23 Lecture 23 – Vector Subspaces and Span

23.1 Vector Subspaces

Throughout this section, V denotes a vector space over $\mathbb{F}$.

Definition 23.1. Let V be a vector space over $\mathbb{F}$, and let $U \subset V$ be a subset of V. Then U is called a **subspace** of V if U is a vector space over $\mathbb{F}$

To check that a subset $U \subset V$ is a subspace, it is suffices to check only a few of the conditions of a vector space.

Lemma 23.1. *Let $U \subset V$ be a subset of a vector space V over $\mathbb{F}$. Then U is a subspace of V iff the following conditions holds:*

(1) ***closure under addition:*** $\mathbf{u}, \mathbf{v} \in U$ *implies* $\mathbf{u} + \mathbf{v} \in U$*;*

(2) ***closure under scalar multiplication:*** $a \in \mathbb{F}$*,* $\mathbf{u} \in U$ *implies that* $a\mathbf{u} \in U$*.*

Exercise 23.1. Prove Lemma 23.1.

Example 23.1. In every vector space V, the subset $U = \{\mathbf{0}\}$ is a vector subspace of V.

Exercise 23.2. Show that the set $\{(x_1, 0) : \ x_1 \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^2$.

Exercise 23.3. Show that the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + 2x_2 - x_3 + 1 = 0\}$ is not a vector subspace of $\mathbb{R}^3$.

Exercise 23.4. Let U_1 and U_2 be a vector subspaces of V. Prove that the intersection $U_1 \cap U_2$ is also a vector subspace. Is the union $U_1 \cup U_2$ a vector subspace?

23.2 Sums and Direct Sums of Vector Subspaces

Let U_1, U_2 be a vector subspaces of V.

Definition 23.2. Let U_1, U_2 be a vector subspaces of V. The set

$$U_1 + U_2 = \{\mathbf{u}_1 + \mathbf{u}_2 : \ \mathbf{u}_1 \in U_1, \ \mathbf{u}_2 \in U_2\}$$

is said to be a **sum of vector subspaces** U_1 and U_2.

Exercise 23.5. Check that a direct sum of two vector subspaces is a vector space.

Example 23.2. Let

$$\begin{aligned} U_1 &= \{(x, 0, 0) \in \mathbb{F}^3 : \ x \in \mathbb{F}\} \\ U_2 &= \{(0, y, 0) \in \mathbb{F}^3 : \ y \in \mathbb{F}\} \\ U_3 &= \{(y, y, 0) \in \mathbb{F}^3 : \ y \in \mathbb{F}\}. \end{aligned}$$

Then

$$U_1 + U_2 = U_1 + U_3 = \{(x, y, 0) \in \mathbb{F}^3 : \ x, y \in \mathbb{F}\}.$$

We remark that $\mathbf{u} \in U = U_1 + U_2$ if and only if there exist vectors $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

Definition 23.3. If every vector $\mathbf{u} \in U = U_1 + U_2$ can be uniquely written as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ for $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$. Then we call the vector space U the **direct sum** of U_1, U_2 and denote by

$$U = U_1 \oplus U_2.$$

Example 23.3. Let

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{R}^3 : \ x, y \in \mathbb{R}\}, \\ U_2 &= \{(0, 0, z) \in \mathbb{R}^3 : \ z \in \mathbb{R}\}, \\ U_3 &= \{(0, y, z) \in \mathbb{R}^3 : \ y, z \in \mathbb{R}\}. \end{aligned}$$

Then $\mathbb{R}^3 = U_1 \oplus U_2$. But $\mathbb{R}^3 = U_1 + U_3$ and $\mathbb{R}^3 \neq U_1 \oplus U_3$ (the vector $(0, 0, 0)$ can be written as $(0, 0, 0) + (0, 0, 0)$ and $(0, -1, 0) + (0, 1, 0)$).

Proposition 23.1. *Let U_1 and U_2 be a vector subspaces of V. Then $V = U_1 \oplus U_2$ iff the following conditions hold:*

(1) $V = U_1 + U_2$;

(2) *If* $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$ *with* $\mathbf{u}_1 \in U_1$ *and* $\mathbf{u}_2 \in U_2$, *then* $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$.

Proof. We assume that $V = U_1 \oplus U_2$. Then Condition (1) follows from the definition. Since $\mathbf{0}$ can be uniquely written as $\mathbf{0} + \mathbf{0}$, we have that Condition (2) is also true.

