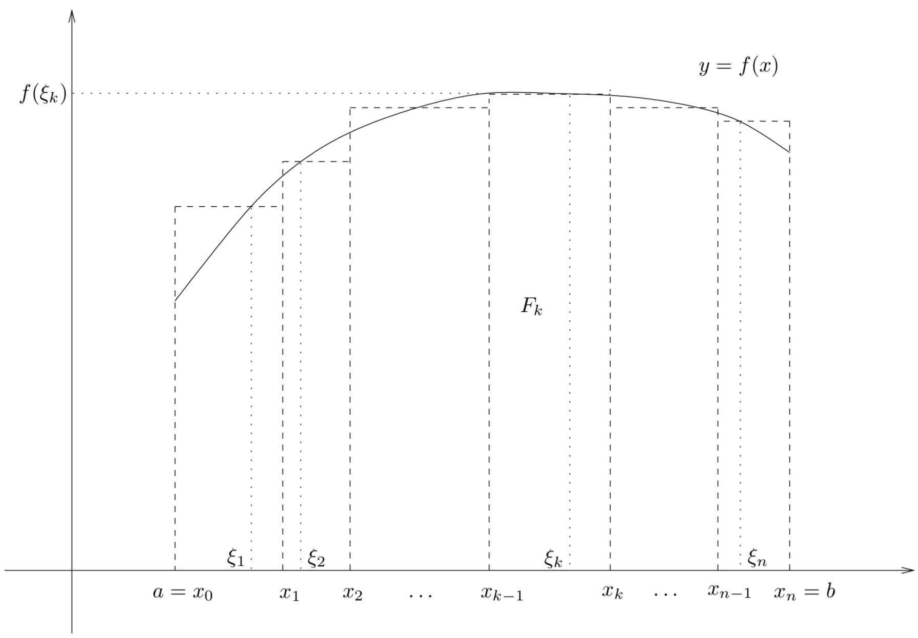
16 Lecture 16 – Riemann Integral

16.1 Area under the Graph of Function

We consider the following problem. Let $f : [a, b] \to \mathbb{R}$ be non-negative continuous function. We want to compute the area of the region under the graph of f, that is, the area of the set

$$F := \{(x, y) : \ y \in [0, f(x)], \ x \in [a, b]\}.$$



For this, we divide the interval $[a, b]$ into smaller subintervals $[x_{k-1}, x_k]$, $k = 1, \dots, n$, where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, and consider the following partition of F to the sets

$$F_k := \{(x, y) : \ y \in [0, f(x)], \ x \in [x_{k-1}, x_k]\},$$

$k = 1, \dots, n$. Since f is a continuous, its values vary little on $[x_{k-1}, x_k]$, if $\Delta x_k = x_k - x_{k-1}$ is small. Consequently, we should expect that the area of F_k should be close to the area of the rectangle with sides Δx_k and $f(\xi_k)$ which equals $f(\xi_k)\Delta x_k$, where ξ_k are points from the intervals $[x_{k-1}, x_k]$. Thus, one can expect that

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \to S(F), \quad \text{as} \quad \max_k |\Delta x_k| \to 0. \tag{21}$$

Limit of the type (21) really exists, and will be studied in the next sections.

16.2 Definition of the Integral

Definition 16.1.
- Let $[a, b]$ be an interval and $n \in \mathbb{N}$. A set of points $x_0, x_1, \dots, x_n$ such that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ is called a **partition of the interval** $[a, b]$ and is denoted by λ.

- The number $|\lambda| = \max\{\Delta x_k : 1 \le k \le n\}$, where $\Delta x_k = x_k - x_{k-1}$, is called the **mesh of a partition** λ.

Let $f : [a, b] \to \mathbb{R}$ be a function, $\lambda = \{x_0, x_1, \dots, x_n\}$ be a partition of the interval $[a, b]$ and $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$. The sum

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{22}$$

is called the **Riemann sum**.

