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SYSTEM OF STICKING DIFFUSION PARTICLES OF VARIABLE MASS

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We construct a mathematical model of an infinite system of diffusion particles with interaction whose masses affect the diffusion coefficient. The particles begin to move from a certain stationary distribution of masses. Their motion is independent up to their meeting. Then the particles become stuck and their masses are added. As a result, the diffusion coefficient varies as a function inversely proportional to the square root of the mass. It is shown that the mass transported by particles is also characterized by a stationary distribution.

In the present paper, we construct a mathematical model for a collection of interacting diffusion particles on a straight line. As the main specific feature of the analyzed system, we can mention the fact the mass of particles affects the diffusion coefficient. As a result of collision, the particles become stuck and their masses are added. In this case, the diffusion coefficient varies as a function inversely proportional to the square root of mass. The following theorem is the main result of the present paper:

Theorem 1. *Let* $\sum_{k\in\mathbb{Z}} a_k \delta_{x_k}$ *be a stationary point measure on* $\mathbb{R}$ *with finitely many atoms on each segment and let* $\mu \neq 0$. *Then there exists a system of processes* $\{x(k, t);\ k\in\mathbb{Z},\ t\geq 0\}$ *satisfying the following conditions:*

(i) $x(k,\cdot) - x_k$ *is a continuous square-integrable local martingale with respect to*

$$(\mathcal{F}_t)_{t\geq 0} = \big(\sigma\left(x(k, s);\ s\leq t,\ k\in\mathbb{Z}\right)\big)_{t\geq 0};$$

(ii) $x(k, 0) = x_k,\ k\in\mathbb{Z}$;

(iii) $x(k, t) \leq x(k+1, t) \quad \forall\, k\in\mathbb{Z} \quad \forall\, t\geq 0$;

(iv) $\langle x(k,\cdot) - x_k\rangle_t = \int_0^t \frac{ds}{m(k, s)} \quad \forall\, t\geq 0$, *where*

$$m(k, t) = \sum_{i\in|A(k,t)|} a_i, \quad A(k, t) = \{j\in\mathbb{Z} : \exists\, s\leq t,\ x(k, s) = x(j, s)\};$$

(v) the consistent characteristic

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$$\langle x(l,\cdot) - x_l,\, x(k,\,\cdot) - x_k\rangle_t \mathbb{I}_{\{t < \tau_{l,k}\}} \;=\; 0,$$

where

$$\tau_{l,k} \;=\; \inf\{t : x(l,t) = x(k,t)\}.$$

Here, μ is the initial distribution of the masses of particles. Condition (i) means that particles participate in the diffusion processes varying according to condition (iv). Namely, as a result of sticking of particles, their masses are added and the diffusion coefficient varies as a function inversely proportional to the square root of the mass. Conditions (ii) and (iii) are responsible for the start and ordering of particles in the process of motion and condition (v) shows that the behavior of particles prior to the collision is independent. Note that conditions (i)–(v) do not contain the condition of sticking of particles after collision. This property is obtained in what follows as a corollary. In the proof of the theorem, we use the martingale methods.

Similar models with interaction have been also studied by the other authors (see, e.g., [1–5]). Thus, the flow constructed in [1] can be interpreted as the description of consistent motions of the Brownian particles on $\mathbb{R}$ originating at each point of the straight line and independent up to their collision as a result of which the particles become stuck and then move together. In other words, the system of processes $\{x(u,\,t);\; u \in \mathbb{R},\; t \ge 0\}$ constructed in the cited work satisfies the following conditions:

(i) $x(u,\,\cdot)$ is a continuous square-integrable local martingale with respect to

$$(\mathcal{F}_t)_{t \ge 0} \;=\; \left(\sigma\left(x(u,\,s);\; s \le t,\; u \in \mathbb{R}\right)\right)_{t \ge 0};$$

(ii) $x(u,\,0) = u,\; u \in \mathbb{R}$;

(iii) $x(u,\,t) \le x(v,\,t) \quad \forall\, u,\, v \in \mathbb{R},\;\; u < v,\; \forall\, t \ge 0$;

(iv) $\langle x(u,\,\cdot)\rangle_t = t \quad \forall\, t \ge 0$;

(v) the consistent characteristic

$$\langle x(u,\cdot),\, x(v,\,\cdot)\rangle_t \mathbb{I}_{\{t < \sigma_{u,v}\}} \;=\; 0,$$

where

$$\sigma_{u,v} \;=\; \inf\{t : x(u,\,t) = x(v,\,t)\}.$$

As follows from condition (iv), the diffusion of particles does not change. The cases finitely and infinitely many particles with masses and velocities are considered in [3]. Their motion obeys the laws of conservation of mass and inertia. The measure-valued equations are presented for the distributions of mass and inertia. The existence of weak solutions of these equations is proved. In the analyzed model, the particles have masses and velocities. At the same time, in the present work, we assume that the velocity is equal to infinity (diffusion case). An empirical distribution of a family of N processes with interaction for fixed t is studied in [5]. It is shown that, under certain conditions, the density of the limit measure obtained as $N \to \infty$ is the solution of a certain evolutionary equation.

As already indicated, we consider the model of diffusion particles whose motion originates from a certain random point measure and is independent up to the collisions of particles. As a result of collisions, the particles become stuck, their masses are added, and they continue their subsequent motion together. The principal problem encountered in the case of countably many starting points is connected with the fact that the motion of each finite subsystem of particles cannot be described without taking into account the influence of the other particles. In the case where the initial distribution of masses is specified in the form of the Lebesgue measure on the straight line, we encounter an additional problem, namely, at the initial time, the particles must begin their motion with infinitely small masses and, hence, with infinitely large diffusion coefficients. To solve this problem, we propose to shift the origin of time by an infinitesimally small period t_0 and assume that, for any segment, all particles originating from this segment have already been stuck to form a finite set of particles for the indicated period of time. In this case, it is necessary to construct the collection of processes for the description of this model on $[t_0, +\infty)$. It is quite natural to assume that, in this case, the motion starts from a random point measure whose probability characteristics are invariant under shifts, i.e., from a stationary measure.

