3 Lecture 3 – Convergence of Sequences

3.1 Limits of Sequences

For more details see [1, Section 2.7].

In this section, we will study some properties of sequences of real numbers which do not depend on finite numbers of their elements. So, we will call a **sequence** any enumerated collection of objects (in our case, real numbers) in which repetitions are allowed. It is often convenient to write the sequence as $(a_m, a_{m+1}, a_{m+2}, \dots)$, $(a_n)_{n\geq m}$ or $(a_n)_{n=m}^{\infty}$, where m is some integer number. Usually, m equals 1.

Definition 3.1. A sequence $(a_n)_{n\geq 1} = (a_1, a_2, \dots, a_n, \dots)$ is called **bounded** if there exists $C > 0$ such that $|a_n| \leq C$ for all $n \geq 1$. In another words, if all elements of the sequence belong to some interval $[-C, C]$.

Example 3.1.
1. The sequence $((-1)^n)_{n\geq 1} = (-1, 1, -1, 1, \dots)$ is bounded and its elements belong to $[-1, 1]$;
2. The sequence $(\sin n)_{n\geq 1}$ is bounded and its elements also belong to $[-1, 1]$;
3. The sequence $(n)_{n\geq 1} = (1, 2, 3, \dots, n, \dots)$ is unbounded, since for each $C > 0$ one can find a number $n \in \mathbb{N}$ larger than C.

Exercise 3.1. Prove the boundedness of the following sequences:

a) $\left(\frac{2^n}{n!}\right)_{n\geq 1}$; b) $\left(a_n = \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2 + \sqrt{2}}}}}_{n \text{ square roots}}\right)_{n\geq 1}$;

c) $\left(a_n = 1 + \frac{2}{2} + \frac{3}{2^2} + \ldots + \frac{n}{2^{n-1}}\right)_{n\geq 1}$ (*Hint:* Use the equality $\frac{1}{2}a_n = a_n - \frac{1}{2}a_n$)

Exercise 3.2. Prove that a sequence $(a_n)_{n\geq 1}$ is bounded iff $(a_n^3 - a_n)_{n\geq 1}$ is.

Definition 3.2. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. A **neighbourhood** or **ε-neighbourhood** of the point x is the interval $(x - \varepsilon, x + \varepsilon) = \{y \in \mathbb{R} : \ |y - x| < \varepsilon\}$.

Exercise 3.3. Check that: a) intersection of a finite number of neighbourhoods of x is again a neighbourhood of x; b) intersection of two neighbourhoods is either $\emptyset$ or a neighbourhood.

Definition 3.3. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to **converge** to a real number a provided that

for each $\varepsilon > 0$ there exists a number N such that $n \geq N$ implies $|a_n - a| < \varepsilon$,

or, shortly,

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{R} \ \forall n \geq N : \quad |a_n - a| < \varepsilon.$$

If $(a_n)_{n\geq 1}$ converges to a, we will write $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$, $n \to \infty$. The number a is called the **limit** of the sequence $(a_n)_{n\geq 1}$. A sequence that does not converge to some real number is said to **diverge**.

Remark 3.1. We note that $a_n \to a$, $n \to \infty$, provided that any ε-neighbourhood of point a contains elements a_n for all $n \geq N$, where N is some number depending on ε.

Exercise 3.4. For which sequences $(a_n)_{n\geq 1}$ the number N from Definition 3.3 could be taken independent of ε.

Answer: If $\exists m \in \mathbb{N}\ \forall n \geq m:\ a_n = a$.

Exercise 3.5. Prove the following statements:

a) $a_n \to a,\ n \to \infty \ \Leftrightarrow\ a_n - a \to 0,\ n \to \infty \ \Leftrightarrow\ |a_n - a| \to 0,\ n \to \infty$;

b) $a_n \to 0,\ n \to \infty \ \Leftrightarrow\ |a_n| \to 0,\ n \to \infty$;

c) $a_n \to a,\ n \to \infty \ \Leftrightarrow\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}:\ \{a_N, a_{N+1}, \dots\} \subset (x - \varepsilon, x + \varepsilon)$;

d) $a_n \to 0,\ n \to \infty \ \Leftrightarrow\ \sup\{|a_k|:\ k \geq n\} \to 0,\ n \to \infty$;

e) $a_n \to a,\ n \to \infty \ \Rightarrow\ |a_n| \to |a|,\ n \to \infty$.

