15}$.

10. Let $\mathbb{R}_4[z]$ denotes the vector space of all polynomials of degree at most 4 with coefficients in $\mathbb{R}$ and let the linear operator $T : \mathbb{R}_4[z] \to \mathbb{R}_4[z]$ is defined as follows $(T\mathbf{p})(z) = \mathbf{p}''(z)$ ($T\mathbf{p}$ is the second order derivative of polynomial $\mathbf{p}$). Identify $\ker T$ and $\operatorname{range} T$. Find a subspace W of $\mathbb{R}_4[z]$ such that $\ker T \oplus W = \mathbb{R}_4[z]$.

11. Formulate the Fermat theorem and the Lagrange (mean value) theorem.

12. Give the definition of subsequential limit of a sequence and the definitions of upper and lower limits.

F Retake Solutions

1. For which $n \in \mathbb{N}$ the following inequality holds?

$$3^n > 5n + 2. \tag{51}$$

Solution. We first we note that inequality (51) is not true for $n = 1$ ($3^1 < 5 \cdot 1 + 2$) and $n = 2$ ($3^2 < 5 \cdot 2 + 2$). If $n = 3$, then the inequality holds because $3^3 = 27 > 5 \cdot 3 + 2 = 17$. In order to show that inequality (51) is true for all $n \geq 3$, we will use the mathematical induction. Let us assume that (51) holds for $n = k$ for some $k \geq 3$, i.e. $3^k > 5k + 2$, and prove it for $n = k + 1$. So, $3^{n+1} = 3^n \cdot 3 > (5k + 2) \cdot 3 = 15k + 6 = 5(k + 1) + 2 + 10k - 1 > 5(k + 1) + 2$. Hence, the inequality $3^n > 5n + 2$ is true for all $n \geq 3$.

2. Compute the following limit

$$\lim_{n\to\infty} \sqrt[n]{n^5 5^n + n 3^n}.$$

Solution. In order to compute the limit we will use the squeeze theorem. For this, we estimate

$$5\left(\sqrt[n]{n}\right)^5 = \sqrt[n]{n^5 5^n} < \sqrt[n]{n^5 5^n + n3^n} < \sqrt[n]{n^5 5^n + n^5 5^n} = 5\sqrt[n]{2n^5} = 5\sqrt[n]{2}\left(\sqrt[n]{n}\right)^5.$$

Since $\lim_{n\to\infty} 5\left(\sqrt[n]{n}\right)^5 = 5 \cdot 1^5 = 5$ and $\lim_{n\to\infty} 5\sqrt[n]{2}\left(\sqrt[n]{n}\right)^5 = 5\lim_{n\to\infty}\sqrt[n]{2} \cdot \lim_{n\to\infty}\left(\sqrt[n]{n}\right)^5 = 5 \cdot 1 \cdot 1^5 = 5$, the squeeze theorem implies that

$$\lim_{n\to\infty} \sqrt[n]{n^5 5^n + n3^n} = 5.$$

3. Show that a sequence $(a_n)_{n\geq 1}$ of real numbers is a Cauchy sequence if and only if $\sup\limits_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$.

Solution. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence and let $\varepsilon > 0$ be fixed. By the definition of Cauchy sequence, there exists a number N such that

$$\forall n, m \geq N \quad |a_n - a_m| < \frac{\varepsilon}{2}.$$

Thus, $\sup\limits_{n\geq k, m\geq k} |a_n - a_m| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $k \geq N$. This implies that $\sup\limits_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$.

Next, we assume that $\sup\limits_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$. Then, by the definition of the convergence, we have that there exists a number N such that

$$\forall k \geq N \quad \sup_{n\geq k, m\geq k} |a_n - a_m| < \varepsilon.$$

In particular, this yields that for all $n, m \geq N$ $|a_n - a_m| < \varepsilon$. So, $(a_n)_{n\geq 1}$ is a Cauchy sequence.

4. For which $a \in \mathbb{R}$ the following function f is differentiable?

$$f(x) = \begin{cases} \frac{\sin x}{e^x - 1}, & x \neq 0, \\ a, & x = 0, \end{cases} \qquad x \in \mathbb{R}.$$

Compute the derivative of f.

