# 5 Lecture 5 – Cauchy Sequences. Base Notion of Functions

## 5.1 Subsequences (continuation)

### 5.1.1 Upper and Lower Limits

**Definition 5.1.**
- Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $A$ be the set of its subsequential limits. The value

$$\varliminf_{n\to\infty} a_n = \begin{cases} -\infty, & \text{if } A \text{ is unbounded below;} \\ \inf A, & \text{if } A \text{ is bounded below and } A \neq \{+\infty\}; \\ +\infty, & \text{if } A = \{+\infty\} \end{cases}$$

  is called the **lower limit** of $(a_n)_{n\geq 1}$.
- The value

$$\varlimsup_{n\to\infty} a_n = \begin{cases} +\infty, & \text{if } A \text{ is unbounded above;} \\ \sup A, & \text{if } A \text{ is bounded above and } A \neq \{-\infty\}; \\ -\infty, & \text{if } A = \{-\infty\} \end{cases}$$

  is called the **upper limit** of $(a_n)_{n\geq 1}$.

**Remark 5.1.** If $(a_n)_{n\geq 1}$ is a bounded sequence, then $\varliminf_{n\to\infty} a_n = \inf A$ and $\varlimsup_{n\to\infty} a_n = \sup A$.

**Example 5.1.** If $a_n \to a$, $n\to\infty$, then $\varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = a$, since $A = \{a\}$ in this case.

**Exercise 5.1.** Prove that $a_n \to a$, $n \to \infty \Leftrightarrow \varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = a$.

**Theorem 5.1.** *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $A$ be the set of its subsequential limits. Then $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$ belong to $A$.*

**Remark 5.2.** If a sequence $(a_n)_{n\geq 1}$ is bounded, then $\inf A = \min A$ and $\sup A = \max A$, by Theorem 5.1, Remark 5.1 and Exercise 2.3. It means that $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$ are the minimal and the maximal subsequential limits of the bounded sequence $(a_n)_{n\geq 1}$, respectively.

**Theorem 5.2.** *The following equalities hold: a) $\varliminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k :\ k \geq n\} =: \lim_{n\to\infty} \inf_{k\geq n} a_k$;*
*b) $\varlimsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k :\ k\geq n\} =: \lim_{n\to\infty} \sup_{k\geq n} a_k$.*

**Exercise 5.2.** Prove Theorem 5.2.

**Exercise 5.3.** For a sequence $(a_n)_{n\geq 1}$ compute $\varliminf_{n\to\infty} a_n$ and $\varlimsup_{n\to\infty} a_n$, if for all $n \geq 1$
a) $a_n = 1 - \frac{1}{n}$; b) $a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$; c) $a_n = \frac{n-1}{n+1}\cos\frac{2n\pi}{3}$; d) $a_n = 1 + n\sin\frac{n\pi}{2}$;
e) $a_n = \left(1+\frac{1}{n}\right)^n \cdot (-1)^n + \sin\frac{n\pi}{4}$.

**Exercise 5.4.** Let $(a_n)_{n\geq 1}$ be a sequence of real numbers and $\sigma_n := \frac{a_1+a_2+\ldots+a_n}{n}$, $n \geq 1$. Prove that

$$\varliminf_{n\to\infty} a_n \leq \varliminf_{n\to\infty} \sigma_n \leq \varlimsup_{n\to\infty} \sigma_n \leq \varlimsup_{n\to\infty} a_n.$$

Compare with the statement from Exercise 3.16.

