# Problem sheet 4

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*Solutions will be collected during the lecture on Monday November 19.*

1. **[1+1+1 points]** Let $f : X \to Y$. Check that
   a) $f(A_1 \cap A_2) \subset (f(A_1) \cap f(A_2))$ for $A_1 \subset X$, $A_2 \subset X$;
   b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$ for $B_1 \subset Y$, $B_2 \subset Y$;
   c) $f(f^{-1}(B)) = B \cap f(X)$ for $B \subset Y$.

2. **[2 points]** Show that the set of all limit points of the set $A = \{r \in [0, 1] : \ r \text{ is rational}\}$ coincides with the interval $[0, 1]$. *(Hint: Use Theorem 2.3)*

3. **[2 point]** Prove that the limit of the function $f(x) = \cos \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, does not exist at the point $a = 0$.

4. **[2+2 point]** Using $\varepsilon - \delta$ definition, show that
   a) $\lim\limits_{x\to 4} \sqrt{x} = 2$; b) $\lim\limits_{x\to +\infty} \frac{\ln x}{x} = 0$.

5. **[2+2+2 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$; b) $\lim\limits_{x\to +\infty} \frac{x^3 - x\sin x + x}{1-3x^3+\ln x}$; c) $\lim\limits_{x\to 1} \frac{x^2-x}{x^2-3x+2}$.

6. **[2 points]** Let $a$ be a limit point of $A \subset \mathbb{R}$ and $f, g : A \to \mathbb{R}$ satisfy the following properties: 1) $f$ is bounded on $A$; 2) $g(x) \to 0$, $x \to a$. Show that $\lim\limits_{x\to a}(f(x) \cdot g(x)) = 0$.
   *(Hint: Use Squeeze theorem for functions)*

7. **[2+2+2 points]** Compute the following limits:
   a) $\lim\limits_{x\to 0-} \frac{x}{\sqrt{1-\cos^2 x}}$; b) $\lim\limits_{x\to 0+} \frac{x}{\sqrt{1-\cos^2 x}}$; c) $\lim\limits_{x\to 0+} e^{-\frac{1}{x}}$.