7 Lecture 7 – Limits of Functions. Left- and Right-Sided Limits

7.1 Limit of Functions via $\varepsilon - \delta$ Approach

Let A be a subset of $\mathbb{R}$. We recall that $B(a,\varepsilon) = (a-\varepsilon, a+\varepsilon)$ denotes the ε-neighbourhood of a.

Theorem 7.1. *(i) Let p be a real number and $a \in \mathbb{R}$ be a limit point of A. Then $\lim_{x\to a} f(x) = p$ is equivalent to*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A \cap B(a,\delta),\ x \neq a:\ |f(x) - p| < \varepsilon.$$

ii) If $p = +\infty$ and $a \in \mathbb{R}$, then $\lim_{x\to a} f(x) = +\infty$ is equivalent to

$$\forall C \in \mathbb{R}\ \exists \delta > 0\ \forall x \in A \cap B(a,\delta),\ x \neq a:\ f(x) > C.$$

iii) If $p \in \mathbb{R}$ and $a = +\infty$, then $\lim_{x\to+\infty} f(x) = p$ is equivalent to

$$\forall \varepsilon > 0\ \exists D \in \mathbb{R}\ \forall x > D:\ |f(x) - p| < \varepsilon.$$

iv) If $p = +\infty$ and $a = +\infty$, then $\lim_{x\to+\infty} f(x) = +\infty$ is equivalent to

$$\forall C \in \mathbb{R}\ \exists D \in \mathbb{R}\ \forall x > D:\ f(x) > D.$$

Example 7.1. $A = \mathbb{R} \setminus \{1\}$, $a = 1$ and $f(x) = \frac{x^2-1}{x-1}$, $x \in A$. Then $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$. Indeed, let us fix an arbitrary $\varepsilon > 0$. Then we can take $\delta := \varepsilon$ because for all $x \in A \cap B(1,\delta)$ we have $\left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1| < \delta = \varepsilon$.

Example 7.2. We show that $\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e$.

By the definition of the number e (see Section 4.2), we have

$$\left(1+\frac{1}{n+1}\right)^n = \left(1+\frac{1}{n+1}\right)^{n+1}\frac{n+1}{n+2} \to e \quad \text{and} \quad \left(1+\frac{1}{n}\right)^{n+1} \to e, \quad n\to\infty.$$

Hence, using the definition of the limit (see Definition 3.3), we obtain that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$

$$e - \varepsilon < \left(1+\frac{1}{n+1}\right)^n, \quad \left(1+\frac{1}{n}\right)^{n+1} < e + \varepsilon.$$

So, taking $D := N$, we can estimate for each $x > D$

$$e - \varepsilon < \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1} < e+\varepsilon,$$

where $\lfloor x\rfloor$ is the greatest integer number less than or equal to x, e.g. $\lfloor 1,7\rfloor = 1$, $\lfloor -\frac{1}{2}\rfloor = -1$, $\lfloor \pi\rfloor = 3$. Consequently, $\left|\left(1+\frac{1}{x}\right)^x - e\right| < \varepsilon$ for all $x > D$. This implies $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, by Theorem 7.1 (iii).

Exercise 7.1. Compute the following limits
a) $\lim_{x\to 0}\left(x \sin\frac{1}{x}\right)$; b) $\lim_{x\to 0}\left(x\left\lfloor\frac{1}{x}\right\rfloor\right)$.

Example 7.3. Let $b > 1$, $A = \mathbb{R}$, $m \in \mathbb{N}$ and $f(x) = x^m b^{-x}$, $x \in \mathbb{R}$.
We show that $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{x^m}{b^x} = 0$.

Solution. Let $\varepsilon > 0$ be given. According to Theorem 3.3, we have $\frac{(n+1)^m}{b^n} = \frac{(n+1)^m}{b^{n+1}} b \to 0$, $n \to \infty$. By the definition of the limit (see Definition 3.3), there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $\frac{(n+1)^m}{b^n} < \varepsilon$. Thus, taking $D := N$, we obtain that for each $x > D$ $\left|\frac{x^m}{b^x} - 0\right| = \frac{x^m}{b^x} < \frac{(\lfloor x\rfloor+1)^m}{b^{\lfloor x\rfloor}} < \varepsilon$. This implies $\lim\limits_{x\to+\infty} \frac{x^m}{b^x} = 0$, by Theorem 7.1 (iii).

