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LARGE DEVIATIONS PRINCIPLE FOR FINITE SYSTEM OF HEAVY DIFFUSION PARTICLES

The large deviation principle for a system of coalescing heavy diffusion particles is proved. Some asymptotic properties of the distribution of the first moment of meeting of two particles are described.

1. Introduction

The present paper is devoted to the large deviation principle (LDP) for the model of interacting diffusion particles system. Suppose that particles start moving from the finite set of points, move independently up to the moment of meeting, then coalesce and move together. Masses of particles are added together at the moment of coalescing. The random process which corresponds to the describing model is said to be the process of heavy diffusion particles and denoted by $X(t) = (X_1(t), \ldots, X_n(t)),\ t \in [0, 1]$. It was proved in [1, 2] that the distribution of such process is uniquely determined by the following conditions

1°) X_k is a continuous square integrable martingale with respect to the filtration
$$\mathcal{F}_t^X = \sigma(X(s),\ s \le t);$$
2°) $X_k(0) = x_k$, $k = 1, \ldots, n$;
3°) $X_k(t) \le X_l(t)$, $k < l$, $t \in [0, 1]$;
4°) $\langle X_k \rangle_t = \int\limits_0^t \frac{ds}{m_k(s)}$, where $m_k(t) = \{j : \ \exists s \le t\ X_j(s) = X_k(s)\}$, $t \in [0, 1]$;
5°) $\langle X_k, X_l \rangle_t \mathbb{I}_{\{t < \tau_{k,l}\}} = 0$, where $\tau_{k,l} = \inf\{t : \ X_k(t) = X_l(t)\}$.

Here $m_k(t)$ is the mass of the particle k at time t. Condition 4°) means that the particle diffusion is inversely proportional to the mass. Condition 5°) means that every two particles move independently up to the moment of meeting. Note that this model is similar to the Arratia model [3, 4, 5, 6]. In the Arratia model Brownian particles move independently up to the moment of meeting, then coalesce and move together as the Brownian motion. The diffusion coefficient of every particle equals one and is not changing. The infinite system of heavy particles with mass addition at the moment of coalescing was studied in [1, 2, 7]. Also some asymptotic results for such system were obtained. In particular, for the evolution of one particle $x(t),\ t \ge 0$
$$\mathbb{P}\left\{ \lim_{t \to +\infty} \frac{|x(t)|}{\sqrt{2t \ln \ln t}} = 0 \right\} = 1.$$
Since the interaction in our system is not smooth, we can not use the methods and tools appropriate for the smooth flows described in [8, 9]. That is why we use some ideas from [10], where the LDP for the coalescing Brownian particles is obtained. Some our methods are close to [11], where the LDP for the discontinuous function from diffusion process is proved. The article is organized as follows. In Section 2 we prove the LDP for

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the process which corresponds to the model of interacting diffusion particles system. In Section 3 the asymptotic properties of the distribution of the first moment of meeting of two particles are described.

2. Large deviations principle

The main result of this section is the LDP for the process of heavy diffusion particles. For the fixed $n \in \mathbb{N}$ let $[n] = \{1, 2, \dots, n\}$.

Definition 2.1. A set $\pi = \{\pi_1, \dots, \pi_p\}$ of non-intersecting subsets of $[n]$ is said to be the ordered partition of $[n]$ if

1) $\bigcup\limits_{i=1}^{p} \pi_i = [n]$;
2) if $l, k \in \pi_i$ and $l < j < k$, then $j \in \pi_i$ for all $i \in [p]$.

The set of all ordered partitions of $[n]$ is denoted by Π^n.

