18 Lecture 18 – Improper Integrals

18.1 Integrals over Unbounded Intervals

18.1.1 Definition and Elementary Properties

In this section, we assume that a function $f : [a, +\infty) \to \mathbb{R}$ is integrable on $[a, z]$ for all $z > a$ and set

$$\varphi(z) := \int_a^z f(x)dx, \quad z \geq a.$$

Definition 18.1. The finite limit

$$\lim_{z\to+\infty} \varphi(z) = \lim_{z\to+\infty} \int_a^z f(x)dx \tag{27}$$

is called the **improper integral of f over** $[a, +\infty)$ and is denoted by

$$\int_a^{+\infty} f(x)dx. \tag{28}$$

In this case, we will say that improper integral (28) **converges**. If limit (27) does not exist or is infinite, then improper integral (28) is said to be **divergent**.

Remark 18.1. If integral (28) converges, then for each $b > a$ the improper integral

$$\int_b^{+\infty} f(x)dx \tag{29}$$

also converges. If for some $b > a$ improper integral (29) converges, then improper integral (28) also converges. These both statements follow from the equality

$$\int_a^z f(x)dx = \int_a^b f(x)dx + \int_b^z f(x)dx, \quad a < b \leq z,$$

and the definition of the improper integral.

Example 18.1. The improper integral $\int_0^{+\infty} e^{-x}dx$ converges and equals 1. Indeed,

$$\int_0^{+\infty} e^{-x}dx = \lim_{z\to+\infty} \int_0^z e^{-x}dx = \lim_{z\to+\infty} \left(-e^{-x}\Big|_0^z\right) = \lim_{z\to+\infty} (1 - e^{-z}) = 1.$$

Example 18.2. The equality $\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$ is true, since

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{z\to+\infty} \int_0^z \frac{dx}{1+x^2} = \lim_{z\to+\infty} \left(\arctan x\Big|_0^z\right) = \lim_{z\to+\infty} (\arctan z - 0) = \frac{\pi}{2}.$$

Example 18.3. Let $p > 0$. The improper integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$. Indeed, for each $z \geq 1$

$$\varphi(z) = \int_1^z \frac{dx}{x^p} = \begin{cases} \ln z - \ln 1, & p = 1, \\ \frac{z^{-p+1}}{-p+1} - \frac{1}{-p+1}, & p \neq 1. \end{cases}$$

Thus,

$$\varphi(z) \to \begin{cases} +\infty, & p \leq 1, \\ \frac{1}{p-1}, & p > 1, \end{cases} \quad z \to +\infty.$$

Example 18.4. The integral $\int_0^{+\infty} \cos x dx$ diverges, since for the function $\varphi(z) = \int_0^z \cos x dx = \sin z$, $z \geq 0$, there is no limit as $z \to +\infty$.

Theorem 18.1 (Elementary properties of improper integrals)**.** *The following properties of improper integrals are true.*

1) *Let integrals $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ converge. Then the integrals $\int_a^{+\infty} cf(x)dx$, $c \in \mathbb{R}$, $\int_a^{+\infty}(f(x) + g(x))dx$ converge and*

$$\int_a^{+\infty} cf(x)dx = c\int_a^{+\infty} f(x)dx,$$
$$\int_a^{+\infty} (f(x) + g(x))dx = \int_a^{+\infty} f(x)dx + \int_a^{+\infty} g(x)dx.$$

2) *Let f have an antiderivative F on $[a, +\infty)$. If the limit*

$$F(+\infty) := \lim_{z\to+\infty} F(z) \tag{30}$$

exists, then $\int_a^{+\infty} f(x)dx = F(+\infty) - F(a)$. If limit (30) does not exists or is infinite, then the integral $\int_a^{+\infty} f(x)dx$ diverges.