Definition 16.2. A function f is said to be **integrable on** $[a, b]$, if there exists a limit J of Riemann sums (22) as $|\lambda| \to 0$ and this limit does not depend on the choice of partitions λ and points ξ_k. More precisely, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each partition $\lambda = \{x_0, x_1, \dots, x_n\}$ with $|\lambda| < \delta$ and points $\xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$,

$$\left| J - \sum_{k=1}^{n} f(\xi_k)\Delta x_k \right| < \varepsilon.$$

The number J is called the **Riemann integral of f over** $[a, b]$ and is denoted by $\int_a^b f(x)dx$.

Shortly, we will write

$$\int_a^b f(x)dx = \lim_{|\lambda|\to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, then we will write $f \in R([a, b])$.

Exercise 16.1. Show that a constant function $f(x) = c$, $x \in [a, b]$, is integrable on $[a, b]$ and compute $\int_a^b c dx$.

Exercise 16.2. Show that the Dirichlet function $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$, is not integrable on any interval $[a, b]$, $a < b$.

Exercise 16.3. Let $f, g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Show that $f + g$ is also integrable on $[a, b]$.

Theorem 16.1. *If a function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, then f is bounded on $[a, b]$.*

Exercise 16.4. Prove Theorem 16.1.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$.

Definition 16.3.

- The **upper Darboux sum** of f with respect to a partition λ is the sum

$$U(f, \lambda) = \sum_{k=1}^{n} M_k \Delta x_k,$$

where $M_k := \sup_{x\in[x_{k-1},x_k]} f(x)$.

- The **lower Darboux sum** of f with respect to a partition λ is the sum

$$L(f, \lambda) = \sum_{k=1}^{n} m_k \Delta x_k,$$

where $m_k := \inf_{x\in[x_{k-1},x_k]} f(x)$.

Theorem 16.2 (Integrability criterion)**.** *A function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists $\lambda = \lambda([a, b])$ such that*

$$U(f, \lambda) - L(f, \lambda) < \varepsilon.$$

Exercise 16.5. Let $f \in R([a, b])$. Show that
a) $|f| \in R([a, b])$; b) $\sin f \in R([a, b])$; c) $f^2 \in R([a, b])$; d) $\max\{0, f\} \in R([a, b])$.

Exercise 16.6. Let $f, g \in R([a, b])$. Show that $fg \in R([a, b])$.

16.3 Classes of Integrable Functions

16.3.1 Integrability of Monotone Functions

Theorem 16.3. *Let $f : [a, b] \to \mathbb{R}$ be a monotone function on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. We assume that f is increasing on $[a, b]$ and $f(a) < f(b)$. To prove the theorem, we are going to use the integrability criterion (see Theorem 16.2). For any $\varepsilon > 0$ we take a partition λ of the interval $[a, b]$ such that $|\lambda| < \frac{\varepsilon}{f(b)-f(a)}$. For such a partition we have

$$\begin{aligned} U(f, \lambda) - L(f, \lambda) &= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))\Delta x_k \\ &\leq |\lambda| \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = |\lambda|(f(x_n) - f(x_0)) = |\lambda|(f(b) - f(a)) < \varepsilon. \end{aligned}$$

□

Exercise 16.7. For any bounded function $f : [a, b] \to \mathbb{R}$ we set $g(x) = \sup_{u\in[a,x]} f(u)$ and $h(x) = \inf_{u\in[a,x]} f(u)$, $x \in [a, b]$. Show that $g, h \in R([a, b])$.