**Exercise 5.5.** Check that

$$\varliminf_{n\to\infty} a_n + \varliminf_{n\to\infty} b_n \le \varliminf_{n\to\infty} (a_n + b_n) \le \varlimsup_{n\to\infty} (a_n + b_n) \le \varlimsup_{n\to\infty} a_n + \varlimsup_{n\to\infty} b_n.$$

## 5.2 Cauchy Sequences

**Definition 5.2.** A sequence $(a_n)_{n\ge1}$ of real numbers is called a **Cauchy sequence** if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N\ \forall m \ge N: \ |a_n - a_m| < \varepsilon.$$

**Example 5.2.**

1. The sequence $\left(\frac{1}{2^n}\right)_{n\ge1}$ is a Cauchy sequence. Indeed, since $\frac{1}{2^n} \to 0$, $n \to \infty$, (see Theorem 3.3), one has that for every given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \ge N$ $\frac{1}{2^n} < \varepsilon$. Consequently, for every $n \ge N$ and $m \ge N$ we can estimate $\left|\frac{1}{2^m} - \frac{1}{2^n}\right| \le \frac{1}{2^k} < \varepsilon$, where $k := \min\{n, m\} \ge N$.

2. The sequence $(a_n = (-1)^n)_{n\ge1}$ is not a Cauchy sequence. To check this, we take $\varepsilon := 1$. Then $\forall N \in \mathbb{N}\ \exists n := N$ and $\exists m := N + 1$ such that $|a_n - a_m| = 2 > \varepsilon$.

**Exercise 5.6.** Prove that a monotone sequence which contains a Cauchy subsequence is also a Cauchy sequence.

**Exercise 5.7.** Show that $(a_n)_{n\ge1}$ is a Cauchy sequence iff $\sup\limits_{m\ge N, n\ge N} |a_m - a_n| \to 0$, $N \to \infty$.

**Lemma 5.1.** *Every convergent sequence is a Cauchy sequence.*

*Proof.* Let $a_n \to a$, $n \to \infty$, and let $\varepsilon > 0$ be given. By the definition of convergence (see Definition 3.3), for the number $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ such that $\forall n \ge N_1$ $|a_n - a| < \frac{\varepsilon}{2}$. Thus we have that $\forall n \ge N := N_1$ and $\forall m \ge N$

$$|a_n - a_m| = |a_n - a + a - a_m| \le |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

by the triangular inequality. □

**Lemma 5.2.** *Every Cauchy sequence is bounded.*

*Proof.* The proof is similar to the proof of Theorem 3.5. □

**Exercise 5.8.** Prove Lemma 5.2.

**Theorem 5.3.** *A sequence converges iff it is a Cauchy sequence.*

*Proof.* The necessity was stated in Lemma 5.1. We will prove the sufficiency. Let $(a_n)_{n\ge1}$ be a Cauchy sequence. By Lemma 5.2, it is bounded. Thus, using the Bolzano-Weierstrass theorem (see Theorem 4.6), there exists a subsequence $(a_{n_k})_{k\ge1}$ which converges to some $a \in \mathbb{R}$.

Next, we are going to show that $a_n \to a$, $n \to \infty$. Let $\varepsilon > 0$ be given. Since $(a_n)_{n\ge1}$ is a Cauchy sequence, for the number $\frac{\varepsilon}{2} > 0$ $\exists N_1 \in \mathbb{N}\ \forall m \ge N\ \forall n \ge N$ such that $|a_m - a_n| < \frac{\varepsilon}{2}$. By the definition of convergence, we have that $\exists K \in \mathbb{N}\ \forall k \ge K$ such that $|a - a_{n_k}| < \frac{\varepsilon}{2}$. Thus, $\forall n \ge N := N_1$

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \le |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where $k$ is any number satisfying $k \ge K$ and $n_k \ge N$. □

**Exercise 5.9.** Show that the sequence $\left(a_n = \frac{\sin 1}{2^1} + \frac{\sin 2}{2^2} + \ldots + \frac{\sin n}{2^n}\right)_{n\ge1}$ is a Cauchy sequence.

**Exercise 5.10.** Let $(a_n)_{n\ge1}$ be a sequence which satisfies the following property: there exists $\lambda \in [0, 1)$ such that $|a_{n+2} - a_{n+1}| \le \lambda |a_{n+1} - a_n|$ for all $n \ge 1$. Prove that $(a_n)_{n\ge1}$ converges.