Every element $\pi = \{\pi_1, \dots, \pi_p\} \in \Pi^n$ generates equivalence between $[n]$ elements. We assume that $i \sim_\pi j$ if there exists a number k such that $i, j \in \pi_k$. Denote by $\widehat{i}_\pi$ an equivalence class containing an element $i \in [n]$, i.e.

$$\widehat{i}_\pi = \{j \in [n] : \ j \sim_\pi i\}.$$

Put

$$i_\pi = \min\{j : \ j \in \widehat{i}_\pi\}.$$

Consider $f = (f_1, \dots, f_n) \in \mathrm{C}\left([0,1]; \mathbb{R}^n\right)$ such that

$$f_1(0) \le \dots \le f_n(0). \tag{1}$$

Suppose that π^0 is an element of Π^n such that

$$i \sim_{\pi^0} j \Leftrightarrow f_i(0) = f_j(0).$$

Let us define the mapping Φ_0 in $C([0;1]; \mathbb{R}^n)$ by the rule $\Phi_0(f) = (g_1^0, \dots, g_n^0)$, where

$$g_k^0(t) = \sum_{i \in \widehat{k}_{\pi^0}} \frac{f_i(t)}{|\widehat{k}_{\pi^0}|}, \quad t \in [0,1], \ k \in [n].$$

If (1) does not hold then one can consider the permutation $\{\sigma_1, \dots, \sigma_n\}$ of the set $[n]$ such that

$$f_{\sigma_1}(0) \le \dots \le f_{\sigma_n}(0)$$

and define

$$\Phi_0(f) = \sigma^{-1} \Phi_0(\sigma f),$$

where

$$\sigma f = (f_{\sigma_1}, \dots, f_{\sigma_n}).$$

It is obvious that Φ_0 is well defined for all $f \in \mathrm{C}\left([0,1]; \mathbb{R}^n\right)$. Set

$$\widehat{\mathrm{C}}[0,1] = \Phi_0\left(\mathrm{C}\left([0,1]; \mathbb{R}^n\right)\right).$$

For some $f \in \widehat{\mathrm{C}}[0,1]$ such that

$$f_1(0) \le \dots \le f_n(0) \tag{2}$$

let us construct $\Phi(f)$ by induction. Denote by

$$\tau^1 = \tau^1(f) = \inf\{t > 0 : \ \exists i, \ f_i(t) = f_{i+1}(t)\} \wedge 1.$$

Consider $\pi^1 = \pi^1(f) \in \Pi^n$ such that

$$i \sim_{\pi^1} j \Leftrightarrow f_i(\tau^1) = f_j(\tau^1).$$

Let

$$g_k^1(t) = \begin{cases} f_k(t), & t \le \tau^1, \\ \sum\limits_{i \in \widehat{k}_{\pi^1}} \frac{f_i(t)}{|\widehat{k}_{\pi^1}|}, & t > \tau^1, \end{cases}$$

$k \in [n]$. Suppose that τ^{p-1}, π^{p-1} and g^{p-1} are defined. Then denote by

$$\tau^p = \tau^p(f) = \inf\{t > \tau^{p-1} : \ \exists i, \ g_i^{p-1}(t) = g_{i+1}^{p-1}(t)\} \wedge 1.$$

For $\pi^p = \pi^p(f) \in \Pi^n$ such that

$$i \sim_{\pi^p} j \Leftrightarrow g_i^{p-1}(\tau^p) = g_j^{p-1}(\tau^p)$$

define

$$g_k^p(t) = \begin{cases} g_k^{p-1}(t), & t \le \tau^p, \\ \sum\limits_{i \in \widehat{k}_{\pi^p}} \frac{f_i(t)}{|\widehat{k}_{\pi^p}|}, & t > \tau^p, \end{cases}$$

$k \in [n]$. Put

$$\Phi(f) = (g_1^{n-1}, \dots, g_n^{n-1}).$$

If (2) does not hold then as in the case of Φ_0 we define Φ by the formula

$$\Phi(f) = \sigma^{-1}\Phi(\sigma f).$$

Denote by

$$\mathrm{E}^n = \{x \in \mathbb{R}^n : \ x_i \le x_{i+1}, \ i \in [n-1]\}.$$

Lemma 2.1. *Let $w(t)$, $t \in [0,1]$, be a Wiener process in $\mathbb{R}^n$ starting from $x \in \mathrm{E}^n$. Then $\Phi \circ \Phi^0(w)$ is the process of heavy diffusion particles.*

The proof of the lemma follows from the construction of the maps Φ_0 and Φ.