Theorem 3.1. *A sequence can have only a unique limit.*

Proof. Let $a_n \to a$, $n \to \infty$, and $a_n \to b$, $n \to \infty$. Then by the definition, $\forall \varepsilon > 0\ \exists N_1 \in \mathbb{R}\ \forall n \geq N_1:$ $|a_n - a| < \varepsilon$ and $\forall \varepsilon > 0\ \exists N_2 \in \mathbb{R}\ \forall n \geq N_2:\ \ |a_n - b| < \varepsilon$. Thus, using the triangular inequality (see Theorem 2.5 1)), we obtain $\forall \varepsilon > 0\ \forall n \geq \max\{N_1, N_2\}:\ \ |a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon$. So, $|a - b| < 2\varepsilon$ for all $\varepsilon > 0$. If $a \neq b$, we set $\varepsilon = \frac{|a-b|}{3} > 0$. Then $|a - b| < \frac{2}{3}|a - b| \Rightarrow \frac{1}{3}|a - b| < 0$, that is impossible. □

3.2 Some Examples

For more examples see [1, Section 2.8].

Theorem 3.2. *The equality* $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ *holds.*

Proof. We note that for each $\varepsilon > 0$ we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ iff $n > \frac{1}{\varepsilon}$. Thus, $\forall \varepsilon > 0\ \exists N := \left(\frac{1}{\varepsilon} + 1\right) \in \mathbb{R}\ \forall n \geq N:\ \left|\frac{1}{n} - 0\right| < \varepsilon$. □

Corollary 3.1. *The equality* $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0$ *holds for each* $\alpha > 0$.

Theorem 3.3. *Let* $a \in \mathbb{R}$, $|a| > 1$, $b \in \mathbb{R}$. *Then* $\lim\limits_{n\to 0} \frac{n^b}{a^n} = 0$.

Proof. We choose $k \in \mathbb{N}$ such that $k \geq b + 1$. By Bernoulli's inequality (see Theorem 2.6), $|a|^n = \left(|a|^{\frac{n}{k}}\right)^k = \left(\left(1 + \left(|a|^{\frac{1}{k}} - 1\right)\right)^n\right)^k > n^k \left(|a|^{\frac{1}{k}} - 1\right)^k$. Hence, $\left|\frac{n^b}{a^n} - 0\right| = \frac{n^b}{|a|^n} \leq \frac{n^{k-1}}{|a|^n} < \frac{1}{n\left(|a|^{\frac{1}{k}} - 1\right)^k} < \varepsilon$. So, $n > \frac{1}{\varepsilon\left(|a|^{\frac{1}{k}} - 1\right)^k}$. Consequently, one can claim

$$\forall \varepsilon > 0\ \exists N := \frac{1}{\varepsilon\left(|a|^{\frac{1}{k}} - 1\right)^k} + 1\ \forall n \geq N:\ \left|\frac{n^b}{a^n} - 0\right| < \varepsilon.$$

□

Theorem 3.4. *The equality* $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ *holds.*

Proof. By Exercise 3.5 a), it is enough to show that $a_n := \sqrt[n]{n} - 1 \to 0$, $n \to \infty$. Since $(1 + a_n)^n = (\sqrt[n]{n})^n = n$, one has

$$n = (1 + a_n)^n \geq 1 + na_n + \frac{1}{2}n(n-1)a_n^2 > \frac{1}{2}n(n-1)a_n^2,$$

by the binomial formula. Thus, $a_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. Next using the standard argument, one has $a_n \to 0$. □

Exercise 3.6. Check the following equalities:
a) $\lim_{n\to\infty} a^n = 0$ for all $0 < a < 1$; b) $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for all $a > 0$; c) $\lim_{n\to\infty} \frac{\lg n}{n^\alpha} = 0$ for all $\alpha > 0$, where $\lg := \log_{10}$.

Definition 3.4.