Exercise 7.2. Prove that $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$.

7.2 Properties of Limits

Let a be a limit point of a set A.

Theorem 7.2. *If* $\lim\limits_{x\to a} f(x) = p_1$ *and* $\lim\limits_{x\to a} f(x) = p_2$, *then* $p_1 = p_2$.

Proof. The theorem immediately follows from the uniqueness of limit for sequences (see Theorem 3.1). Indeed, let $\{x_n\}_{n\geq 1}$ be an arbitrary sequence from A such that $x_n \neq a$, for all $n \geq 1$ and $x_n \to a$, then by the definition of the limit (see Definition 6.6), $f(x_n) \to p_1$, $n \to \infty$, and $f(x_n) \to p_2$, $n \to \infty$. By the uniqueness of limit for sequences (see Theorem 3.1), one has $p_1 = p_2$. □

Theorem 7.3. *Let functions* $f, g : A \to \mathbb{R}$ *satisfy the following properties: a)* $f(x) \leq g(x)$ *for all* $x \in A$; *2)* $\lim\limits_{x\to a} f(x) = p$ *and* $\lim\limits_{x\to a} g(x) = q$. *Then* $p \leq q$, *that is,* $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$.

Proof. The theorem immediately follows from Theorem 3.6. □

Exercise 7.3. Prove Theorem 7.3.

Theorem 7.4 (Squeeze theorem for functions). *Let* $f, g, h : A \to \mathbb{R}$ *satisfy the following conditions:*

a) $f(x) \leq h(x) \leq g(x)$ *for all* $x \in A$;

b) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = p$.

Then $\lim\limits_{x\to a} h(x) = p$.

Proof. The theorem follows from the Squeeze theorem for sequences (see Theorem 3.7). □

Exercise 7.4. Prove Theorem 7.4.

Theorem 7.5. *We assume that for functions* $f, g : A \to \mathbb{R}$ *there exists limits* $\lim\limits_{x\to a} f(x) = p \in \mathbb{R}$ *and* $\lim\limits_{x\to a} g(x) = q \in \mathbb{R}$. *Then*

a) $\lim\limits_{x\to a} (C \cdot f(x)) = C \cdot \lim\limits_{x\to a} f(x)$ *for all* $C \in \mathbb{R}$;

b) $\lim\limits_{x\to a} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$;

c) $\lim\limits_{x\to a} (f(x) \cdot g(x)) = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$;

d) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$, *if* $q\neq 0$.

Proof. The theorem follows from Theorem 3.8. □

Exercise 7.5. Prove Theorem 7.5.

Exercise 7.6. Let $a\notin\{\pi n:\ n\in\mathbb{Z}\}$. Prove that $\lim\limits_{x\to a}\cot x=\cot a$. (*Hint:* Use Example 6.6)

Example 7.4. Let $\alpha\in\mathbb{R}$, and $b>1$. Show that $\lim\limits_{x\to+\infty}\frac{x^\alpha}{b^x}=0$.

Exercise 7.7. Show that for every $a\geq 0$ $\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}$.

Exercise 7.8. Compute the following limits:
a) $\lim\limits_{x\to+\infty}\frac{x^2+\cos x+1}{\sqrt{x^4+1}+x+3}$; b) $\lim\limits_{x\to+\infty}\left(x(\sqrt{x^2+2x+2}-x-1)\right)$; c) $\lim\limits_{x\to 0}\left(\frac{2}{\sin^2 x}-\frac{1}{1-\cos x}\right)$;
d) $\lim\limits_{x\to 0}\frac{x^2+x}{\sqrt[3]{1+\sin x}-1}$; e) $\lim\limits_{x\to+\infty}(\sqrt{ax+1}-\sqrt{x})$, for some $a>0$.