3) *(Integration by parts) We assume that functions u, v are continuously differentiable on $[a, +\infty)$. If the integral $\int_a^{+\infty} u(x)v'(x)dx = \int_a^{+\infty} u(x)dv(x)$ converges and the limit $u(+\infty)v(+\infty) := \lim_{z\to+\infty} u(z)v(z)$ exists, then the integral $\int_a^{+\infty} u'(x)v(x)dx = \int_a^{+\infty} v(x)du(x)$ converges and*

$$\int_a^{+\infty} v(x)du(x) = u(x)v(x)\Big|_a^{+\infty} - \int_a^{+\infty} u(x)dv(x).$$

Example 18.5. The integral $\int_0^{+\infty} xe^{-x}dx$ converges and equals 1, since

$$\int_0^{+\infty} xe^{-x}dx = -\int_0^{+\infty} xde^{-x} = -xe^{-x}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-x}dx = 1,$$

according to examples 7.3 and 18.1.

18.1.2 Convergence of Improper Integrals of Non-Negative Functions

Theorem 18.2. *The improper integral $\int_a^{+\infty} f(x)dx$ of a non-negative function f converges iff there exists $C \in \mathbb{R}$ such that $\varphi(z) = \int_a^z f(x)dx \leq C$ for all $z \geq a$.*

Proof. Since f is non-negative function, the function φ non-decreases. Consequently, the upper boundedness of φ is equivalent to the existence of the limit $\lim_{z\to+\infty} \varphi(z)$, by Theorem 7.9 (i). □

Theorem 18.3. *Let $f : [a, +\infty) \to \mathbb{R}$ and $g : [a, +\infty) \to \mathbb{R}$ satisfy $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then the convergence of the improper integral $\int_a^{+\infty} g(x)dx$ implies the convergence of $\int_a^{+\infty} f(x)dx$.*

Proof. The statement follows from Theorem 18.2 and the estimate

$$\varphi(z) = \int_a^z f(x)dx \le \int_a^z g(x)dx \le \lim_{z\to+\infty} \int_a^z g(x)dx = \int_a^{+\infty} g(x)dx =: C, \quad z \ge a.$$

Here the inequality for integrals follows from Theorem 16.6. □

Example 18.6. The integral $\int_0^{+\infty} \frac{\cos^2 x}{1+x^2}dx$ converges because we can apply Theorem 18.3 with $a = 0$, $f(x) = \frac{\cos^2 x}{1+x^2}$, $x \ge 0$, and $g(x) = \frac{1}{1+x^2}$, $x \ge 0$. For the convergence of the integral $\int_0^{+\infty} g(x)dx = \int_0^{+\infty} \frac{dx}{1+x^2}$ see Example 18.2.

Exercise 18.1. Show that the following improper integrals converge:
a) $\int_1^{+\infty} e^{-x^2}dx$; b) $\int_1^{+\infty} e^{-x}\ln x dx$; c) $\int_1^{+\infty} \frac{\ln x}{1+x^2}dx$.

Corollary 18.1. *We assume that for some numbers $0 < C < +\infty$ and $p > 0$ $f(x) \sim \frac{C}{x^p}$, $x \to +\infty$, i.e. $\lim_{x\to+\infty} x^p f(x) = C$. Then the integral $\int_a^{+\infty} f(x)dx$ converges for $p > 1$ and diverges for $p \le 1$.*

Proof. Let $p > 1$. By Theorem 7.1 (iii), there exists $D \ge a$ such that $x^p f(x) \le 2C$ for all $x \ge D$. Thus, $f(x) \le \frac{2C}{x^p}$, $x \ge D$. Now applying Theorem 18.3 with $a = D$, $g(x) = \frac{2C}{x^p}$, $x \ge D$, and using Example 18.3, we obtain that $\int_D^{+\infty} f(x)dx$ converges. Hence, $\int_a^{+\infty} f(x)dx$ also converges, by Remark 18.1.

Let $p \le 1$. Similarly, there exists $D \ge a$ such that $f(x) \ge \frac{C}{2x^p}$ for all $x \ge D$. Since the integral $\int_D^{+\infty} \frac{Cdx}{2x^p}$ diverges (see Example 18.3), the integral $\int_D^{+\infty} f(x)dx$ also diverges. □

18.1.3 Absolute and conditional convergence

Definition 18.2. An improper integral

$$\int_a^{+\infty} f(x)dx \tag{31}$$

is said to be **absolutely convergent**, if the integral

$$\int_a^{+\infty} |f(x)|dx \tag{32}$$

converges. If integral (31) converges but integral (32) diverges, then (31) is called **conditionally convergent**.