Suppose that H is the set of absolutely continuous functions $g \in \mathrm{C}([0,1];\mathbb{R}^n)$ such that $\dot{g}_k \in L_2[0,1]$, $k \in [n]$. For $x \in \mathbb{R}^n$ define the rate function $\widetilde{I}_x$ on $\mathrm{C}([0,1];\mathbb{R}^n)$ as follows

$$\widetilde{I}_x(g) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt, & g(0) = x, \ g \in H \cap \mathrm{C}([0,1];\mathrm{E}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

The main result of this section is the following statement.

Theorem 2.1. *Let $X(t)$, $t \in [0,1]$, be the process of heavy diffusion particles, $X(0) = x \in \mathrm{E}^n$. Then the family $\{X(\varepsilon\cdot), \ \varepsilon > 0\}$ satisfies the LDP in the space $\mathrm{C}([0,1];\mathbb{R}^n)$ with the good rate function $\widetilde{I}_x$.*

Set

$$I_x(g) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt, & g(0) = x, \ g \in H \cap \Phi(\widehat{\mathrm{C}}[0,1]), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that a function f belongs to the set $\Phi(\widehat{\mathrm{C}}[0,1])$ iff for all $k, l \in [n]$, $f_k(s) = f_l(s)$, $s \in [t,1]$, whenever $f_k(t) = f_l(t)$.

It must be mentioned that the function I_x is not lower semicontinuous but the family $\{X(\varepsilon\cdot), \ \varepsilon > 0\}$ satisfies the LDP in the space $\mathrm{C}([0,1];\mathbb{R}^n)$ with the rate function I_x, i.e. for any open set G in $\mathrm{C}([0,1];\mathbb{R}^n)$

$$\varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{X(\varepsilon\cdot) \in G\} \ge -\inf_G I_x \tag{3}$$

and for any closed set F

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{X(\varepsilon\cdot) \in F\} \le -\inf_F I_x. \tag{4}$$

Since $\widetilde{I}_x \le I_x$, then for every closed set F and open set G the following relations hold $\inf\limits_F \widetilde{I}_x \le \inf\limits_F I_x$, $\inf\limits_G \widetilde{I}_x = \inf\limits_G I_x$. That is why to prove the theorem it suffices to check (3) and (4). Since the function Φ is discontinuous, one can not use the contraction principle as in the case of Φ_0. Let us modify the contraction principle for discontinuous functions described in [11] to our case. To prove the theorem, we need the following lemmas.

Lemma 2.2. *Let $w_x(t)$, $t \in [0,1]$, be a Wiener process in $\mathbb{R}^n$ with $w_x(0) = x \in \mathrm{E}^n$. Let $X^0 = \Phi_0(w_x)$. Then the family $\{X^0(\varepsilon\cdot),\ \varepsilon > 0\}$ satisfies the LDP in $\widehat{\mathrm{C}}[0,1]$ with the rate function*

$$I_x^0(g) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt, & g(0) = x,\ g \in H \cap \widehat{\mathrm{C}}[0,1], \\ +\infty & \textit{otherwise}. \end{cases}$$

Proof of Lemma 2.2. For a fixed $x \in \mathrm{E}^n$ denote by

$$\mathrm{C}_x[0,1] = \{f \in \mathrm{C}\left([0,1];\mathbb{R}^n\right) :\ f(0) = x\}.$$

Since $\mathrm{C}_x[0,1]$ is a closed subspace of $\mathrm{C}\left([0,1];\mathbb{R}^n\right)$ and $\{w_x(\varepsilon\cdot),\ \varepsilon > 0\}$ satisfies the LDP in $\mathrm{C}\left([0,1];\mathbb{R}^n\right)$ with the rate function

$$I_x^w(g) = \begin{cases} \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt, & g(0) = x,\ g \in H, \\ +\infty & \text{otherwise}, \end{cases}$$

then by Lemma 4.1.5 [12] the family $\{w_x(\varepsilon\cdot),\ \varepsilon > 0\}$ satisfies the LDP in $\mathrm{C}_x[0,1]$ with the rate function $\widehat{I}_x^w = I_x^w\big|_{\mathrm{C}_x[0,1]}$.

