17 Lecture 17 – Fundamental Theorem of Calculus and Application of Riemann Integral

17.1 Fundamental Theorem of Calculus

We set $\int_a^a f(x)dx := 0$ and $\int_b^a f(x)dx := -\int_a^b f(x)dx$ for $a < b$.

Theorem 17.1. *Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then the function $\varphi(x) := \int_a^x f(u)du$, $x \in [a, b]$, is continuous on $[a, b]$.*

Proof. For every $x', x'' \in [a, b]$ we have

$$|\varphi(x') - \varphi(x'')| = \left| \int_a^{x'} f(x)dx - \int_a^{x''} f(x)dx \right| = \left| \int_{x'}^{x''} f(x)dx \right| \leq \int_{x'}^{x''} |f(x)|dx \leq \sup_{x\in[a,b]} |f(x)||x' - x''|,$$

by Theorem 16.5 (iii) and corollaries 16.1, 16.2. Consequently, φ is uniformly continuous on $[a, b]$. □

Theorem 17.2. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then the function $\varphi(x) := \int_a^x f(u)du$, $x \in [a, b]$, is differentiable on $[a, b]$ and $\varphi'(x) = f(x)$, $x \in [a, b]$, that is, φ is an antiderivative of f on $[a, b]$.*

Proof. Let $x_0 \in [a, b]$ and $h \neq 0$. By the mean value theorem (see Theorem 16.7), there exists θ_h between x_0 and $x_0 + h$ such that

$$\frac{\varphi(x_0 + h) - \varphi(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} f(x)dx = f(\theta_h).$$

Since $\theta_h \to x_0$, $h \to 0$, and f is continuous, we obtain

$$\lim_{h\to 0} \frac{\varphi(x_0 + h) - \varphi(x_0)}{h} = \lim_{h\to 0} f(\theta_h) = f(x_0).$$

□

Theorem 17.3 (Fundamental Theorem of Calculus)**.** *We assume that $f : [a, b] \to \mathbb{R}$ satisfies the following properties:*

1) f is integrable on $[a, b]$;

2) f has an antiderivative F on $[a, b]$.

Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

We will also denote $F(x)\Big|_a^b := F(b) - F(a)$.

Proof. We first prove the theorem in the case $f \in C([a, b])$. The function $\varphi(x) := \int_a^x f(u)du$, $x \in [a, b]$, is an antiderivative of f on $[a, b]$, by Theorem 17.2. Thus, using Remark 15.1, there exists $C \in \mathbb{R}$ such that $\varphi(x) = F(x) + C$, $x \in [a, b]$. In particular, $\varphi(a) = F(a) + C = 0$. Thus, $C = -F(a)$. Consequently, $\int_a^b f(x)dx = \varphi(b) = F(b) + C = F(b) - F(a)$.

Next, we give the second proof of the theorem in the general case. Let $\lambda = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We first note that

$$F(b) - F(a) = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1})) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})).$$

We apply the Lagrange theorem (see Theorem 11.4) to the function F on $[x_{k-1}, x_k]$ for each $k = 1, \ldots, n$. So, there exists $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$, such that

$$F(b) - F(a) = \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} F'(\xi_k)\Delta x_k = \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

Making $|\lambda| \to 0$, we have

$$F(b) - F(a) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \to \int_a^b f(x)dx,$$

since f is integrable on $[a, b]$. □

Exercise 17.1. Compute the following integrals:
a) $\int_{-1}^{8} \sqrt[3]{x}dx$; b) $\int_0^{\pi} \sin x dx$; c) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$; d) $\int_0^2 |1 - x|dx$; e) $\int_{-1}^{1} \frac{dx}{x^2 - 2x\cos\alpha + 1}$ for $\alpha \in (0, \pi)$.

