2 Lecture 2 – Completeness of the Set of Real Numbers and some Inequalities

2.1 Real Numbers

2.1.1 Definition of Real Numbers

Very often the set of rational numbers needs an extension. For example, the length of a diagonal of a square with side 1 can not be given as a rational number.

Exercise 2.1. Prove that there does not exist a rational number x solving the equation $x^2 = 2$.

Definition 2.1. A **real number** is an infinite sequence of numerical digits with the comma between them, that is,

$$a = \alpha_0, \alpha_1 \alpha_2 \ldots \alpha_n \ldots,$$

where $\alpha_0 \in \mathbb{Z}$ and $\alpha_n \in \{0, 1, \ldots, 9\}$ for all $n \in \mathbb{N}$.

The **set of all real numbers** is denoted by $\mathbb{R}$.

Definition 2.2. Numbers from $\mathbb{R} \setminus \mathbb{Q}$ is called **irrational**.

Remark 2.1. We will identify of two real numbers of the form

$$a = \alpha_0, \alpha_1 \ldots \alpha_n 99999 \ldots$$

and

$$a = \alpha_0, \alpha_1 \ldots (\alpha_n + 1)00000 \ldots,$$

where $\alpha_n < 9$. Further, we will avoid numbers, where 9 is in the period.

The order relations "$<, \leq, >, \geq$" between real numbers can be introduced by the natural way as well as the notions of positive and negative real numbers.

Definition 2.3. The **absolute value** of a real number a is defined as follows

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

2.1.2 Supremum and Infimum of Subsets of Real Numbers

Let A be a non-empty subset of $\mathbb{R}$.

Definition 2.4.
- If A contains a larger element a_0, then we call a_0 the **maximum** of A and write $a_0 = \max A$.
- If A contains a smallest element, then we call the smallest element the **minimum** of A and write it as $\min S$.

Example 2.1.
a) $\max\{1, 2, 3, 4, 5\} = 5$, $\min\{1, 2, 3, 4, 5\} = 1$;

b) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Then $\max A = 1$ but $\min A$ does not exist.

Definition 2.5.
- If a real number M satisfies $a \leq M$ for all $a \in A$, then M is called an **upper bound** of A and the set A is said to be **bounded above**.
- If a real number m satisfies $m \leq a$ for all $a \in A$, then m is called a **lower bound** of A and the set A is said to be **bounded below**.
- The set A is said to be **bounded** if it is bounded above and bounded below.

Example 2.2.
1. The set $A = \left\{\frac{1}{n} : \ n \in \mathbb{N}\right\}$ is bounded.
2. The set $\mathbb{N}$ is bounded below but not above.
3. The set $\mathbb{R}$ is neither bounded below nor above.

Exercise 2.2. Prove that the following sets are bounded:

a) $\left\{\frac{n}{n+1} : \ n \in \mathbb{N}\right\}$;

b) $\left\{\frac{(-1)^n n+1}{n-(-1)^n} : \ n \in \mathbb{N}\right\}$.

Definition 2.6.
- If A is bounded above and A has a least upper bound, then we will call it the **supremum** of A and denote it by $\sup A$.
- If A is bounded below and A has a greatest lower bound, then we will call it the **infimum** of A and denote it by $\inf A$.

Exercise 2.3. If $\min A$ exists, then $\min A = \inf A$. Similarly, if $\max A$ exists, then $\max A = \sup A$. Check these statements.

Theorem 2.1. *(i) The number a^* is the supremum of a subset A of $\mathbb{R}$ iff*
- *a^* is an upper bound of A;*
- *$\forall a < a^* \ \ \exists x \in A \ \ x > a$.*

(ii) The number a_* is the supremum of a subset A of $\mathbb{R}$ iff
- *a_* is an lower bound of A;*
- *$\forall a > a^* \ \ \exists x \in A \ \ x < a$.*

Exercise 2.4. For each $a < b$ prove that $\inf[a, b] = \inf(a, b] = a$ and $\sup[a, b] = \sup[a, b) = b$.

Theorem 2.2. *(i) For every non-empty subset A of $\mathbb{R}$ that is bounded above $\sup A$ exists and is a real number.*

(ii) For every non-empty subset A of $\mathbb{R}$ that is bounded below $\inf A$ exists and is a real number.

The latter theorem states the completeness of the set of real numbers, which is not true e.g. for rational numbers. Indeed, the set $A = \{r \in \mathbb{Q} : 0 \leq r \text{ and } r^2 \leq 2\}$ is a set of rational numbers and it is bounded above by some rational numbers but A has no least upper bound that is a rational number.

Theorem 2.3. *For each positive real number* $a = \alpha_0, \alpha_1\alpha_2 \dots \alpha_n \dots$, *we have*

$$a = \sup\{a_n : \ n \in \mathbb{N}\},$$

where $a_n = \alpha_0, \alpha_1\alpha_2 \dots \alpha_n$.

Exercise 2.5. Prove Theorem 2.3.