## 5.3 Base Notion of Functions

Let $X$ and $Y$ be two sets.

**Definition 5.3.**
- A **function** $f$ is a process or a relation that associates each element $x$ of $X$ to a single element $y$ of $Y$. The set $X$ is called the **domain** of the function $f$ and is denoted by $D(f)$. The set $Y$ is said to be the **codomain** of $f$. We will use the notation $f : X \to Y$.
- The element $y \in Y$ which is associated to $x \in Y$ by a function $f$ is called the **value of** $f$ **applied to the argument** $x$ or the **image of** $x$ **under** $f$ and is denoted by $f(x)$. We will also write $x \mapsto f(x)$.
- The set
$$R(f) := \{y \in Y : \ \exists x \in X \ y = f(x)\}$$
is called the **range** or the **image** of the function $f$.
- If $Y \subset \mathbb{R}$, then $f$ is called a **real valued function**.

In further sections, we will usually consider real valued functions with $D(f) \subset \mathbb{R}$.

**Exercise 5.11.** Determine domains $X \subset \mathbb{R}$ for which the following functions $f : X \to \mathbb{R}$ are well-defined:
a) $f(x) = \frac{x^2}{x+1}$; b) $f(x) = \sqrt{3x - x^3}$; c) $f(x) = \ln(x^2 - 4)$; d) $\sqrt{\cos(x^2)}$; e) $f(x) = \frac{\sqrt{x}}{\sin \pi x}$.

**Exercise 5.12.** Compute $f(-1)$, $f(-0,001)$ and $f(100)$, if $f(x) = \lg(x^2)$.

**Exercise 5.13.** Compute $f(-2)$, $f(-1)$, $f(0)$, $f(1)$ and $f(2)$, if

$$f(x) = \begin{cases} 1 + x, & \text{if } x \leq 0, \\ 2^x, & \text{if } x > 0. \end{cases}$$

**Exercise 5.14.** Define the range $R(f)$ of the following functions:

a) $X = \mathbb{Z}$, $Y = \mathbb{Z}$ and $f(x) = |x| - 1$, $x \in \mathbb{Z}$;

b) $X = \mathbb{R}$, $Y = \mathbb{R}$ and $f(x) = x^2 + x$, $x \in \mathbb{R}$;

c) $X = (0, \infty)$, $Y = \mathbb{R}$ and $f(x) = (x - 1) \ln x$, $x > 0$.

**Exercise 5.15.** Let $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, where $a, b, c$ are some numbers. Show that

$$f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x) = 0.$$

**Exercise 5.16.** Find a function of the form $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, which satisfies the following properties: $f(-2) = 0$, $f(0) = 1$, $f(1) = 5$.

**Definition 5.4.** We will say that a function $f_1 : X_1 \to Y_1$ equals a function $f_2 : X_2 \to Y_2$, if $X_1 = X_2$ and $f_1(x) = f_2(x)$ for all $x \in X_1$. We will use the notation $f_1 = f_2$.

**Definition 5.5.** Let $f : X \to Y$ be a function and $A$ be a subset of $X$. The function $f|_A : A \to Y$ defined by $f|_A(x) = f(x)$ for all $x \in A$ is called the **restriction of** $f$ **to** $A$.

**Definition 5.6.** For sets $A$ and $B$, we will denote the new set $A \times B$ that consists of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$, that is,

$$A \times B := \{(a, b) : \ a \in A, \ b \in B\}.$$

The set $A \times B$ is called the **Cartesian product** of $A$ and $B$.

**Definition 5.7.** The set $G(f) = \{(x, f(x)) : \ x \in X\}$ is said to be the **graph** of a function $f : X \to Y$.

## References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus.* Undergraduate Texts in Mathematics. Springer New York, 2013.