Solution. It is clear that the function f is differentiable on $\mathbb{R} \setminus \{0\}$. So, we have to check whether f is differentiable at $x = 0$. For this we first find a for which f is continuous. Only for that a the function could be differentiable. We compute

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{e^x - 1} = \lim_{x\to 0} \frac{x \sin x}{x(e^x - 1)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{x}{e^x - 1} = 1.$$

So, only for $a = 1$ $\lim_{x\to 0} f(x) = f(0)$. This implies that f is continuous on $\mathbb{R}$ for $a = 1$. Let us check that f is differentiable at $x = 0$ for $a = 1$.

We compute for $a = 1$

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\sin x}{e^x - 1} - 1}{x} = \lim_{x\to 0} \frac{\sin x - e^x + 1}{(e^x - 1)x} \\ &= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{3!} + o(x^3) - \cancel{1} - \cancel{x} - \frac{x^2}{2!} - o(x^2) + \cancel{1}}{(\cancel{1} + x + o(x) - \cancel{1})x} \\ &= \lim_{x\to 0} \frac{\cancel{x^2}\left(-\frac{x}{3!} + \frac{o(x^3)}{x^2} - \frac{1}{2!} - \frac{o(x^2)}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{o(x)}{x}\right)} = -\frac{1}{2!} = -\frac{1}{2}. \end{aligned}$$

Consequently, $f'(0) = -\frac{1}{2}$ for $a = 1$. If $a \neq 1$, then the function f is not differentiable at $x = 0$ because it is not continuous.

It remains to compute

$$f'(x) = \left(\frac{\sin x}{e^x - 1}\right)' = \frac{\cos x\,(e^x - 1) - e^x \sin x}{(e^x - 1)^2}, \quad x \neq 0.$$

5. Prove that the function $f(x) = x^x$ is increasing on $\left(\frac{1}{e}, \infty\right)$. Is it convex on $\left(\frac{1}{e}, \infty\right)$?

   Solution. In order to show that f is increasing, it is enough to show that its derivative is positive. So, we compute

   $$f'(x) = (x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x \ln x}\right)' = e^{x \ln x}\left(\ln x + \frac{x}{x}\right) = x^x\,(\ln x + 1) > 0$$

   for $x \in \left(\frac{1}{e}, \infty\right)$. Hence the function f is strictly increasing on $\left(\frac{1}{e}, \infty\right)$. To check the convexity, we compute the second derivative:

   $$f''(x) = (x^x\,(\ln x + 1))' = (x^x)'\,(\ln x + 1) + x^x\,(\ln x + 1)' = x^x\,(\ln x + 1)^2 + x^x \frac{1}{x} > 0$$

   on $\left(\frac{1}{e}, \infty\right)$. Thus, the function f is strictly convex on $\left(\frac{1}{e}, \infty\right)$.

6. Compute the area of the region bounded by the graphs of the following functions $2x = y^2$ and $2y = x^2$.

   Solution. We have to compute the region between two parabolas which intersect each other at points $x = 0$ and $x = 2$ (the points of intersection can be found from the equation $2x = \left(\frac{x^2}{2}\right)^2$). Thus, the area can be computed by the formula

   $$\int_0^2 \left(\sqrt{2x} - \frac{x^2}{2}\right) dx = \sqrt{2} \int_0^2 x^{\frac{1}{2}} dx - \frac{1}{2} \int_0^2 x^2 dx = \sqrt{2} \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 - \frac{1}{6} x^3 \Big|_0^2 = \frac{4}{3}.$$

7. Compute the improper integral $\int_0^\infty |x-1|e^{-x}dx$.

   Solution.

$$\int_0^\infty |x-1|e^{-x}dx = \int_0^1 |x-1|e^{-x}dx + \int_1^\infty |x-1|e^{-x}dx = -\int_0^1 (x-1)e^{-x}dx + \int_1^\infty (x-1)e^{-x}dx.$$

   Let us compute the indefinite integral

$$\begin{aligned}\int (x-1)e^{-x}dx &= -\int (x-1)de^{-x} = -(x-1)e^{-x} + \int e^{-x}d(x-1) \\ &= -(x-1)e^{-x} + \int e^{-x}dx = -(x-1)e^{-x} - e^{-x} + C = -xe^{-x} + C.\end{aligned}$$