Note that Φ_0 is a continuous function in $\mathrm{C}_x[0,1]$. By Theorem 27.11 [13] the family $\{X^0(\varepsilon\cdot),\ \varepsilon > 0\}$ satisfies the LDP in $\mathrm{C}_x[0,1]$ with the rate function

$$I_x^0(g) = \inf_{\Phi_0^{-1}(g)} \widehat{I}_x^w, \quad g \in \mathrm{C}_x[0,1].$$

It is obvious that $I_x^0(g) = +\infty$ if $g \notin H \cap \Phi_0\left(\mathrm{C}_x[0,1]\right)$. Therefore, let $g \in H \cap \Phi_0\left(\mathrm{C}_x[0,1]\right)$. Take $f \in H \cap \mathrm{C}_x[0,1]$ such that $\Phi_0(f) = g$ and estimate $\widehat{I}_x^w(f)$ as follows

$$\widehat{I}_x^w(f) = \frac{1}{2}\int\limits_0^1 \sum_{l=1}^n \dot{f}_l^2(t)dt = \frac{1}{2}\int\limits_0^1 \sum_{\alpha\in\pi^0}\sum_{i\in\alpha} \dot{f}_i^2(t)dt \ge \frac{1}{2}\int\limits_0^1 \sum_{\alpha\in\pi^0} |\alpha| \left(\sum_{i\in\alpha} \frac{\dot{f}_i^2(t)}{|\alpha|}\right)^2 dt =$$

$$= \frac{1}{2}\int\limits_0^1 \sum_{l=1}^n \left(\sum_{i\in\widehat{l}_{\pi^0}} \frac{\dot{f}_i^2(t)}{|\widehat{l}_{\pi^0}|}\right)^2 dt = \frac{1}{2}\int\limits_0^1 \sum_{l=1}^n \dot{g}_l^2(t)dt = \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt.$$

Since $\Phi_0(g) = g$ and $\widehat{I}_x^w(g) = \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt$, then

$$\inf_{\Phi_0^{-1}(g)} \widehat{I}_x^w = \frac{1}{2}\int\limits_0^1 \|\dot{g}(t)\|^2 dt.$$

Lemma 4.1.5 [12] ends the proof. □

Lemma 2.3. *For any open set G and closed set F in $\mathrm{C}\left([0,1];\mathbb{R}^n\right)$ the following relations hold*

$$\inf_G I_x = \inf_{\Phi^{-1}(G)^\circ} I_x^0, \quad \inf_F I_x = \inf_{\overline{\Phi^{-1}(F)}} I_x^0,$$

where A° and $\overline{A}$ denote an interior and a closure of A in the space $\widehat{\mathrm{C}}[0,1]$.

Proof. For an open set G in $\mathrm{C}([0,1];\mathbb{R}^n)$ let us prove that

$$\inf_G I_x = \inf_{\Phi^{-1}(G)^\circ} I_x^0.$$

First, let us check that $\inf_G I_x \geq \inf_{\Phi^{-1}(G)^\circ} I_x^0$. If $f \in G$ such that $I(f) < +\infty$, then $f \in \widehat{\mathrm{C}}[0,1]$ and $\Phi^{-1}(f) \neq \emptyset$. Moreover, $\Phi(f) = f$. Since the set G is open in $\widehat{\mathrm{C}}[0,1]$ and $f \in G$, then G contains an open ball $B(f,\varepsilon)$ with center f and radius $\varepsilon > 0$.

