15 Lecture 15 – Antiderivative and Indefinite Integral

15.1 Definitions and Elementary Properties

In this section, J denotes one of the following intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b), $(-\infty, a]$, $(-\infty, a)$, $[b, +\infty)$, $(b, +\infty)$ or $(-\infty, +\infty)$. Moreover, for a function $f : [a, b] \to \mathbb{R}$, we set $f'(a) := f'_+(a)$ and $f'(b) := f'_-(b)$.

Definition 15.1. A function $F : J \to \mathbb{R}$ is said to be an **antiderivative** or a **primitive function** of a function $f : J \to \mathbb{R}$, if for each $x \in J$ there exists $F'(x)$ and $F'(x) = f(x)$.

Example 15.1. An antiderivative of the function $f(x) = x$, $x \in \mathbb{R}$, is the function $F(x) = \frac{1}{2}x^2$, $x \in \mathbb{R}$, since $\left(\frac{1}{2}x^2\right)' = x$ for all $x \in \mathbb{R}$.

The function $G(x) = \frac{1}{2}x^2 + 1$ is also an antiderivative of f because $\left(\frac{1}{2}x^2 + 1\right)' = x$ for all $x \in \mathbb{R}$.

Example 15.2. An antiderivative of the function $f(x) = \begin{cases} 0, & x < 0, \\ x, & x \geq 0, \end{cases}$ is the function $F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{2}, & x \geq 0. \end{cases}$
Indeed, for each $x < 0$, $F'(x) = 0$ and for each $x > 0$ $F'(x) = x$. Moreover, $F'_-(0) = 0$, $F'_+(0) = 0$ and, thus, $F'(0) = 0$, by Remark 10.2.

Remark 15.1. We note *if f has an antiderivative, then it is not unique.* Indeed, if F is an antiderivative of f, then for any constant $C \in \mathbb{R}$ the function $F + C$ is also an andiderivative of f because for each $x \in J$ $(F(x) + C)' = F'(x) = f(x)$. Moreover, if F and G are antiderivatives of f, then there exists a constant $C \in \mathbb{R}$ such that $F = G + C$, by Corollary 12.2.

Definition 15.2. The indefinite integral of a function $f : J \to \mathbb{R}$ is the expression $F(x) + C$, $x \in J$, where F is an antiderivative of f and C denotes an arbitrary constant. The indefinite integral of a function f is denoted by $\int f(x)dx$, $x \in J$.

Exercise 15.1. Find antiderivatives of the following functions:
a) $f(x) = |x|$, $x \in \mathbb{R}$; b) $f(x) = \max\{1, x^2\}$, $x \in \mathbb{R}$; c) $f(x) = |\sin x|$, $x \in \mathbb{R}$;
d) $f(x) = \sin x + |\sin x|$, $x \in \mathbb{R}$.

Exercise 15.2. Let a function $f : \mathbb{R} \to \mathbb{R}$ has an antiderivative $F : \mathbb{R} \to \mathbb{R}$. Find f, if for each $x \in \mathbb{R}$:
a) $F(x) = f(x)$; b) $F(x) = \frac{1}{2}f(x)$; c) $F(x) = f(x) + 1$; d) $2xF(x) = f(x)$.

Theorem 15.1 (Properties of indefinite integral)**.** *Indefinite integral satisfies the following properties:*

1) $\frac{d}{dx}\int f(x)dx = f(x)$, $x \in J$;

2) $\int f'(x)dx = f(x) + C$, $x \in J$;

3) $\int (af(x))dx = a\int f(x)dx$, $x \in J$, *for all* $a \in \mathbb{R}$, $a \neq 0$;

4) $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$, $x \in J$.

From definitions 15.2 and 15.1 we have that $F'(x) = f(x)$, $x \in J$, provided $\int f(x)dx = F(x) + C$, $x \in J$. Using this relationship, we can get the following list of important indefinite integrals.