   By the fundamental theorem of calculus,

$$\int_0^\infty |x-1|e^{-x}dx = -\left(-xe^{-x}\right)\Big|_0^1 + \left(-xe^{-x}\right)\Big|_1^\infty = e^{-1} - 0 + 0 + e^{-1} = 2e^{-1} = \frac{2}{e}.$$

8. Does the following series converges?

$$\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n + 1\right)}.$$

   Solution. Since the sequence $\frac{1}{n\left(\ln^2 n+1\right)}$ decreases, we can use the integral criterion. According to that criterion, the convergence of the series $\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n+1\right)}$ is equivalent to the convergence of the improper integral

$$\int_1^\infty \frac{dx}{x\left(\ln^2 x+1\right)} = \int_1^\infty \frac{d\ln x}{\ln^2 x+1} \overset{y=\ln x}{=} \int_0^\infty \frac{dy}{y^2+1} = \arctan y\Big|_0^\infty = \frac{\pi}{2} < \infty.$$

   Hence, the series $\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n+1\right)}$ converges.

9. Write the following complex numbers in algebraic form: $\frac{i}{(1-i)^2}$, $(1-\sqrt{3}i)^{15}$.

   Solution.

$$\frac{i}{(1-i)^2} = \frac{i}{1-2i+i^2} = \frac{i}{1-2i-1} = \frac{i}{-2i} = -\frac{1}{2}.$$

   In order to compute $(1-\sqrt{3}i)^{15}$, we will use de Moivre's formula. For this, we need to rewrite the complex number $1-\sqrt{3}i$ in polar form. So, the absolute value r of $1-\sqrt{3}i$ is given by the formula $r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$. The argument θ of $1-\sqrt{3}i$ can be found from the equalities $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$. Hence, $\theta = -\frac{\pi}{3}$. Consequently, we can compute

$$\begin{aligned}(1-\sqrt{3}i)^{15} &= \left(2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right)^{15} = 2^{15}\left(\cos\left(-\frac{15\pi}{3}\right) + i\sin\left(-\frac{15\pi}{3}\right)\right) \\ &= 2^{15}\left(\cos(-5\pi) + i\sin(-5\pi)\right) = -2^{15}.\end{aligned}$$

10. Let $\mathbb{R}_4[z]$ denotes the vector space of all polynomials of degree at most 4 with coefficients in $\mathbb{R}$ and let the linear operator $T : \mathbb{R}_4[z] \to \mathbb{R}_4[z]$ is defined as follows $(T\mathbf{p})(z) = \mathbf{p}''(z)$ ($T\mathbf{p}$ is the second order derivative of polynomial $\mathbf{p}$). Identify $\ker T$ and $\operatorname{range} T$. Find a subspace W of $\mathbb{R}_4[z]$ such that $\ker T \oplus W = \mathbb{R}_4[z]$.

    Solution. We take $\mathbf{p} \in \mathbb{R}_n[z]$. Then $\mathbf{p}$ can be written as $\mathbf{p}(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$, where $a_i \in \mathbb{R}$, $i = 0, \ldots, 4$. So, by the definition of T,

$$(T\mathbf{p})(z) = 12a_4z^2 + 6a_3z + 2a_2 \in \mathbb{R}_2[z]. \tag{52}$$

    This implies that $T\mathbf{p} = \mathbf{0}$ if and only if $a_2 = a_3 = a_4 = 0$. Hence,

$$\ker T = \{\mathbf{p} \in \mathbb{R}_4[z] : \ T\mathbf{p} = \mathbf{0}\} = \{\mathbf{p}(z) = a_1z + a_0 : \ a_0, a_1 \in \mathbb{R}\} = \mathbb{R}_1[z].$$

    Next, we compute

$$\operatorname{range} T = \{\mathbf{q} \in \mathbb{R}_4[z] : \ \exists \mathbf{p} \in \mathbb{R}_4[z] \ \text{ such that } \ T\mathbf{p} = \mathbf{q}\} = \mathbb{R}_2[z].$$

    Indeed, if $\mathbf{q}(z) = b_2z^2 + b_1z + b_0 \in \mathbb{R}_2[z]$, then for $\mathbf{p}(z) = \frac{b_2}{12}z^4 + \frac{b_1}{6}z^3 + \frac{b_1}{2}z^2$ we trivially have $T\mathbf{p} = \mathbf{q}$. So, $\operatorname{range} T \supset \mathbb{R}_2[z]$. Moreover, equality (52) implies $\operatorname{range} T \subset \mathbb{R}_2[z]$.