Now, we are ready to introduce operation or real numbers. Let a, b be a positive real numbers and a_n and b_n, $n \in \mathbb{N}$, be defined as in Theorem 2.3.

Definition 2.7. We set by the definition $a + b := \sup\{a_n + b_n : \ n \in \mathbb{N}\}$; $a \cdot b := \sup\{a_n \cdot b_n : \ n \in \mathbb{N}\}$; $\frac{a}{b} := \sup\left\{\frac{a_n}{b_n} : \ n \in \mathbb{N}\right\}$; for $a > b$, $a - b := \sup\{a_n - b_n : \ n \in \mathbb{N}\}$.

We note that all numbers a_n and b_n, $n \geq \mathbb{N}$, in the definition are rational and for them all arithmetic operations are defined. All known properties of arithmetic operations on integer numbers are also valid for real numbers but now they have to be proved.

Exercise 2.6. Show that

a) $a \cdot b = b \cdot a$ and $a + b = b + a$;

b) $a + (b + c) = (a + b) + c =: a + b + c$

c) $a \cdot (b \cdot c) = (a \cdot b) \cdot c =: a \cdot b \cdot c$.

2.1.3 n-th Root of a Positive Real Number

Theorem 2.4. *Let a be a positive real number and $n \in \mathbb{N}$. Then there exist a unique positive real number x satisfying $x^n = a$, where* $x^n := \underbrace{x \cdot \ldots \cdot x}_{n \text{ times}}$.

Remark 2.2. The number x can be constructed as the supremum of the set $\{y > 0 : \ y^n < a\}$, which is a non-empty bounded above set.

Definition 2.8. Let $a > 0$ and $n \in \mathbb{N}$. The unique positive solution of the equation $x^n = a$, which exists according to Theorem 2.4, is called the **n-th root** of the positive real number a. We use the notation for x: $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Definition 2.9. Let $a > 0$ and $r \in \mathbb{Q}$, $r > 0$. We define

$$a^r := (a^m)^{\frac{1}{n}},$$

where $r = \frac{m}{n}$, $m, n \in \mathbb{N}$.

Definition 2.10. Let $a > 1$ and $b > 0$. We define

$$a^b := \sup\{a^{b_n} : \ n \in \mathbb{N}\},$$

where $b := \beta_0, \beta_1\beta_2 \dots \beta_n \dots$ and $b_n := \beta_0, \beta_1\beta_2 \dots \beta_n$.

Exercise 2.7. Give a definition of a^b in the case $0 < a < 1$ and $b > 0$.

2.2 Some important inequalities

We recall that the absolute value of a real number a is given by

$$|a| = \begin{cases} a, & \text{if } a \geq 0, \\ -a, & \text{if } a < 0. \end{cases}$$

We note that $-|a| \leq a \leq |a|$ and also $|a| < c \iff -c < a < c$. Moreover, $|a| = |-a|$.

Theorem 2.5. *For all $a, b \in \mathbb{R}$ the inequalities*

$$1)\ |a+b| \leq |a| + |b| \quad 2)\ ||a| - |b|| \leq |a-b|$$

holds. For every $a_1, \dots a_n \in \mathbb{R}$ one has

$$|a_1 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|.$$

Proof. Since $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, we obtain $-(|a| + |b|) \leq a + b \leq |a| + |b|$. This implies inequality 1). Now, applying 1), we obtain $|a| = |a - b + b| \leq |a - b| + |b|$. Hence, $|a| - |b| \leq |a - b|$. Since $|a - b| = |b - a| \geq |b| - |a|$, we obtain 2). The latter inequality trivially follows from 1). □

Inequality 1) from Theorem 2.5 is called the **triangular inequality**.

Theorem 2.6 (Bernoulli's inequality)**.** *For each real number $x > -1$ and $n \in \mathbb{N}$ the inequality*

$$(1+x)^n \geq 1 + nx$$

holds. Moreover, $(1+x)^n = 1 + nx$ iff $x = 0$ or $n = 1$.

Proof. If $n = 1$ or $x = 0$, then the equality holds. We assume that $x \neq 0$ and use mathematical induction to prove $(1+x)^n > 1 + nx$ for all $n \geq 2$. So, for $n = 2$ one has

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x.$$

Next, we assume that the strict inequality holds for some $n \geq 2$. Then

$$(1+x)^{n+1} = (1+x)(1+x)^n > (1+x)(1+nx) = 1 + (n+1)x + nx^2 > 1 + (n+1)x.$$

□

Exercise 2.8. Show that
a) $2^n \geq n + 1$, $n \in \mathbb{N}$; b) $3^n \geq 2n + 1$, $n \in \mathbb{N}$; c) $2^n > (\sqrt{2} - 1)^2 n^2$, $n \in \mathbb{N}$.

Exercise 2.9. Let $x_1, \ldots, x_n$ be a positive real numbers. Prove that

$$(1 + x_1) \cdot \ldots \cdot (1 + x_n) \geq 1 + x_1 + \ldots + x_n.$$

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus.* Undergraduate Texts in Mathematics. Springer New York, 2013.