- $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$, $x \in (0, +\infty)$, for all $\alpha \in \mathbb{R} \setminus \{-1\}$;
  $\int x^n dx = \frac{x^n}{n+1} + C$, $x \in \mathbb{R}$, for all $n \in \mathbb{N} \cup \{0\}$;

- $\int \frac{1}{x} dx = \ln |x| + C$ on each interval $(-\infty, 0)$ and $(0, +\infty)$;
- $\int a^x dx = \frac{a^x}{\ln a} + C$, $x \in \mathbb{R}$ for all $a > 0$, $a \neq 1$;
- $\int e^x dx = e^x + C$, $x \in \mathbb{R}$;
- $\int \cos x dx = \sin x + C$, $x \in \mathbb{R}$;
- $\int \sin x dx = -\cos x + C$, $x \in \mathbb{R}$;
- $\int \frac{dx}{\cos^2 x} = \tan x + C$ on each interval $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$, $n \in \mathbb{Z}$;
- $\int \frac{dx}{\sin^2 x} = -\cot x + C$ on each interval $(n\pi, \pi + n\pi)$, $n \in \mathbb{Z}$;
- $\int \frac{dx}{1+x^2} = \arctan x + C$, $x \in \mathbb{R}$;
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$, $x \in (-1, 1)$.

15.2 Computation of Indefinite Integrals

An elementary function is the compositions of rational, exponential, trigonometric functions and their inverse functions. A function is called elementary integrable if it has an elementary antiderivative. "Most" functions are not elementary integrable. For example, antiderivatives of $f_1(x) = e^{-x^2}$, $x \in \mathbb{R}$; $f_2(x) = \frac{e^x}{x}$, $x > 0$; $f_3(x) = \frac{\sin x}{x}$, $x > 0$; $f_4(x) = \sin x^2$, $x \in \mathbb{R}$; $f_5(x) = \cos x^2$, $x \in \mathbb{R}$, cannot be expressed as elementary functions.

In the following subsections, we will consider some approaches which allow to compute antiderivatives of some classes of functions.

15.2.1 Substitution rule

Definition 15.3. The **differential** $df(x)$ **of a differentiable function** f is defined by $df(x) = f'(x)dx$.

According to Definition 15.3, we set $\int f(x)d\varphi(x) := \int f(x)\varphi'(x)dx$.

Theorem 15.2. *Let a function $f : J_1 \to \mathbb{R}$ be continuous on J_1, $g : J \to J_1$ be continuously differentiable on J (i.e. g has the continuous derivative on J) and let $\int f(t)dx = F(t) + C$, $t \in J_1$. Then $\int f(g(x))g'(x)dx = \int f(g(x))dg(x) = F(g(x)) + C$, $x \in J$.*

Proof. Indeed, $(F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x)$, $x \in J$, by the chain rule. □

Example 15.3. Compute $\int \sin 5x dx$, $x \in \mathbb{R}$.

Solution. According to Theorem 15.2, we have

$$\int \sin 5x dx = \frac{1}{5} \int \sin 5x d(5x) = |5x = t| = \frac{1}{5} \int \sin t dt = -\frac{1}{5} \cos t + C = -\frac{1}{5} \cos 5x + C, \quad x \in \mathbb{R}.$$

Example 15.4. Compute $\int 2xe^{x^2} dx$, $x \in \mathbb{R}$.

Solution. By Theorem 15.2, we obtain

$$\int 2xe^{x^2} dx = \int e^{x^2} dx^2 = \left|x^2 = t\right| = \int e^t dt = e^t + C = e^{x^2} + C, \quad x \in \mathbb{R}.$$

Exercise 15.3. Compute the following indefinite integrals:
a) $\int \sin^2 x dx$, $x \in \mathbb{R}$; b) $\int \sin 2x \sin 3x dx$, $x \in \mathbb{R}$; c) $\int \sin^3 x dx$, $x \in \mathbb{R}$; d) $\int \frac{dx}{\sin x \cos^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$;
e) $\int x \cos x^2 dx$, $x \in \mathbb{R}$; f) $\int \frac{dx}{1-x}$ on $(-\infty, 1)$ and $(1, +\infty)$.

Theorem 15.3. *Let a function $f : J \to \mathbb{R}$ be continuous, $\varphi : J_0 \to J$ be continuously differentiable on J_0 and let φ have an inverse function φ^{-1}. Let also G be an antiderivative for the function $g(t) = f(\varphi(t))\varphi'(t)$, $t \in J_0$. Then*

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt = G(t) + C = G(\varphi^{-1}(x)) + C, \quad x \in J.$$

Proof. Let F be an antiderivative of f on J. Then according to the chain rule, we have

$$(F(\varphi(t)))' = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi(t) \quad t \in J_0.$$

Thus, there exists a constant C such that $G(t) = F(\varphi(t)) + C$, $t \in J_0$, or $G(\varphi^{-1}(x)) = F(x) + C$, $x \in J$. □

Example 15.5. Compute $\int \sqrt{1 - x^2} dx$, $x \in [-1, 1]$.