1. $\lim_{n\to\infty} a_n = +\infty \iff \forall C \in \mathbb{R}\ \exists N \in \mathbb{R}\ \forall n \geq N:\ a_n \geq C$.
2. $\lim_{n\to\infty} a_n = -\infty \iff \forall C \in \mathbb{R}\ \exists N \in \mathbb{R}\ \forall n \geq N:\ a_n \leq C$.

Exercise 3.7. Prove that for a sequence $(a_n)_{n\geq 1}$ with $a_n \neq 0$ the equality $\lim_{n\to\infty} |a_n| = +\infty$ is equivalent to $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Exercise 3.8. Let $(a_n)_{n\geq 1}$ be a sequence such that $\frac{a_n}{n} \to 0$, $n \to \infty$. Prove that $\frac{\max\{a_1,a_2,\ldots,a_n\}}{n} \to 0$, $n \to \infty$.

Exercise 3.9. Assume that $a_n \to a$, $n \to \infty$, and $b_n \to b$, $n \to \infty$. Show that $\max\{a_n, b_n\} \to \max\{a, b\}$, $n \to \infty$.

3.3 Limit Theorems for Sequences

See also [1, Section 2.9].

In this section, we will prove some properties of convergent sequences and their limits. We recall that a sequence $(a_n)_{n\geq 1}$ of real numbers is said to be bounded if there exists a constant C such that $|a_n| \leq C$ for all n.

Theorem 3.5. *Any convergent sequence is bounded.*

Proof. Let $a_n \to a$, $n \to \infty$. We have to show that $(a_n)_{n\geq 1}$ is bounded. By the definition of convergence (see Definition 3.3), for each $\epsilon > 0$, in particular for $\varepsilon = 1$, there exists a number N, which can be taken from $\mathbb{N}$, such that $|a_n - a| < \varepsilon = 1$ for all $n \geq N$. Thus, setting $C := \max\{|a_1|, \ldots, |a_{N-1}|, |a| + 1\}$, one trivially obtains for $n \in \{1, 2, \ldots, N-1\}$

$$|a_n| \leq C.$$

Next, using the triangular inequality (inequality 1) of Theorem 2.5), we have

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \leq C,$$

for all $n \geq N$. □

Exercise 3.10. Give an example of a bounded divergent sequence.

Theorem 3.6. *Let $a_n \to a \in \mathbb{R}$, $n \to \infty$, $b_n \to b$, $n \to \infty$, and let $a_n \leq b_n$ for all $n \geq 1$. Then $a \leq b$.*

Exercise 3.11. Prove Theorem 3.6.

Remark 3.2. We note that replacing the inequality $a_n \leq b_n$ by the strong one, i.e. $a_n < b_n$, it does not imply $a < b$. Indeed, for $a_n := 0$ and $b_n := \frac{1}{n}$, $n \geq 1$, one has $a_n < b_n$ but $a_n \to 0$, $b_n \to 0$, $n \to \infty$.

Remark 3.3. Theorem 3.6 remains valid, if the inequality $a_n \leq b_n$ holds only for all $n \geq M$, where M is some number N.

Theorem 3.7 (Squeeze theorem)**.** *Let sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ satisfy the following conditions:*

a) $a_n \leq b_n \leq c_n$ for all $n \geq 1$;

b) $a_n \to a$, $n \to \infty$, and $c_n \to a$, $n \to \infty$.

Then $b_n \to a$, $n \to \infty$.

Proof. According to Remark 3.1, for each $\varepsilon > 0$ there exists N_1 and N_2 from $\mathbb{R}$ such that a_n belongs to the ε-neighbourhood $(a-\varepsilon, a+\varepsilon)$ of the point a for all $n \geq N_1$ and c_n belongs to $(a-\varepsilon, a+\varepsilon)$ for all $n \geq N_2$. Thus, for all $n \geq \max\{N_1, N_2\}$ elements b_n also belong to $(a-\varepsilon, a+\varepsilon)$ due to property a). □

Example 3.2. Show that $\lim\limits_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} = 1$.