16.3.2 Integrability of Continuous Functions

Theorem 16.4. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is integrable on $[a, b]$.*

Proof. We will use the integrability criterion again, to prove the theorem. By the Cantor theorem (see Theorem 9.4), f is uniformly continuous on $[a, b]$. Thus, for a number $\frac{\varepsilon}{b-a} > 0$ there exists $\delta > 0$ such that for each $x', x'' \in [a, b]$, $|x' - x''| < \delta$ it follows $|f(x') - f(x'')| < \frac{\varepsilon}{b-a}$. Next, we choose a partition λ of $[a, b]$ with $|\lambda| < \delta$. Thus, by the 2nd Weierstrass theorem (see Theorem 9.2), for each $k = 1, \ldots, n$

$$M_k - m_k = \sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x) = f(x^*) - f(x_*) < \frac{\varepsilon}{b-a},$$

where x^* and x_* are points where f takes its maximum and minimum value on $[x_{k-1}, x_k]$, respectively. Consequently,

$$U(f, \lambda) - L(f, \lambda) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \varepsilon.$$

□

16.4 Properties of Riemann Integral

Theorem 16.5 (Linearity and addidivity)**.** *(i) Let $f \in R([a,b])$ and $c \in \mathbb{R}$. Then $cf \in R([a,b])$ and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

(ii) Let $f, g \in R([a,b])$. Then $f+g \in R([a,b])$ and

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(iii) Let $f \in R([a,b])$ and $c \in (a,b)$. Then $f \in R([a,c])$ and $f \in R([c,b])$. Moreover,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Exercise 16.8. Prove (i) and (ii) of Theorem 16.5.

Exercise 16.9. Let $c \in (a,b)$. Show that $f \in R([a,b])$, if $f \in R([a,c])$ and $f \in R([c,b])$.

Theorem 16.6. *Let $f, g \in R([a,b])$ and $f(x) \leq g(x)$, $x \in [a,b]$. Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*

Proof. The statement immediately follows from the definition of the integral. □

Exercise 16.10. Prove Theorem 16.6.

Corollary 16.1. *Let $f \in R([a,b])$ and $m := \inf\limits_{x\in[a,b]} f(x)$, $M := \sup\limits_{x\in[a,b]} f(x)$. Then*

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a). \tag{23}$$

Proof. We first note that m and M exists, since f is bounded (see Theorem 16.1). Inequality (23) follows from the inequality $m \leq f(x) \leq M$, $x \in [a,b]$, and Theorem 16.6. □

Corollary 16.2. *Let $f \in R([a,b])$. Then $|f| \in R([a,b])$ and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Exercise 16.11. Prove Corollary 16.2.

Theorem 16.7 (Mean value theorem for integrals)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then there exists $\theta \in [a,b]$ such that $\int_a^b f(x)dx = f(\theta)(b-a)$.*

Proof. By Corollary 16.1,

$$m \leq L := \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

Since f is continuous, we can apply the 2nd Weierstrass theorem (see Theorem 9.2) to f. Thus, there exist $x_*, x^* \in [a,b]$ such that $m = f(x_*)$ and $M = f(x^*)$. Consequently, $f(x_*) \leq L \leq f(x^*)$. By the intermediate value theorem (see Theorem 9.3), there exists θ between x^* and x_* such that $f(\theta) = L$. □

Exercise 16.12. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function on $[a, b]$ such that $f(x_0) > 0$ for some $x_0 \in [a, b]$. Show that $\int_a^b f(x)dx > 0$.

Exercise 16.13. Let $f \in C([a, b])$, $g \in R([a, b])$ and $g(x) \geq 0$, $x \in [a, b]$. Show that there exists $\theta \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(\theta) \int_a^b g(x)dx$.

Exercise 16.14. For functions $f, g \in R([a, b])$ compute the limit

$$\lim_{|\lambda| \to 0} \sum_{k=1}^{n} f(\xi_k) \int_{x_{k-1}}^{x_k} g(x)dx.$$

Exercise 16.15. For a function $f \in R([0, 1])$ prove the equality

$$\lim_{n \to \infty} \int_{\frac{1}{n}}^{1} f(x)dx = \int_0^1 f(x)dx.$$

Exercise 16.16 (Cauchy inequality)**.** For $f, g \in R([a, b])$ prove the following inequality

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$$