Next, let conditions (1) and (2) hold. By Condition (1), for every vector $\mathbf{u} \in V$ there exist $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$ such that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. We assume that $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{v}_1 \in U_1$ and $\mathbf{v}_2 \in U_2$. Subtracting the two equations, we obtain

$$0 = (\mathbf{u}_1 - \mathbf{v}_1) + (\mathbf{u}_2 - \mathbf{v}_2),$$

where $\mathbf{u}_1 - \mathbf{v}_1 \in U_1$ and $\mathbf{u}_2 - \mathbf{v}_2 \in U_2$. By Condition (2), we have that $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{v}_2$. This implies that $V = U_1 \oplus U_2$. □

Proposition 23.2. *Let U_1 and U_2 be a vector subspaces of V. Then $V = U_1 \oplus U_2$ iff the following conditions hold:*

(1) $V = U_1 + U_2$;

(2) $U_1 \cap U_2 = \{\mathbf{0}\}$.

Proof. We assume that $V = U_1 \oplus U_2$. Then Condition (1) follows from the definition. Next, we suppose that $\mathbf{u} \in U_1 \cap U_2$. Then by Exercise 23.4, $-\mathbf{u}$ also belongs to $U_1 \cap U_2$ because $U_1 \cap U_2$ is a vector space. Thus, $\mathbf{0} = \mathbf{u} + (-\mathbf{u})$, where $\mathbf{u} \in U_1 \cap U_2 \subset U_1$ and $-\mathbf{u} \in U_1 \cap U_2 \subset U_2$. By Proposition 23.1, $\mathbf{u} = \mathbf{0}$.

Next, we assume that conditions (1) and (2) hold. In order to prove that $V = U_1 \oplus U_2$, we show that $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$, implies $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$. Since $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1 = -\mathbf{u}_2$. So, $\mathbf{u}_1 = -\mathbf{u}_2 \in U_2$ because U_2 is a vector space. Thus, $u_1 \in U_1 \cap U_2$ and, consequently, $\mathbf{u}_1 = -\mathbf{u}_2 = \mathbf{0}$, according to Condition (2). USsing Proposition 23.1, we obtain that V is the direct sum of U_1 and U_2. □

Exercise 23.6. Prove or give a counterexample to the following claim:

1) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are vector subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.

2) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are vector subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.

Exercise 23.7. Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let

$$U = \{az^2 + bz^5 : \ a, b \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}[z]$ such that $F[z] = U \oplus W$.

23.3 Linear Span

In order to give a definition of one of the main notion of the linear algebra: basis of a vector space, we need to introduce a notion of a linear span of vectors.

Definition 23.4. A vector $\mathbf{v} \in V$ is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, if there exists scalars $a_1, a_2, \ldots, a_n$ from $\mathbb{F}$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n.$$

Definition 23.5. The set

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} := \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n : \ a_1, a_2, \ldots, a_n \in \mathbb{F}\}$$

is called a **linear span** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The following lemma follows from the definitions of a vector spaces and linear span.

Proposition 23.3. *Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Then*

(i) *the vector $\mathbf{v}_i$ belongs to* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$*;*

(ii) $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is a subspace of V;*

(iii) *If U is a subspace of V such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in U$, then* $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset U$.

Proposition 23.3 implies that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the smallest vector space of V which contains the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Definition 23.6. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$, then we say that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ **span** V and we call V **finite-dimensional**. If a vector space is not finite dimensional, then we call it **infinite-dimensional**.

Example 23.4. The vectors $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, ..., $\mathbf{e}_n = (0, \ldots, 0, 1)$ span $\mathbb{F}^n$. According to the previous definition the space $\mathbb{F}^n$ is finite-dimensional.

Example 23.5. Let $\mathbf{p}_k(z) = z^k$, for $k = 0, \ldots, n$. Then the set $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ span $\mathbb{F}_n[z]$. It is easy to see that the space $\mathbb{F}[z]$ of all polynomials is infinite-dimensional.

Exercise 23.8. Consider the complex vector space $V = \mathbb{C}^3$ and the list $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of vectors in V, where $\mathbf{v}_1 = (i, 0, 0)$, $\mathbf{v}_2 = (i, 1, 0)$ and $\mathbf{v}_3 = (i, i, -1)$.
a) Prove that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = V$.