Problem sheet 14

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Solutions will be collected during the lecture on Wednesday February 6.

Points for solved exercises have to be included as bonus points for the homework

1. [**1 point**] Let V be a vector space over $\mathbb{F}$. Then, given $a \in \mathbb{F}$ and $v \in V$ such that $av = 0$, prove that either $a = 0$ or $v = 0$.

2. [**2x3 points**] Prove or give a counterexample to the following claim:

   1) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.
   2) Let V be a vector space over $\mathbb{F}$ and suppose that W_1, W_2 and W_3 are subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.

3. [**2 points**] Let $\mathbb{F}[z]$ denote the vector space of all polynomials with coefficients in $\mathbb{F}$ and let
$$U = \{az^2 + bz^5 : \ a, b \in \mathbb{F}\}.$$
   Find a subspace W of $\mathbb{F}[z]$ such that $F[z] = U \oplus W$.

4. [**2x2 points**] Consider the complex vector space $V = \mathbb{C}^3$ and the list $\{v_1, v_2, v_3\}$ of vectors in V, where $v_1 = (i, 0, 0)$, $v_2 = (i, 1, 0)$ and $v_3 = (i, i, -1)$.
   a) Prove that $\operatorname{span}\{v_1, v_2, v_3\} = V$.
   b) Prove or disprove that $\{v_1, v_2, v_3\}$ is a basis of V.

5. [**2 points**] Let V be a vector space over $\mathbb{F}$, and suppose that $v_1, v_2, \dots, v_n \in V$ are linearly independent. Let w be a vector from V such that the vectors $v_1 + w, v_2 + w, \dots, v_n + w$ are linearly dependent. Prove that $w \in \operatorname{span}\{v_1, v_2, \dots, v_n\}$.

6. [**3 points**] Let $p_0, p_1, \dots, p_n \in \mathbb{F}_n[z]$ satisfy $p_j(2) = 0$ for all $j = 0, 1, \dots, n$. Prove that $p_0, p_1, \dots, p_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

7. [**3x1 points**] Define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$.
   a) Show that T is linear; b) show that T is surjective; c) find $\dim(\ker T)$.

8. [**3 points**] Show that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ is surjective if
$$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1 = 5x_2, \ x_3 = 7x_4\}.$$

9. [**3 points**] Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional, and let U be any subspace of V. Given a linear map $S \in \mathcal{L}(U, W)$, prove that there exists a linear map $T \in \mathcal{L}(V, W)$ such that, for every $u \in U$, $S(u) = T(u)$.

10. [**3 points**] Let V and W be vector spaces over $\mathbb{F}$ with V finite-dimensional. Given $T \in \mathcal{L}(V, W)$, prove that there is a subspace U of V such that $U \cap \ker T = \{0\}$ and $\operatorname{range} T = \{T(u) : \ u \in U\}$.

11. [**3 points**] Let U, V and W be finite-dimensional vector spaces over $\mathbb{F}$ with $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that
$$\dim(\ker(TS)) \le \dim(\ker T) + \dim(\ker S).$$