Let $\gamma_\delta : [0,1] \to [0,\delta]$ be a continuously differentiable function satisfying the properties

1) $\gamma_\delta(0) = 0$;
2) $\gamma_\delta(t) = \delta$, for all $t \in [\delta, 1]$;
3) $|\dot\gamma_\delta(t)| \leq 2$, for all $t \in [0,1]$.

For sufficiently small $0 < \delta < \varepsilon$, $i \not\sim_{\pi^0(f)} j$ and $t \in [0,\delta]$ we have $|f_i(t) - f_j(t)| > \delta$. Put

$$h_i^\rho(t) = \gamma_{\frac{\rho i_{\pi^0}}{n}}(t) + f_i(t), \quad t \in [0,1],\ i \in [n],\ \rho \in (0,\delta].$$

It is clear that the family of functions $h^\rho = (h_1^\rho, \dots, h_n^\rho)$ satisfies the following properties

1) $B\left(h^\rho, \frac{\rho}{2n}\right) \subseteq G$, for all $\rho \in (0,\delta]$;
2) $\Phi(h) = h$, for all $h \in B\left(h^\rho, \frac{\rho}{2n}\right)$ and $\rho \in (0,\delta]$;
3) $h^\rho \to f$ in $\widehat{\mathrm{C}}[0,1]$ as $\rho \to 0$;
4) $I_x(h^\rho) \to I_x(f)$ as $\rho \to 0$.

From properties 1) and 2) we conclude that $B\left(h^\delta, \frac{\delta}{2n}\right) \subseteq \Phi^{-1}(G)$. Hence $\Phi^{-1}(G)^\circ \neq \emptyset$. Since $h^\rho \in \Phi^{-1}(G)^\circ$, $\rho \in (0,\delta]$, then it follows from 3), 4) that

$$I_x(f) \geq \inf_{\Phi^{-1}(G)^\circ} I_x^0.$$

Further let us prove that $\inf_G I_x \leq \inf_{\Phi^{-1}(G)^\circ} I_x^0$. It must be noted that $\inf_G I_x = \inf_{G \cap \Phi(\widehat{\mathrm{C}}[0,1])} I_x$. For $g \in \Phi(\widehat{\mathrm{C}}[0,1]) \cap H$ and $f \in \widehat{\mathrm{C}}[0,1] \cap H$ such that $\Phi(f) = g$ we have

$$I_x^0(f) = \frac{1}{2}\int_0^1 \sum_{l=1}^n \dot f_l^2(t)dt = \frac{1}{2}\sum_{k=1}^n \int_{\tau^{k-1}}^{\tau^k} \sum_{l=1}^n \dot f_l^2(t)dt =$$

$$= \frac{1}{2}\sum_{k=1}^n \int_{\tau^{k-1}}^{\tau^k} \sum_{\alpha \in \pi^k} \sum_{i \in \alpha} \dot f_i^2(t)dt \geq \frac{1}{2}\sum_{k=1}^n \int_{\tau^{k-1}}^{\tau^k} \sum_{\alpha \in \pi^k} |\alpha| \left(\sum_{i \in \alpha} \frac{\dot f_i^2(t)}{|\alpha|}\right)^2 dt =$$

$$= \frac{1}{2}\sum_{k=1}^n \int_{\tau^{k-1}}^{\tau^k} \sum_{l=1}^n \left(\sum_{i \in \widehat{l}_{\pi^k}} \frac{\dot f_i^2(t)}{|\widehat{l}_{\pi^k}|}\right)^2 dt = \frac{1}{2}\int_0^1 \sum_{l=1}^n \dot g_l^2(t)dt = \frac{1}{2}\int_0^1 \|\dot g(t)\|^2 dt.$$