    In order to find a vector subspace W of $\mathbb{R}_4[z]$ such that $\ker T \oplus W = \mathbb{R}_4[z]$, we recall that it should be a vector subspace such that $\ker T + W = \mathbb{R}_4[z]$ and $\ker T \cap W = \{\mathbf{0}\}$. We set

$$W = \left\{\mathbf{q}(z) = b_4z^4 + b_3z^3 + b_2z^2 : \ b_2, b_3, b_4 \in \mathbb{R}\right\}.$$

    It is easily to see that W is a vector subspace of $\mathbb{R}_4[z]$ and $\ker T + W = \mathbb{R}_2[z] + W = \mathbb{R}_4[z]$. Moreover, only zero polynomial belongs to both $\ker T = \mathbb{R}_2[z]$ and W. Hence, $\ker T \oplus W = \mathbb{R}_4[z]$.

G Some Important Limits

G.1 Limits of sequences

- $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0,\ \alpha > 0;$ *(Corollary 3.1)*
- $\lim\limits_{n\to\infty} \frac{n^b}{a^n} = 0,\ a \in \mathbb{R},\ |a| > 1,\ b \in \mathbb{R};$ *(Theorem 3.3)*
- $\lim\limits_{n\to\infty} \frac{\lg n}{n^\alpha} = 0,\ \alpha > 0;$ *(Exercise 3.6 c))*
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1;$ *(Theorem 3.4)*
- $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0,\ a > 0;$ *(Example 4.2 for $a = 10$)*
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e;$ *(Section 4.2)*
- $\lim\limits_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right) = 1.$ *(Exercise 4.9)*

G.2 Limits of functions

- $\lim\limits_{x\to+\infty} \frac{x^b}{a^x} = 0,\ a \in \mathbb{R},\ |a| > 1,\ b \in \mathbb{R};$ *(Example 7.3 for $b \in \mathbb{N}$ and Exercise 7.6 for the general case)*
- $\lim\limits_{x\to+\infty} \frac{\ln x}{x^\alpha} = 0,\ \alpha > 0;$ *(Exercise 7.2 for $\alpha = 1$)*
- $\lim\limits_{x\to0} \frac{\sin x}{x} = 1;$ *(Example 6.5)*
- $\lim\limits_{x\to0} \frac{\tan x}{x} = 1;$ *(Example 7.8 a))*
- $\lim\limits_{x\to0} \frac{\arcsin x}{x} = 1;$ *(Exercise 8.11 c))*
- $\lim\limits_{x\to0} \frac{\arctan x}{x} = 1;$ *(Exercise 8.11 e))*
- $\lim\limits_{x\to0} \frac{1-\cos x}{x^2} = \frac{1}{2};$ *(Exercise 7.8 d))*
- $\lim\limits_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = \lim\limits_{x\to0} (1+x)^{\frac{1}{x}} = e;$ *(Example 7.2)*
- $\lim\limits_{x\to0} \frac{e^x-1}{x} = 1;$ *(A partial case of Theorem 8.7)*
- $\lim\limits_{x\to0} \frac{a^x-1}{x} = \ln a,\ a > 0;$ *(Theorem 8.7)*
- $\lim\limits_{x\to0} \frac{\ln(1+x)}{x} = 1;$ *(A partial case of Theorem 8.6)*
- $\lim\limits_{x\to0} \frac{\log_a(1+x)}{x} = \log_a e,\ a > 0,\ a \neq 1;$ *(Theorem 8.6)*
- $\lim\limits_{x\to0} \frac{(1+x)^\alpha-1}{x} = \alpha,\ \alpha \in \mathbb{R}.$ *(Theorem 8.8)*