Theorem 18.4. *If an improper integral absolutely converges, then it converges.*

Proof. Let integral (32) converge. We consider the following functions

$$f_-(x) := \frac{|f(x)| - f(x)}{2}, \quad x \ge a,$$
$$f_+(x) := \frac{|f(x)| + f(x)}{2}, \quad x \ge a,$$

and note that $0 \le f_-(x) \le |f(x)|$, $0 \le f_+(x) \le |f(x)|$ and $f(x) = f_+(x) - f_-(x)$ for all $x \ge a$.

By Theorem 18.3, the integrals $\int_a^{+\infty} f_-(x)dx$ and $\int_a^{+\infty} f_+(x)dx$ converge. Thus, using Theorem 18.1 1), we have that the improper integral $\int_a^{+\infty} f(x)dx = \int_a^{+\infty}(f_+(x) - f_-(x))dx$ also converges. □

Example 18.7. The integral $\int_1^{+\infty} \frac{\sin x}{x} dx$ is conditionally convergent. Indeed, according to the integration by parts formula, we have

$$\int_1^{+\infty} \frac{\sin x}{x} dx = \lim_{z\to+\infty} \int_1^z \frac{\sin x}{x} dx = -\lim_{z\to+\infty} \int_1^z \frac{1}{x} d\cos x = -\lim_{z\to+\infty} \left[\frac{\cos x}{x}\Big|_1^z - \int_1^z \cos x d\frac{1}{x} \right]$$
$$= -\lim_{z\to+\infty} \left[\frac{\cos z}{z} - \cos 1 + \int_1^z \frac{\cos x}{x^2} dx \right] = \cos 1 - \int_1^{+\infty} \frac{\cos x}{x^2} dx.$$

The integral $\int_1^{+\infty} \frac{\cos x}{x^2} dx$ absolutely converges because $\frac{|\cos x|}{x^2} \leq \frac{1}{x^2}$, $x \geq 1$, and the integral $\int_1^{+\infty} \frac{dx}{x^2}$ converges (see Example 18.3). Thus, $\int_1^{+\infty} \frac{\cos x}{x^2} dx$ converges, by Theorem 18.4. This implies the convergence of the integral $\int_1^{+\infty} \frac{\sin x}{x} dx$.

Next, we show that $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ diverges. We estimate

$$\int_\pi^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=2}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| dx = \frac{2}{\pi} \sum_{k=2}^n \frac{1}{k}.$$

In the next lecture, we will show that $\sum_{k=2}^n \frac{1}{k} \to +\infty$, $n \to \infty$. Thus, $\int_\pi^{+\infty} \frac{|\sin x|}{x} dx$ diverges and, consequently, $\int_1^{+\infty} \frac{|\sin x|}{x} dx$ also diverges.

Theorem 18.5 (Dirichlet's test)**.** *Let functions f and g satisfy the following properties:*

1) *there exists $C \in \mathbb{R}$ such that $\left|\int_a^z f(x)dx\right| \leq C$ for all $z \geq a$;*

2) *the function g is monotone on $[a, +\infty)$;*

3) *$g(x) \to 0$, $x \to +\infty$.*

Then the integral $\int_a^{+\infty} f(x)g(x)dx$ converges.

Example 18.8. The integral $\int_1^{+\infty} \frac{\sin x}{x} dx$ converges, since the functions $f(x) = \sin x$, $x \geq 1$, and $g(x) = \frac{1}{x}$, $x \geq 1$, satisfy conditions 1)-3) of Theorem 18.5 with $C = 2$.

Example 18.9. The integral $\int_1^{+\infty} \sin x^3 dx$ converges, since the functions $f(x) = x^2 \sin x^3$, $x \geq 1$, and $g(x) = \frac{1}{x^2}$, $x \geq 1$, satisfy conditions 1)-3) of Theorem 18.5 with $C = \frac{2}{3}$.