It implies that

$$\inf_{\Phi^{-1}(G)^\circ} I_x^0 \geq \inf_{\Phi^{-1}(G)} I_x^0 \geq \inf_G I_x.$$

For a closed set F in $\mathrm{C}([0,1];\mathbb{R}^n)$ let us prove that

$$\inf_F I_x = \inf_{\overline{\Phi^{-1}(F)}} I_x^0.$$

Suppose that $f \in \overline{\Phi^{-1}(F)}$ such that $I_x^0(f) < \infty$. Let us check that $f \in \Phi^{-1}(F)$. Consider a sequence $\{f_n,\ n \geq 1\} \subseteq \Phi^{-1}(F)$ converging to f. Using the relative compactness of $\{f_n,\ n \geq 1\} \subseteq \Phi^{-1}(F)$, the Arzela-Ascoli theorem and the properties of the map Φ one can check that the sequence $\{\Phi(f_n),\ n \geq 1\}$ contains the subsequence $\{\Phi(f_{n_i}),\ i \geq 1\}$ which converges to an element $g \in F$. Since $\Phi(f_{n_i}) \to g$ and $f_{n_i} \to f$, then one can conclude that $f \in \Phi^{-1}(F)$.

As in the case of the function Φ_0

$$I_x(g) = \inf_{\Phi^{-1}(g)} I_x^0.$$

Therefore,

$$\inf_F I_x = \inf_{\Phi^{-1}(F)} I_x^0 = \inf_{\overline{\Phi^{-1}(F)}} I_x^0.$$

□

Proof of Theorem 2.1. Since $X = \Phi(X^0)$ is the process of heavy diffusion particles, then using Lemma 2.3 and the LDP for the family $\{X^0(\varepsilon\cdot),\ \varepsilon > 0\}$ (see Lemma 2.2) one can conclude that for every open set G in $\mathrm{C}([0,1];\mathbb{R}^n)$ the following relation holds

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X(\varepsilon\cdot) \in G\} = \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^0(\varepsilon\cdot) \in \Phi^{-1}(G)\} \geq$$
$$\geq \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^0(\varepsilon\cdot) \in \Phi^{-1}(G)^\circ\} \geq - \inf_{\Phi^{-1}(G)^\circ} I_x^0 = - \inf_G I_x.$$

Similarly, for every closed set F

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X(\varepsilon\cdot) \in F\} = \varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^0(\varepsilon\cdot) \in \Phi^{-1}(F)\} \leq$$
$$\leq \varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X^0(\varepsilon\cdot) \in \overline{\Phi^{-1}(F)}\} \leq - \inf_{\overline{\Phi^{-1}(F)}} I_x^0 = - \inf_F I_x.$$

□

3. Some asymptotic behaviour of the distribution function of the meeting time of two particles

For fixed $k, l \in [n],\ x \in \mathrm{E}^n$ put $r = \min\{j :\ x_j = x_k \wedge x_l\},\ R = \max\{j :\ x_j = x_k \vee x_l\}$. Let

$$S_x^2 = \frac{1}{R - r + 1} \sum_{j=r}^{R} \left(\sum_{i=r}^{R} \frac{x_i}{R - r + 1} - x_j \right)^2.$$

The main result of this section is the following theorem.

Theorem 3.1. *Let $X(t), t \in [0,1]$, be the process of heavy diffusion particles starting from $x \in \mathrm{E}^n$. Let $\tau_{k,l} = \inf\{t :\ x_k(t) = x_l(t)\} \wedge 1$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{\tau_{k,l} \leq \varepsilon\} = -\frac{A_{k,l}}{2} S_x^2,$$

where $A_{k,l}$ is the number of elements of $\{x_r, \ldots, x_R\}$.

To prove the theorem we need the following statement.