Example 17.1 (Leibniz's rule)**.** Let a function $f : \mathbb{R} \to \mathbb{R}$ have an antiderivative on $\mathbb{R}$ and be integrable on each finite interval. Let functions $a, b\mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Then

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(u)du = f(b(x))b'(x) - f(a(x))a'(x), \quad x \in \mathbb{R}.$$

Indeed, let F be an antiderivative of f on $\mathbb{R}$. By the fundamental theorem of calculus,

$$\int_{a(x)}^{b(x)} f(u)du = F(b(x)) - F(a(x)), \quad x \in \mathbb{R}. \tag{25}$$

Moreover, the right hand side of (25) is differentiable and

$$\frac{d}{dx}(F(b(x)) - F(a(x))) = F'(b(x))b'(x) - F'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x), \quad x \in \mathbb{R},$$

by the chain rule.

Exercise 17.2. Compute the following derivatives:
a) $\frac{d}{dx} \int_a^b \sin x^2 dx$; b) $\frac{d}{da} \int_a^b \sin x^2 dx$; c) $\frac{d}{dx} \int_0^{x^2} \sqrt{1 + t^2}dt$; d) $\frac{d}{dx} \int_{x^2}^{x^3} \frac{dt}{1+t^4}$.

Exercise 17.3. Compute the following limits:
a) $\lim\limits_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x}$; b) $\lim\limits_{x\to+\infty} \frac{\int_0^x (\arctan t)^2 dt}{\sqrt{x^2+1}}$; c) $\lim\limits_{x\to+\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt}$.

17.2 Some Corollaries

Theorem 17.4 (Substitution rule)**.** *We assume that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $u : [\alpha,\beta] \to [a,b]$ is continuously differentiable on $[\alpha,\beta]$. Then the following equality*

$$\int_\alpha^\beta f(u(t))u'(t)dt = \int_\alpha^\beta f(u(t))du(t) = \int_{u(\alpha)}^{u(\beta)} f(x)dx$$

holds.

Proof. Since the function f is continuous on $[u(\alpha), u(\beta)]$, it has an antiderivative F on $[u(\alpha), u(\beta)]$, by Theorem 17.2. Using the fundamental theorem of calculus,

$$\int_{u(\alpha)}^{u(\beta)} f(x)dx = F(u(\beta)) - F(u(\alpha)).$$

Moreover, the function $F(u)$ is an antiderivative of $f(u)u'$ on $[\alpha,\beta]$. Thus, by the fundamental theorem of calculus,

$$\int_\alpha^\beta f(u(t))u'(t)dt = F(u(\beta)) - F(u(\alpha)).$$

This proves the theorem. □

Exercise 17.4. Using the substitution rule, compute the following integrals:
a) $\int_0^{\sqrt{\pi}} x \sin x^2 dx$; b) $\int_0^1 e^{2x-1}dx$; c) $\int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$; d) $\int_0^{\ln 2} \sqrt{e^x - 1}dx$; e) $\int_0^{\frac{\pi}{6}} \frac{dx}{\cos x}$.

Theorem 17.5 (Integration by parts)**.** *Let $u, v : [a,b] \to \mathbb{R}$ be continuously differentiable functions on $[a,b]$. Then*

$$\int_a^b u(x)dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)du(x),$$

i.e.

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)dx.$$

Proof. Since the function uv is an antiderivative of $uv' + u'v$ on $[a,b]$,

$$\int_a^b (u(x)v'(x) + u'(x)v(x))dx = u(b)v(b) - u(a)v(a),$$

by the fundamental theorem of calculus. Using Theorem 16.5 (ii), we obtain the integration by parts formula. □

Exercise 17.5. Using the integration by parts formula, compute the following integrals:
a) $\int_0^{\ln 2} xe^{-x}dx$; b) $\int_0^\pi x \sin x dx$; c) $\int_0^{2\pi} x^2 \cos x dx$; d) $\int_{\frac{1}{e}}^e |\ln x| dx$; e) $\int_0^1 \arccos x dx$.