Solution. Using Theorem 15.3, we have

$$\int \sqrt{1 - x^2} dx = \left| \begin{array}{c} x = \sin t \\ dx = d \sin t = \cos t dt \end{array} \right| = \int \cos^2 t dt = \int \frac{1 + \cos 2t}{2} dt$$
$$= \frac{1}{2}t + \frac{1}{4}\sin 2t + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1 - x^2} + C.$$

Here, we have used that $t = \arcsin x$ and $\sin 2t = 2 \sin t \cos t = 2 \sin t\sqrt{1 - \cos^2 t} = x\sqrt{1 - x^2}$, for $x = \sin t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Exercise 15.4. Compute the following indefinite integrals:
a) $\int \frac{dx}{\cos x}$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; b) $\int \frac{1}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$; c) $\int \frac{e^{\frac{1}{x}} dx}{x^2}$, $x > 0$; d) $\int \frac{(2x+1)dx}{\sqrt[3]{1+x+x^2}}$, $x \in \mathbb{R}$;
e) $\int \sqrt{1 - 3x} dx$, $x < \frac{1}{3}$; f) $\int \frac{\sqrt{\tan x}}{\cos^2 x} dx$, $x \in \left(0, \frac{\pi}{2}\right)$; g) $\int \frac{dx}{x \ln x}$, $x > 0$; h) $\int \cos^2 x \sin^3 x dx$, $x \in \mathbb{R}$;
i) $\int \frac{dx}{x^2+x+1}$, $x \in \mathbb{R}$; j) $\int \frac{dx}{1-x^2}$ on $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.

15.2.2 Integration by Parts Formula

Theorem 15.4. *Let $u, v : J \to \mathbb{R}$ be differentiable on J and the function uv' has an antiderivative on J. Then the function $u'v$ also has an antiderivative on J and the following equality*

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx, \quad x \in J, \tag{20}$$

holds.

Proof. The function uv is antiderivative of the function $u'v + uv'$ on J, by Theorem 10.3 3). Thus,

$$\int (u'(x)v(x) + u(x)v'(x))dx = u(x)v(x) + C,$$

which implies equality (20). □

Remark 15.2. According to Definition 15.3, the integration by parts formula (20) can be written as follows
$$\int v(x)du(x) = u(x)v(x) - \int u(x)dv(x), \quad x \in J.$$

Example 15.6. Compute $\int x \sin^x dx$, $x \in \mathbb{R}$.

Solution. Using Theorem 15.4 and Remark 15.2, we have
$$\int x \sin x dx = -\int x d\cos x = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C, \quad x \in \mathbb{R}.$$

Example 15.7. Compute $\int \ln x dx$, $x > 0$.

Solution. Using Theorem 15.4 and Remark 15.2, we get
$$\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - \int dx = x\ln x - x + C, \quad x > 0.$$

Exercise 15.5. Compute $\int e^x \sin x dx$, $x \in \mathbb{R}$.

Solution. Applying Theorem 15.4 and Remark 15.2, we obtain
$$\begin{aligned}\int e^x \sin x dx = \int \sin x de^x &= e^x \sin x - \int e^x d\sin x = e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - \int \cos x de^x = e^x \sin x - e^x \cos x + \int e^x d\cos x \\ &= e^x(\sin x - \cos x) - \int e^x \sin x dx \quad x \in \mathbb{R}.\end{aligned}$$

Thus, $\int e^x \sin x dx = \frac{1}{2} e^x(\sin x - \cos x) + C$, $x \in \mathbb{R}$.

Exercise 15.6. Compute the following indefinite integrals:
a) $\int x \sin x dx$, $x \in \mathbb{R}$; b) $\int x^2 \sin x dx$, $x \in \mathbb{R}$; c) $\int (\ln x)^2 dx$, $x > 0$; d) $\int \ln(x^2 + x + 1)dx$, $x \in \mathbb{R}$.

Exercise 15.7. Find a mistake in the following reasoning.

Using Theorem 15.4 and Remark 15.2, we have
$$\int \frac{dx}{x} = x \cdot \frac{1}{x} - \int x d\frac{1}{x} = 1 - \int x \cdot \left(-\frac{1}{x^2}\right) dx = 1 + \int \frac{dx}{x}, \quad x > 0.$$

Thus, $0 = 1$!