7.3 Left- and Right-Sided Limits

Let A be a subset of $\mathbb{R}$ and a is a limit point of A satisfying the following property

$$\begin{aligned}&\text{there exists a sequence}\quad (x_n)_{n\geq 1}\quad\text{such that}\\ &x_n\in A,\quad x_n<a\quad\text{for all}\quad n\geq 1\quad\text{and}\quad x_n\to a,\quad n\to\infty.\end{aligned}\tag{7}$$

Definition 7.1. A number $p\in\mathbb{R}$ is the **left-sided limit** of a function $f:A\to\mathbb{R}$ at the point a if for each sequence $(x_n)_{n\geq 1}$ such that 1) $x_n\in A$, $x_n<a$ for all $n\geq 1$; 2) $x_n\to a$, $n\to\infty$, it follows that $f(x_n)\to p$, $n\to\infty$. We will use the notation $p=f(a-)$ or $p=\lim\limits_{x\to a-}f(x)$.

Theorem 7.6. *We assume that $a\in\mathbb{R}$ and $(a-\gamma,a)\subset A$ for some $\gamma>0$. Then $p=\lim\limits_{x\to a-}f(x)$ iff*

$$\forall\varepsilon>0\ \exists\delta>0\ \forall x\in(a-\delta,a):\ |f(x)-p|<\varepsilon.$$

Next, if a is a limit point of A satisfying the following property

$$\begin{aligned}&\text{there exists a sequence}\quad (x_n)_{n\geq 1}\quad\text{such that}\\ &x_n\in A,\quad x_n>a\quad\text{for all}\quad n\geq 1\quad\text{and}\quad x_n\to a,\quad n\to\infty,\end{aligned}\tag{8}$$

then we can introduce the right-sided limit of a function.

Definition 7.2. A number $p\in\mathbb{R}$ is the **right-sided limit** of a function $f:A\to\mathbb{R}$ at the point a if for each sequence $(x_n)_{n\geq 1}$ such that 1) $x_n\in A$, $x_n>a$ for all $n\geq 1$; 2) $x_n\to a$, $n\to\infty$, it follows that $f(x_n)\to p$, $n\to\infty$. We will use the notation $p=f(a+)$ or $p=\lim\limits_{x\to a+}f(x)$.

Theorem 7.7. *We assume that $a\in\mathbb{R}$ and $(a,a+\gamma)\subset A$ for some $\gamma>0$. Then $p=\lim\limits_{x\to a+}f(x)$ iff*

$$\forall\varepsilon>0\ \exists\delta>0\ \forall x\in(a,a+\delta):\ |f(x)-p|<\varepsilon.$$

Example 7.5. For the function

$$\operatorname{sgn}(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0, \end{cases}$$

one has $\operatorname{sgn}(0-) = -1$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(0+) = 1$.

Theorem 7.8. *Let $f : A \to \mathbb{R}$ and a be a limit point of A which satisfies properties (7) and (8). Then the limit $\lim\limits_{x\to a} f(x)$ exists iff $f(a-)$ and $f(a+)$ exist and are equal to each other. In this case, $\lim\limits_{x\to a} f(x) = f(a-) = f(a+)$.*

Proof. The necessity of the theorem immediately follows from the definition of the limit of f at a. Next we prove the sufficiency. Setting $p := f(a-) = f(a+)$, we are going to show that $\lim\limits_{x\to a} f(x) = p$. Let $(x_n)_{n\geq 1}$ be as in Definition 6.6, i.e. it satisfies the properties: 1) $x_n \in A$, $x_n \neq a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$. If all elements of the sequence are from one hand side of a starting from some number N, that is, $x_n < a$ for all $n \geq N$ or $x_n > a$ for all $n \geq N$, then $f(x_n) \to f(a-) = p$, $n \to \infty$, or $f(x_n) \to f(a+) = p$, $n \to \infty$, respectively. Next, we assume that infinitely many elements of $(x_n)_{n\geq 1}$ are from both hand sides of a. We construct two subsequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ of $(x_n)_{n\geq 1}$, where $(y_n)_{n\geq 1}$ consists of all elements of $(x_n)_{n\geq 1}$ which are less than a and $(z_n)_{n\geq 1}$ consists of all elements of $(x_n)_{n\geq 1}$ which are grater than a. Then $f(y_n) \to f(a-) = p$, $n \to \infty$, and $f(z_n) \to f(a-) = p$, $n \to \infty$. This implies $f(x_n) \to p$, $n \to \infty$. ☐

Exercise 7.9. Compute the following limits:
a) $\lim\limits_{x\to\frac{\pi}{2}-} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$; b) $\lim\limits_{x\to\frac{\pi}{2}+} \frac{x-\frac{\pi}{2}}{\sqrt{1-\sin x}}$; c) $\lim\limits_{x\to 0+} e^{-\frac{1}{x}}$; d) $\lim\limits_{x\to 0+} \frac{e^{-\frac{1}{x}}}{x}$.