Theorem 18.6 (Abel's test)**.** *Let functions f and g satisfy the following properties:*

1) *the integral $\int_a^{+\infty} f(x)dx$ converges;*

2) *the function g is monotone on $[a, +\infty)$;*

3) *the function g is bounded on $[a, +\infty)$.*

Then the integral $\int_a^{+\infty} f(x)g(x)dx$ converges.

Exercise 18.2. Prove the convergence of the following integrals:
a) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}} dx$; b) $\int_1^{+\infty} \cos x^2 dx$; c) $\int_0^{+\infty} \sin x^2 dx$; d) $\int_1^{+\infty} \frac{\sin 2x \cdot \sin x}{x} dx$.

Remark 18.2. The definition and properties of the improper integral $\int_{-\infty}^a f(x)dx$ are similar to ones of $\int_a^{+\infty} f(x)dx$. The integral $\int_{-\infty}^{+\infty} f(x)dx$ is defined as $\int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)dx$ for any $a \in \mathbb{R}$.

18.2 Improper Integrals of Unbounded Functions

In this section, we will consider a function $f[a,b) \to \mathbb{R}$ such that for all $c \in (a,b)$ it is integrable on $[a,c]$ and unbounded on (c,b). The case of a function $f(a,b] \to \mathbb{R}$, which is unbounded near a can be considered similarly. We set

$$\varphi(z) = \int_a^z f(x)dx, \quad z \in (a,b).$$

Definition 18.3. The finite limit

$$\lim_{z\to b-} \varphi(z)dz = \lim_{z\to b-} \int_a^z f(x)dx \tag{33}$$

is called the **improper integral** of f over $[a,b)$ and is denoted by

$$\int_a^b f(x)dx. \tag{34}$$

In this case, we will say that the improper integral (34) **converges**. If limit (33) does not exist or is infinite, then the improper integral (34) is said to be **divergent**.

Exercise 18.3. The improper integral $\int_0^1 \frac{dx}{\sqrt{1-x}}$ converges, since

$$\int_0^z \frac{dx}{\sqrt{1-x}} = -\int_0^z \frac{d(1-x)}{\sqrt{1-x}} = -\int_0^z (1-x)^{-\frac{1}{2}} d(1-x) = -2(1-x)^{\frac{1}{2}}\Big|_0^z = -2(1-z)^{\frac{1}{2}}+2 \to 2, \quad z \to 1-.$$

Example 18.10. The improper integral $\int_0^1 \ln x dx$ converges, since

$$\int_z^1 \ln x dx = (x\ln x - x)\Big|_z^1 = -1 - z\ln z + z \to -1, \quad z \to 0+,$$

by Exeample 13.1 b). For the computation of an antiderivative of $\ln x$ see Example 15.7.

Exercise 18.4. Prove that the improper integral $\int_a^b \frac{dx}{(b-x)^p}$, $p > 0$, converges for $p < 1$ and diverges for $p \geq 1$.

The following properties of improper integrals of unbounded functions can be proved similarly as properties of improper integrals over unbounded intervals.

1. Let $f(x) \geq 0$, $x \in [a,b)$. The improper integral $\int_a^b f(x)dx$ converges iff there exists $C \in \mathbb{R}$ such that $\int_a^z f(x)dx \leq C$ for all $z \in [a,b)$.

2. Let $0 \leq f(x) \leq g(x)$, $x \in [a,b)$. If the improper integral $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.

3. If for some $p > 0$ and $0 < C < +\infty$ $f(x) \sim \frac{C}{(b-x)^p}$, $x \to b-$, that is, $\lim_{x\to b-} (b-x)^p f(x) = C$, then the integral $\int_a^b f(x)dx$ converges for $p < 1$ and diverges for $p \geq 1$.

4. If a function f has an antiderivative F on $[a,b)$ and there exists a limit $F(b-) := \lim_{x\to b-} F(x)$, then $\int_a^b f(x)dx = F(b-) - F(a)$.

Exercise 18.5. Prove the convergence of the integral $\int_0^1 \frac{dx}{\sqrt[3]{1-x^2}}$.