Problem sheet 11

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Solutions will be collected during the lecture on Wednesday January 23.

1. [**2x3 points**] Compute the following improper integrals:
   a) $\int_0^{+\infty} \frac{dx}{x^2+x+1}$; b) $\int_0^1 \frac{dx}{\sqrt{1-x}}$; c) $\int_0^{+\infty} x^2 e^{-x} dx$.

2. [**3 points**] Identify all $p \in \mathbb{R}$ for which the improper integral $\int_1^{+\infty} x^p e^{-x} dx$ converges. Justify your answer.

3. [**2x4 points**] Show that the following improper integrals converge:
   a) $\int_1^{+\infty} e^{-x^2} dx$; b) $\int_1^{+\infty} \frac{x-2}{x^3+x+1} dx$; c) $\int_0^{+\infty} \frac{\sin x}{1+x^2} dx$; d) $\int_1^{+\infty} \frac{\cos x}{\sqrt{x}} dx$.

4. [**2 points**] Show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots = 1$.
   (*Hint:* Use the equality $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$)

5. [**3 points**] Identify all $p > 0$ for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges. Justify your answer.

6. [**2x3 points**] Prove the convergence of the following series:
   a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n^2+1}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)^2$.