7.4 Existence of Limit of Function

Let A be a subset of $\mathbb{R}$.

Definition 7.3. A function $f : A \to \mathbb{R}$ is said to be **increasing** (**decreasing**) **on** A if for all $x_1, x_2 \in A$ the inequality $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$).

Example 7.6. The function $f(x) = x^2$, $x \in \mathbb{R}$, decreases on $(-\infty, 0]$ and increases on $[0, +\infty)$.

Definition 7.4. A function $f : A \to \mathbb{R}$ is called a **monotone function on** A if it is either increasing or decreasing on A.

Definition 7.5. A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if the set $f(A)$ is bounded, that is, there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in A$.

Theorem 7.9. *(i) If $f : A \to \mathbb{R}$ be a monotone and bounded function, then for each limit point a of A which satisfies (7) the left-sided limit $\lim\limits_{x\to a-} f(x)$ exists and belongs to $\mathbb{R}$.*

(ii) If $f : A \to \mathbb{R}$ be a monotone and bounded function, then for each limit point a of A which satisfies (8) the right-sided limit $\lim\limits_{x\to a+} f(x)$ exists and belongs to $\mathbb{R}$.

Proof. We will prove only Part (i). Let $f : A \to \mathbb{R}$ increase and be bounded. We consider the set $B := \{x \in A : x < a\}$. By (7), it is non-empty. Consequently, the set $f(B)$ is also non-empty. Moreover, it is bounded, by the boundedness of the function f. We set

$$p := \sup f(B) = \sup_{x<a} f(x),$$

which exists according to Theorem 2.2.

We are going to show that $f(a-) = p$. Let $(x_n)_{n\geq 1}$ be an arbitrary sequence such that 1) $x_n \in A$, $x_n < a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$. Since for each $n \geq 1$ $x_n < a$, we have $f(x_n) \leq p$ for each $n \geq 1$, by the definition of supremum (see Definition 2.6).

Next, we fix $\varepsilon > 0$ and show that there exists $N \in \mathbb{N}$ such that $|p - f(x_n)| = p - f(x_n) < \varepsilon$ for all $n \geq N$. By Theorem 2.1 (i), there exists $b < a$ such that $p - \varepsilon < f(b)$. Since $x_n \to a$, $n \to \infty$, for $\varepsilon_1 := a - b > 0$ there exists N such that for all $n \geq N$ $|a - x_n| = a - x_n < \varepsilon_1 = a - b$. Hence, $x_n > b$ for all $n \geq N$. Consequently, using the increasing of f, we obtain $|p - f(x_n)| = p - f(x_n) \leq p - f(b) < \varepsilon$. This proves that $f(x_n) \to p$, $n \to \infty$, and, thus, $f(a-) = p$.

If the function f decreases and is bounded, then $f(a-) := \inf_{x<a} f(x)$. The proof is similar. □

Exercise 7.10. Prove Part (ii) of Theorem 7.9.

Exercise 7.11. Let f be an increasing function on an interval $[a, b]$.
a) For each $c \in (a, b)$ show that the one-sided limits $f(a+)$, $f(c-)$, $f(c+)$, $f(b-)$ exist.
b) Check the inequalities

$$f(a) \leq f(a+) \leq f(c-) \leq f(c) \leq f(c+) \leq f(b-) \leq f(b),$$

for all $c \in (a, b)$.
c) Prove that $\lim_{x\to c+} f(x-) = f(c+)$ and $\lim_{x\to c-} f(x+) = f(c-)$ for all $c \in (a, b)$.

Theorem 7.10 (Cauchy Criterion)**.** *Let $a \in \mathbb{R}$ be a limit point of A and $f : A \to \mathbb{R}$. A (finite) limit of f at the point a exists iff*

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall x, y \in A \cap B(a, \delta),\ x \neq a,\ y \neq a :\ |f(x) - f(y)| < \varepsilon.$$

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.