6 Lecture 6 – Limits of Functions

6.1 Base Notion of Functions (continuation)

Definition 6.1. Let $f : X \to Y$ and $g : Y \to Z$ be functions. The function $h : X \to Z$ defined by $h(x) = f(g(x))$ for all $x \in X$ is called the composition of f and g and it is denoted by $h = f \circ g$.

Definition 6.2. Let $f : X \to Y$, $A \subset X$ and $B \subset Y$. The set

$$f(A) := \{f(x) : \ x \in X\}$$

is said to be the **image** of A by f. The set

$$f^{-1}(B) := \{x : \ f(x) \in B\}$$

is called the **preimage** of B by f.

Be note that $f(A)$ is a subset of Y and $f^{-1}(B)$ is a subset of X.

Example 6.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2$, $x \in \mathbb{R}$. Then $f([0,1)) = f((-1,1)) = [0,1)$; $f^{-1}([-4,4]) = f^{-1}([0,4]) = [-2,2]$; $f^{-1}((1,9]) = [-3,-1) \cup (1,3]$; $f((-\infty,0)) = \emptyset$.

Exercise 6.1. Let $f : X \to Y$ and $A_1 \subset X$, $A_2 \subset X$. Check that
a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$; b) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$; c) $(f(A_1) \setminus f(A_2)) \subset f(A_1 \setminus A_2)$;
d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$; e) $A_1 \subset f^{-1}(f(A_1))$; f) $(f(X) \setminus f(A_1)) \subset f(X \setminus A_1)$.

Exercise 6.2. Let $f : X \to Y$ and $B_1 \subset Y$, $B_2 \subset Y$. Show that
a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$; b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
c) $f^{-1}(B_1 \backslash B_2) = f^{-1}(B_1) \backslash f^{-1}(B_2)$; d) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$; e) $f(f^{-1}(B_1)) = B_1 \cap f(X)$;
f) $f^{-1}(B_1^c) = (f^{-1}(B_1))^c$.

Definition 6.3.
- A function $f : X \to Y$ is **surjective** or a **surjection**, if $f(X) = Y$, i.e. for every element y in Y there is at least one element x in X such that $f(x) = y$.
- A function $f : X \to Y$ is **injective** or an **injection**, if for each $x_1, x_2 \in X$ $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.
- A function $f : X \to Y$ is **bijective** or a **bijection** or an **one-to-one function**, if it is surjective and injective, that is, for each $y \in Y$ there exists a unique element $x \in X$ such that $f(x) = y$. We set $f^{-1}(y) := x$. The function $f^{-1} : Y \to X$ is called the inverse function to f.

Exercise 6.3. Prove that the composition of two bijective functions is a bijection.

Exercise 6.4. Check the following statements:
a) $f : X \to Y$ is a surjection iff for all $y \in Y$ $f^{-1}(\{y\}) \neq \emptyset$.
b) $f : X \to Y$ is an injection iff for all $y \in Y$ the set $f^{-1}(\{y\})$ is either empty or contains only one element.
c) $f : X \to Y$ is a bijection iff for all $y \in Y$ the set $f^{-1}(\{y\})$ contains only one element.

Exercise 6.5. a) Let functions $f : X \to Y$ and $g : Y \to X$ satisfy the following property $g(f(x)) = x$ for all $x \in X$. Prove that f is an injection and g is a surjection.
b) Let additionally $f(g(y)) = y$ for all $y \in Y$. Show that f, g are bijections and $g = f^{-1}$.

Remark 6.1. Every sequence $(a_n)_{n \geq 1}$ of real numbers can be considered as a function $f : \mathbb{N} \to \mathbb{R}$, namely, $f(n) := a_n$ for all $n \in \mathbb{N}$.

6.2 Limit Points of a Set

Definition 6.4. Let a be a real number or the symbol $+\infty$ or $-\infty$. Then a is called a **limit point** of a subset A of $\mathbb{R}$, if there exists a sequence $(a_n)_{n\geq 1}$ satisfying the following properties: 1) $a_n \in A$ and $a_n \neq a$ for all $n \geq 1$; 2) $a_n \to a$, $n \to \infty$.

Example 6.2.

- For the set $A = [0, 1]$, the set of its limit points is A.
- For the set $A = (0, 1] \cup \{2\}$, the set of its limit points is $[0, 1]$.
- The set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ has only one limit point 0.
- The limit points of $A = \mathbb{Z}$ are $+\infty$ and $-\infty$.
- The set $A = \{1, 2, 3, \ldots, 10\}$ has no limit points.

For convenience, we will denote the ε-neighbourhood of a point a by

$$B(a, \varepsilon) := (a - \varepsilon, a + \varepsilon) = \{y \in \mathbb{R} : |a - y| < \varepsilon\}.$$

Theorem 6.1. *(i) A real number $a \in \mathbb{R}$ is a limit point of a subset A of $\mathbb{R}$ iff*

$$\forall \varepsilon > 0\ \exists y \in A,\ y \neq a :\ |y - a| < \varepsilon, \tag{5}$$

that is, each ε-neighbourhood $B(a, \varepsilon)$ of the point a contains at least one point different from a.