G.3 Derivatives of Elementary Functions

- $(x^\alpha)' = \alpha x^{\alpha-1}$, $x > 0$, for $\alpha \in \mathbb{R}$; *(Example 10.6)*
- $(x^n)' = mx^{m-1}$, $x \in \mathbb{R} \setminus \{0\}$, for $m \in \mathbb{Z}$; *(Exercise 10.6)*
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, for $n \in \mathbb{N}$; *(Exercise 10.6)*
- $(a^x)' = a^x \ln a$, $x \in \mathbb{R}$, for $a > 0$; *(Example 10.7)*
- $(e^x)' = e^x$, $x \in \mathbb{R}$; *(Example 10.7)*
- $(\sin x)' = \cos x$, $x \in \mathbb{R}$; *(Example 10.8)*
- $(\cos x)' = -\sin x$, $x \in \mathbb{R}$; *(Example 10.8)*
- $(\tan x)' = \frac{1}{\cos^2 x}$, $x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi k : \ k \in \mathbb{Z}\right\}$; *(Example 10.8)*
- $(\cot x)' = -\frac{1}{\sin^2 x}$, $x \in \mathbb{R} \setminus \{\pi k : \ k \in \mathbb{Z}\}$; *(Example 10.8)*
- $(\log_\alpha x)' = \frac{1}{x \ln \alpha}$, $x > 0$, for $\alpha > 0$, $\alpha \neq 1$; *(Example 11.1)*
- $(\ln x)' = \frac{1}{x}$, $x > 0$; *(Example 11.1)*
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$; *(Example 11.3)*
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$; *(Exercise 11.3)*
- $(\arctan x)' = \frac{1}{1+x^2}$, $x \in \mathbb{R}$; *(Example 11.2)*
- $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$, $x \in \mathbb{R}$; *(Exercise 11.2)*
- $(\sinh x)' = \cosh x$, $x \in \mathbb{R}$; *(Exercise 10.9)*
- $(\cosh x)' = \sinh x$, $x \in \mathbb{R}$; *(Exercise 10.9)*
- $(\tanh x)' = \frac{1}{\cosh^2 x}$, $x \in \mathbb{R}$; *(Exercise 10.9)*
- $(\coth x)' = -\frac{1}{\sinh^2 x}$, $x \in \mathbb{R} \setminus \{0\}$. *(Exercise 10.9)*

G.4 Taylor's Expansion of Elementary Functions

- $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + o(x^n), \quad x \to 0;$ *(Example 13.5)*
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n), \quad x \to 0;$ *(Example 13.6)*
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)x^n}{n!} + o(x^n), \quad x \to 0, \quad \alpha \in \mathbb{R};$ *(Example 13.7)*
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0;$ *(Exercise 13.8)*
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0;$ *(Exercise 13.8)*
- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0;$ *(Exercise 13.9)*
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$ *(Exercise 13.9)*

G.5 Indefinite Integrals of Elementary Functions

- $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$, $x \in (0, +\infty)$, for all $\alpha \in \mathbb{R} \setminus \{-1\}$; *(Example 10.6)*
  $\int x^n dx = \frac{x^n}{n+1} + C$, $x \in \mathbb{R}$, for all $n \in \mathbb{N} \cup \{0\}$; *(Exercise 10.6)*
- $\int \frac{1}{x} dx = \ln|x| + C$ on each interval $(-\infty, 0)$ and $(0, +\infty)$; *(Example 11.1)*
- $\int a^x dx = \frac{a^x}{\ln a} + C$, $x \in \mathbb{R}$ for all $a > 0$, $a \neq 1$; *(Example 10.7)*
- $\int e^x dx = e^x + C$, $x \in \mathbb{R}$; *(Example 10.7)*
- $\int \cos x dx = \sin x + C$, $x \in \mathbb{R}$; *(Example 10.8)*
- $\int \sin x dx = -\cos x + C$, $x \in \mathbb{R}$; *(Example 10.8)*
- $\int \frac{dx}{\cos^2 x} = \tan x + C$ on each interval $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, $n \in \mathbb{Z}$; *(Example 10.8)*
- $\int \frac{dx}{\sin^2 x} = -\cot x + C$ on each interval $(n\pi, \pi + n\pi)$, $n \in \mathbb{Z}$; *(Example 10.8)*
- $\int \frac{dx}{1+x^2} = \arctan x + C$, $x \in \mathbb{R}$; *(Example 11.2)*
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$, $x \in (-1, 1)$; *(Example 11.3)*
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$, $x \in \mathbb{R}$, $a \neq 0$;
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right| + C$ on $(-\infty, -a)$, $(-a, a)$ and $(a, +\infty)$, $a > 0$;
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$, $x \in (-a, a)$, $a > 0$;
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$ on $(-\infty, -a)$ and $(a, +\infty)$, $a > 0$;
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C$, $x \in \mathbb{R}$, $a \neq 0$.