Lemma 3.1. *Let X be the process of heavy diffusion particles starting from $x \in \mathrm{E}^n$. Then for all $t \in [0,1]$ and $k, l \in [n]$,*

$$\mathbb{P}_x\{X_k(t) = X_l(t)\} \geq \mathbb{P}\left\{ |w(t)| > |x_l - x_k| \sqrt{\frac{n}{2}} \right\} \geq C_1 e^{-\frac{C_2 (x_l - x_k)^2}{t}},$$

where w is a Wiener process and C_1, C_2 are positive constants.

To prove the statement we use the ideas of the paper [1].

Proof of Lemma 3.1. Assume that $x_k < x_l$. Denote by

$$\sigma = \inf\{t :\ X_k(t) = X_l(t)\} \wedge 1.$$

Consider

$$y(t) = X_l(t) - x_l - X_k(t) + x_k, \quad t \in [0,1].$$

Let us note that $y(t),\ t \in [0,1]$ is a square integrable martingale with

$$\langle y\rangle_t \geq \frac{2t}{n}, \quad t \in [0,\sigma].$$

It follows from the theorem 2.7.2 [14] that there exists a Wiener process $\widehat{w}$ such that $y(t) = \widehat{w}(\langle y\rangle_t)$. Denote by

$$\widetilde{\sigma} = \inf\left\{t:\ \widehat{w}\left(\frac{2t}{n}\right) = x_l - x_k\right\}.$$

Then

$$x_l - x_k = y(\sigma) = \widehat{w}(\langle y\rangle_\sigma) = \widehat{w}\left(\frac{2\widetilde{\sigma}}{n}\right).$$

Using the monotonicy of $\langle y\rangle_t$, $t \in [0,1]$, we have

$$\frac{2\widetilde{\sigma}}{n} = \langle y\rangle_\sigma \geq \frac{2\sigma}{n}$$

or

$$\widetilde{\sigma} \geq \sigma.$$

Hence

$$\mathbb{P}\{\sigma < t\} \geq \mathbb{P}\{\widetilde{\sigma} < t\} = \mathbb{P}\left\{\max_{s\in[0,t]} \widehat{w}\left(\frac{2s}{n}\right) \geq x_l - x_k\right\} =$$

$$= \mathbb{P}\left\{\max_{s\in[0,t]} \widehat{w}(s) \geq \sqrt{\frac{n}{2}}(x_l - x_k)\right\} = \mathbb{P}\left\{|\widehat{w}(t)| \geq \sqrt{\frac{n}{2}}(x_l - x_k)\right\} \geq C_1 e^{-\frac{C_2(x_l - x_k)^2}{t}}.$$

□

Proof of Theorem 3.1. Suppose that $x_k < x_l$. Denote by

$$F = \{f \in \mathrm{C}([0,1];\mathbb{R}^n):\ \exists t \in [0,1]\ f_k(t) = f_l(t)\}.$$

Note that F is a closed set. To estimate $\inf_F I_x$ let us take $f \in F \cap \Phi(\widehat{\mathrm{C}}[0,1]) \cap H$ such that $f(0) = x$. Then

$$(f_j(1) - x_j)^2 = \left(\int_0^1 \dot{f}_j(t)dt\right)^2 \leq \int_0^1 \dot{f}_j^2(t)dt.$$

Hence

$$\sum_{j=r}^{R}(f_j(1) - x_j)^2 \leq 2I_x(f).$$

Since the function $h(z) = \sum_{j=r}^{R}(z - x_j)^2$, $r \in \mathbb{R}$, reaches its minimum at $z = \sum_{i=r}^{R}\frac{x_i}{R-r+1}$ and $f_j(1) = f_i(1)$, $i,j = r,\ldots,R$, then

$$\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1} - x_j\right)^2 \leq I_x(f).$$

Consequently

$$\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1} - x_j\right)^2 \leq \inf_F I_x.$$

Using the LDP for the process of heavy diffusion particles we have

$$\varlimsup_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{\tau_{k,l}\le\varepsilon\}=\varlimsup_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{X(\varepsilon\cdot)\in F\}\le$$
$$\le-\inf_F I_x\le-\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)^2.$$