(ii) The symbol $a = +\infty$ ($a = -\infty$) is a limit point of a subset A of $\mathbb{R}$ iff

$$\forall C \in \mathbb{R}\ \exists y \in A :\ y > C\ (y < C).$$

Proof. We will prove only Part (i). If a is a limit point of A, then (5) immediately follows from the definition of the limit of a sequence and the definition of a limit point (see definitions 3.3 and 6.4).

Next, let (5) hold. Then for each $\varepsilon := \frac{1}{n}$ there exists $a_n \in A$ and $a_n \neq a$ such that $|a_n - a| < \varepsilon = \frac{1}{n}$. By theorems 3.7 and 3.2 and Exercise 3.5 a), $a_n \to a$, $n \to \infty$. So, a is a limit point of A. □

Exercise 6.6. Prove that the set of all limit points of $\mathbb{Q}$ equals $\mathbb{R} \cup \{-\infty, +\infty\}$.

Exercise 6.7. Let a be a limit point of A. Show that every neighbourhood of the point a contains infinitely many points from A.

Definition 6.5. A point $a \in A$ is an **isolated point** of a set A, if it is not a limit point of A.

Remark 6.2. A point $a \in A$ is an isolated point of A iff $\exists \varepsilon > 0$ such that $B(a, \varepsilon) \cap A = \{a\}$.

Example 6.3.

- The set $A = [0, 1]$ has no isolated points.
- The set $A = (0, 1] \cup \{2\}$ has only one isolated point 2.
- For the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, the set of its isolated points is A.

6.3 Limits of Functions

In this section, we will assume that A is any subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$.

Definition 6.6. Let a be a limit point of A. The value p (maybe $p = -\infty$ or $p = +\infty$) is called a **limit of the function** f **at the point** a, if for every sequence $(x_n)_{n\geq 1}$ satisfying the properties: 1) $x_n \in A$, $x_n \neq a$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, implies $f(x_n) \to p$, $n \to \infty$. In this case, we will write $\lim_{x\to a} f(x) = p$ or $f(x) \to p$, $x \to a$.

Example 6.4. Let $A = \mathbb{R}$, $f(x) = x^2$, $x \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = a^2$ for each $a \in \mathbb{R}$. Indeed, let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers such that $x_n \neq a$ for all $n \geq 1$ and $x_n \to a$, $n \to \infty$. Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = a \cdot a = a^2$, by Theorem 3.8 c).

Example 6.5. Let $A = \mathbb{R} \setminus \{0\}$, $a = 0$, and $f(x) = \frac{\sin x}{x}$, $x \in A$. Then $\lim_{x\to 0} \frac{\sin x}{x} = 1$. To show this, we will compare areas of triangles and a sector of a circle with radius 1. So, we obtain for each $x \in \left(0, \frac{\pi}{2}\right)$

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x.$$

This yields

$$\cos x < \frac{\sin x}{x} < 1, \tag{6}$$

for all x satisfying $0 < x < \frac{\pi}{2}$, and, consequently, for all $0 < |x| < \frac{\pi}{2}$ because each function in the latter inequalities is even. Thus, if $\{x_n\}_{n\geq 0}$ is any sequence such that $x_n \neq 0$ for all $n \geq 1$ and $x_n \to 0$, then inequality (6) and the Squeeze theorem (see Theorem 3.7) implies that $\lim_{n\to\infty} \frac{\sin x_n}{x_n} = 1$.

Remark 6.3. Inequality (6) implies that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$. Moreover, $|\sin x| = |x|$ iff $x = 0$.

Exercise 6.8. Prove that $\frac{1}{f(x)} \to 0$, $x \to a$, if $f(x) \to +\infty$, $x \to a$.

Example 6.6. Show that for every $a \in \mathbb{R}$ $\lim_{x\to a} \sin x = \sin a$ and $\lim_{x\to a} \cos x = \cos a$.

Solution. We prove only the first equality. The proof of the second one is similar. So, using properties of sin and cos and Remark 6.3, we can estimate

$$|\sin x - \sin a| = 2\left|\cos\frac{x+a}{2}\right| \cdot \left|\sin\frac{x-a}{2}\right| \leq 2 \cdot 1 \cdot \frac{|x-a|}{2} = |x-a|,$$

for all $x \in \mathbb{R}$. Thus, if $(x_n)_{n\geq 1}$ is any sequence which convergences to a, one has $\sin x_n \to \sin a$, by the Squeeze theorem (see Theorem 3.7).

Exercise 6.9. Prove that the limit of the function $f(x) = \sin\frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exists at the point $a = 0$.

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.