17.3 Application of the Integral

17.3.1 Area of the Region under the Graph of Function

Theorem 17.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and $f(x) \geq 0$, $x \in [a, b]$. Then the area of the region*

$$F = \{(x, y) : \ 0 \leq y \leq f(x), \ a \leq x \leq b\}$$

under the graph of f is equal to

$$S(F) = \int_a^b f(x)dx.$$

Proof. We first note that f is integrable on $[a, b]$ because it is continuous (see Theorem 16.4). Thus, the formula for the area follows from the discussion in Section 16.1 and definition of the integral (see (23)). □

Example 17.2. The area of the region under the graph of the function $f(x) = x^2$, $x \in [0, 1]$, is equal

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

Example 17.3. Compute the area of the region G enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > 0$, $b > 0$. In order to compute the area of G, it is enough to compute the area of

$$F = \left\{(x, y) : \ 0 \leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}, \quad 0 \leq x \leq a\right\}.$$

By Theorem 17.6,

$$S(G) = 4S(F) = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}dx = \left| \begin{array}{c} x = a\sin t \\ dx = a\cos t dt \end{array} \right| = 4ab\int_0^{\frac{\pi}{2}} \cos^2 t dt$$
$$= 4ab\int_0^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} dt = 2abt\Big|_0^{\frac{\pi}{2}} + ab\sin 2t\Big|_0^{\frac{\pi}{2}} = \pi ab.$$

Exercise 17.6. Compute the area of regions bounded by the graphs of the following functions:
a) $2x = y^2$ and $2y = x^2$; b) $y = x^2$ and $x + y = 2$; c) $y = 2^x$, $y = 2$ and $x = 0$;
d) $y = \frac{a^3}{a^2+x^2}$ and $y = 0$, where $a > 0$.

17.3.2 Length of a Curve

Definition 17.1. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$. The set of points

$$\Gamma := \{(x, y) \in \mathbb{R}^2 : \ x = \varphi(t), \ y = \psi(t), \ t \in [a, b]\} \tag{26}$$

is called a **continuous (plane) curve**.

We first give a definition of the length of the continuous curve Γ. Let $\lambda = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. We consider the polygonal curve Γ_λ with vertices $(\varphi(t_k), \psi(t_k))$, $k = 0, \ldots, n$. Its length equals

$$l(\Gamma_\lambda) = \sum_{k=1}^n \sqrt{(\varphi(t_k) - \varphi(t_{k-1}))^2 + (\psi(t_k) - \psi(t_{k-1}))^2}.$$

Definition 17.2. The curve Γ is said to be a **rectifiable curve**, if there exists a finite limit

$$\lim_{|\lambda|\to 0} l(\Gamma_\lambda) =: l(\Gamma),$$

that is, if there exists a real number $l(\Gamma)$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall \lambda \;\; |\lambda| < \delta : \quad |l(\Gamma_\lambda) - l(\Gamma)| < \varepsilon.$$

The limit $l(\Gamma)$ is called the **length of rectifiable curve** Γ.

Theorem 17.7. *Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be continuously differentiable on $[a, b]$. Then Γ, defined by (26), is a rectifiable curve and its length equals*

$$l(\Gamma) = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2} dt.$$

Proof. Using the Lagrange theorem (see Theorem 11.4), we have

$$l(\Gamma_\lambda) = \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\eta_k))^2} \Delta t_k = \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2} \Delta t_k + r_\lambda,$$

where $\xi_k, \eta_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$, and

$$r_\lambda := \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\eta_k))^2} \Delta t_k - \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2} \Delta t_k.$$

Since the function $f(t) = \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}$, $t \in [a, b]$, is continuous on $[a, b]$, it is integrable on $[a, b]$, by Theorem 16.4. Thus,

$$\lim_{|\lambda|\to 0} \sum_{k=1}^n \sqrt{(\varphi'(\xi_k))^2 + (\psi'(\xi_k))^2} \Delta t_k = \int_a^b \sqrt{(\varphi'(t))^2 + (\psi'(t))^2} dt.$$

Moreover, using the inequality

$$|\sqrt{u^2 + v^2} - \sqrt{u^2 + w^2}| \le |v - w|, \quad u, v, w \in \mathbb{R},$$