For an open set

$$G^\delta=\{f\in \mathrm{C}([0,1];\mathbb{R}^n):\ \exists t\in[0,1]\ |f_l(t)-f_k(t)|<\delta\}$$

let us estimate $\inf_{G^\delta} I_x$. To do that consider $f\in \mathrm{C}([0,1];\mathbb{R}^n)$ of the following form

$$f_j(t)=\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)t+x_j,\quad t\in[0,1],\ j=r,\ldots,R,$$

with

$$f_j(t)=x_j,\quad t\in[0,1],\ j\ne r,\ldots,R.$$

Then for small $\delta>0$, $f\in G^\delta\cap\Phi(\widehat{\mathrm{C}}[0,1])\cap H$ and $f(0)=x$. Hence

$$\inf_{G^\delta} I_x\le I_x(f)=\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)^2.$$

Denote by

$$\tau_\delta=\inf\{t:\ X_l(t)-X_k(t)<\delta\}\wedge 1.$$

Using the LDP for the process of heavy diffusion particles one can get

$$\varliminf_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{\tau_\delta<\varepsilon\}=\varliminf_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{X(\varepsilon\cdot)\in G^\delta\}\ge$$
$$\ge-\inf_{G^\delta} I_x\ge-\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)^2.$$

Using the strong Markov property of the process of heavy diffusion particles [7] and Lemma 3.1 one can conclude that

$$\mathbb{P}\{\tau_{k,l}\le\varepsilon\}\ge\mathbb{P}\{\tau_{k,l}-\tau_\delta<(1-\lambda)\varepsilon,\ \tau_\delta<\lambda\varepsilon\}=$$
$$=\mathbf{E}\left(\mathbb{P}\left\{\tau_{k,l}-\tau_\delta<(1-\lambda)\varepsilon,\ \tau_\delta<\lambda\varepsilon|\mathcal{F}^X_{\tau_\delta}\right\}\right)=$$
$$=\mathbf{E}\left(\mathbb{I}_{\{\tau_\delta<\lambda\varepsilon\}}\mathbb{P}\left\{\tau_{k,l}-\tau_\delta<(1-\lambda)\varepsilon|\mathcal{F}^X_{\tau_\delta}\right\}\right)=$$
$$=\mathbf{E}\left(\mathbb{I}_{\{\tau_\delta<\lambda\varepsilon\}}\mathbb{P}_{X(\tau_\delta)}\left\{X_k((1-\lambda)\varepsilon)=X_l((1-\lambda)\varepsilon)\right\}\right)\ge$$
$$\ge\mathbb{P}\left\{\tau_\delta<\lambda\varepsilon\right\}C_1e^{-\frac{C_2\delta^2}{(1-\lambda)\varepsilon}},\ \lambda,\varepsilon\in(0,1).$$

Therefore,

$$\varliminf_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{\tau_{k,l}\le\varepsilon\}\ge\frac{1}{\lambda}\varliminf_{\varepsilon\to 0}\lambda\varepsilon\ln\mathbb{P}\{\tau_\delta<\lambda\varepsilon\}-\frac{C_2\delta^2}{(1-\lambda)}\ge$$
$$\ge-\frac{1}{2\lambda}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)^2-\frac{C_2\delta^2}{(1-\lambda)}.$$

Passing to the limit as $\delta\to 0$ and $\lambda\to 1$ we obtain

$$\varliminf_{\varepsilon\to 0}\varepsilon\ln\mathbb{P}\{\tau_{k,l}\le\varepsilon\}\ge-\frac{1}{2}\sum_{j=r}^{R}\left(\sum_{i=r}^{R}\frac{x_i}{R-r+1}-x_j\right)^2.$$

□

Corollary 3.1. *If the process of heavy diffusion particles starts from* $(1, \ldots, n)$, *then*

$$\lim_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{\tau_{1,n} \le \varepsilon\} = -\frac{n^3 - n}{24}.$$

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