Solution. We take $a_n := \sqrt[n]{1} = 1$ and $c_n := \underbrace{\sqrt[n]{1+1+1+\ldots+1}}_{n \text{ times}} = \sqrt[n]{n}$. Then

$$a_n \leq \sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} \leq c_n$$

for all $n \geq 1$. Moreover, $a_n \to 1$, $n \to \infty$, and $c_n \to 1$, $n \to \infty$, by Theorem 3.4. Hence, Theorem 3.7 implies $\lim\limits_{n\to\infty} \sqrt[n]{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}} = 1$.

Theorem 3.8. *Let $a_n \to a \in \mathbb{R}$, $n \to \infty$, and $b_n \to b \in \mathbb{R}$, $n \to \infty$. Then*

a) $\lim\limits_{n\to\infty}(c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n$ for all $c \in \mathbb{R}$;

b) $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$;

c) $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$;

d) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $b \neq 0$.

Proof. For proof of the theorem see Section 2.9 [1]. □

Example 3.3. Compute the limit $\lim\limits_{n\to\infty} \frac{2n^2+\lg n}{3n^2+n\cos n+5}$.

Solution. We cannot apply Theorem 3.8 directly, since the numerator and denominator of $\frac{2n^2+\lg n}{3n^2+n\cos n+5}$ tend to infinity. So, first we rewrite them as follows:

$$\frac{2n^2+\lg n}{3n^2+n\cos n+5} = \frac{n^2\cdot\left(2+\frac{\lg n}{n^2}\right)}{n^2\cdot\left(3+\frac{\cos n}{n}+\frac{5}{n^2}\right)} = \frac{2+\frac{\lg n}{n^2}}{3+\frac{\cos n}{n}+\frac{5}{n^2}}.$$

Now, we can use Theorem 3.8 d) to the right hand side of the latter equality. Indeed, we first compute

$$\lim_{n\to\infty}\left(2+\frac{\lg n}{n^2}\right) = 2+\lim_{n\to\infty}\frac{\lg n}{n^2} = 2,$$

by, Theorem 3.8 b) and Exercise 3.6 c). Next, due to the inequality

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}, \quad n \ge 1,$$

theorems 3.7 and 3.2, one has $\lim\limits_{n\to\infty}\frac{\cos n}{n} = 0$. Thus, by Theorem 3.8 a), b)

$$\lim_{n\to\infty}\left(3+\frac{\cos n}{n}+\frac{5}{n^2}\right) = 3+\lim_{n\to\infty}\frac{\cos n}{n}+5\lim_{n\to\infty}\frac{1}{n^2} = 3 \ne 0.$$

So, we can apply Theorem 3.7 d) and obtain

$$\lim_{n\to\infty}\frac{2n^2+\lg n}{3n^2+n\cos n+5} = \lim_{n\to\infty}\frac{2+\frac{\lg n}{n^2}}{3+\frac{\cos n}{n}+\frac{5}{n^2}} = \frac{2}{3}.$$

Exercise 3.12. Compute the following limits:
a) $\lim\limits_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}}$; b) $\lim\limits_{n\to\infty}\frac{n^2+\sin n}{n^2+n\cos n}$; c) $\lim\limits_{n\to\infty}\sqrt[n]{n^2 2^n+3^n}$; d) $\lim\limits_{n\to\infty}\frac{2^n+n^3}{3^n+1}$; e) $\sqrt[n+1]{n}$.

Exercise 3.13. Let $(a_n)_{n\ge1}$ be a bounded sequence and $b_n \to 0$, $n \ge \infty$. Prove that $a_n b_n \to 0$, $n\to\infty$.

Exercise 3.14. Let $(a_n)_{n\ge1}$ be a bounded sequence and $b_n \to +\infty$, $n \ge \infty$. Prove that $a_n+b_n \to +\infty$, $n\to\infty$.

Exercise 3.15. Let $a_n \ge 0$ for all $n \ge 1$ and $a_n \to a$, $n\to\infty$. Show that for all $k\in\mathbb{N}$ one has $\sqrt[k]{a_n} \to \sqrt[k]{a}$, $n\to\infty$.

Exercise 3.16. Let $a_n \to a \in \mathbb{R}$, $n\to\infty$. Prove that $\frac{a_1+\ldots+a_n}{n} \to a$, $n\to\infty$.

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.