(see Exercise 12.5 b)), we have

$$|r_\lambda| \le \sum_{k=1}^n |\psi'(\xi_k) - \psi'(\eta_k)| \Delta t_k \le \sum_{k=1}^n (M_k - m_k) \Delta t_k,$$

where $M_k := \sup_{t \in [t_{k-1}, t_k]} \psi'(t)$ and $m_k := \inf_{t \in [t_{k-1}, t_k]} \psi'(t)$, $k = 1, \ldots, n$. Using theorems 16.2 and 16.4, we obtain

$$|r_\lambda| \le \sum_{k=1}^n (M_k - m_k) \Delta t_k \to 0, \quad |\lambda| \to 0.$$

□

Remark 17.1. If a curve Γ is given by the graph of a continuously differentiable function $f : [a, b] \to \mathbb{R}$, that is,

$$\Gamma = \{(x, y) : \ y = f(x), \ \ x \in [a, b]\},$$

then its length equals

$$l(\Gamma) = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Example 17.4. We compute the length of the circle $x^2 + y^2 = r^2$, $r > 0$, that is, the length of the curve

$$\Gamma = \left\{(x, y) : \ x^2 + y^2 = r^2\right\} = \{(x, y) : \ x = r\cos t, \ y = r\sin t, \ t \in [0, 2\pi)\}.$$

By Theorem 17.7,

$$l(\Gamma) = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt = \int_0^{2\pi} r dt = 2\pi r.$$

Exercise 17.7. Compute the length of continuous curves defined by the following functions:
a) $y = x^{\frac{3}{2}}$, $x \in [0, 4]$; b) $y = e^x$, $0 \le x \le b$; c) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.

17.3.3 Volume of Solid of Revolution

Definition 17.3. Let $f : [a, b] \to \mathbb{R}$ be a positive continuous function. A **solid of revolution** G is a set of points in $\mathbb{R}^3$ obtained by rotating of the region under the graph of f around the x-axis, that is,

$$G = \left\{(x, y, z) : \ y^2 + z^2 \le f^2(x), \ x \in [a, b]\right\}.$$

Theorem 17.8. *Let $f : [a, b] \to \mathbb{R}$ be a positive continuous function. Then the volume of solid of revolution G is equal to*

$$V(G) = \pi \int_a^b f^2(x) dx.$$

Idea of Proof. We consider a partition $\lambda = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$ and split G into smaller sets

$$G_k = \left\{(x, y, z) : \ y^2 + z^2 \le f^2(x), \ x \in [x_{k-1}, x_k]\right\}, \quad k = 1, \ldots, n.$$

Then the volume of G_k is approximately equal the volume of the cylinder

$$\left\{(x, y, z) : \ y^2 + z^2 \le f^2(\xi_k), \ x \in [x_{k-1}, x_k]\right\},$$

where $\xi_k \in [x_{k-1}, x_k]$. Thus,

$$V(G) = \sum_{k=1}^n V(G_k) \approx \sum_{k=1}^n \pi f^2(\xi_k) \Delta x_k.$$

Passing to the limit as $|\lambda| \to 0$, we obtain

$$V(G) = \pi \int_a^b f^2(x) dx.$$

□

Example 17.5. The volume of the cone

$$G = \{(x, y, z) : \; y^2 + z^2 \le x^2, \quad x \in [0, 1]\}.$$

equals

$$V(G) = \pi \int_0^1 x^2 dx = \pi \frac{x^3}{3}\Big|_0^1 = \frac{\pi}{3},$$

since G can be obtained by rotating of the region under the graph of the function $f(x) = x$, $x \in [0, 1]$, around the x-axis.

Exercise 17.8. Compute the volume of the paraboloid of revolution

$$G = \{(x, y, z) : \; y^2 + z^2 \le x, \quad x \in [0, 1]\}.$$

(*Hint:* G can be obtained by rotating of the region under the graph of the function $f(x) = \sqrt{x}$, $x \in [0, 1]$, around the x-axis)