The present paper is organized as follows:

In the first section, we use the martingale methods to construct a collection of processes for the description of the consistent motion of particles originating from a σ-finite deterministic point measure. This motion is independent up to the collisions of particles as a result of which the particles remain stuck, their diffusion coefficient changes, and the particles continue their motion together. We present sufficient conditions that should be imposed on the initial measure to guarantee the existence of this collection.

In the second section, we analyze some properties of stationary point measures because later they are regarded as starting measures. In conclusion, we construct a system of processes for the description of the motion of particles changing their mass as a result of sticking and such that the distribution of masses at the initial time is described by a stationary measure.

1. Deterministic Measure

Theorem 2. *Assume that the sequences of real numbers* $\{x_k;\ k \in \mathbb{Z}\}$ *and* $\{a_k;\ k \in \mathbb{Z}\}$ *satisfies the conditions:*

(1°) $x_k < x_{k+1},\ a_k > 0\quad \forall\, k \in \mathbb{Z}$;

(2°) there exist sequences $\{n_i;\ i \in \mathbb{Z}\}$ *and a constant* $C > 0$ *such that, for any* $i \in \mathbb{Z}$, $a_{n_i+1} \wedge a_{n_i} \geq C,\ x_{n_i+1} - x_{n_i} \geq C,\ n_0 = 0$.

Then there exists a system of random processes $\{x(k, t);\ k \in \mathbb{Z},\ t \geq 0\}$ *such that*

(i) $x(k, \cdot)$ *is a continuous square-integrable local martingale with respect to*

$$(\mathcal{F}_t)_{t \geq 0} = \big(\sigma\left(x(k, s);\ s \leq t,\ u \in \mathbb{Z}\right)\big)_{t \geq 0};$$

(ii) $x(k, 0) = x_k,\ k \in \mathbb{Z}$;

(iii) $x(k, t) \le x(k+1, t) \quad \forall\, k \in \mathbb{Z} \quad \forall\, t \ge 0$;

(iv) $\langle x(k, \cdot)\rangle_t = \int_0^t \frac{ds}{m(k, s)} \quad \forall\, t \ge 0$,

where

$$m(k, t) = \sum_{i \in |A(k,t)|} a_i, \quad A(k,t) = \{ j \in \mathbb{Z} : \exists\, s \le t,\ x(k, s) = x(j, s)\};$$

(v) *the consistent characteristic*

$$\langle x(l, \cdot),\ x(k, \cdot)\rangle_t \mathbb{I}_{\{t < \tau_{l,k}\}} = 0,$$

where

$$\tau_{l,k} = \inf \{ t : x(l, t) = x(k, t)\}.$$

Conditions (i)–(v) uniquely define the distribution $(\ldots, x(-n, \cdot), \ldots, x(n, \cdot), \ldots)$ *in the space* $\left((\mathrm{C}(\mathbb{R}^+))^\infty, \mathcal{B}\left((\mathrm{C}(\mathbb{R}^+))^\infty \right) \right)$.

Proof. The proof of the theorem is split into several parts. First, we construct a finite collection of processes $\{ x_n(k, t);\ k = \overline{-n, n},\ t \in [0, 1]\}$ satisfying conditions (i)–(v). Then we pass to the limit as $n \to +\infty$ and extend the limit transition from $[0, 1]$ onto the entire half line. Finally, we show that conditions (i)–(v) indeed uniquely determine the distribution of the analyzed system.

In a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a collection of independent Wiener processes $\{ w_k(t);\ k \in \mathbb{Z},\ t \ge 0\}$. By using this collection, for any $n \in \mathbb{N}$, we construct a system of processes $\{ x_n(k, t);\ k = \overline{-n, n},\ t \in [0, 1]\}$.

Let $\tau^{(0)} = 0$, let $\mathfrak{S}^{(0)} = \{ \{i\};\ i = \overline{-n, n}\}$, and let

$$w_k^{(0)}(t) = x_k + \frac{1}{\sqrt{a_k}} w_k(t), \quad k = \overline{-n, n}, \quad t \in [0, 1].$$

By induction, we construct a system of processes $\{ w_k^{(p)};\ k = \overline{-n, n}\}$, $p = \overline{1, 2n}$. For $k = \overline{-n, n-1}$, we consider

$$\tau_k^{(p)} = \inf \left\{ t : w_k^{(p-1)}(t) = w_{k+1}^{(p-1)}(t)\right\} \wedge 1$$

and set

$$\tau^{(p)} = \inf\left\{\tau_k^{(p)} > \tau^{(p-1)};\ k = \overline{-n,\, n-1}\right\} \wedge 1 .$$

We consider a class of subsets $\mathfrak{S}^{(p)}$ of the set $\{-n, \ldots, n\}$ with the following properties:

(i) if $l \le i \le k$, then $i \in A \quad \forall A \in \mathfrak{S}^{(p)} \quad \forall k, l \in A$;

(ii) $w_l^{(p-1)}\left(\tau^{(p)}\right) = w_k^{(p-1)}\left(\tau^{(p)}\right) \quad \forall A \in \mathfrak{S}^{(p)} \quad \forall k, l \in A$;

(iii) $w_l^{(p-1)}\left(\tau^{(p)}\right) \ne w_k^{(p-1)}\left(\tau^{(p)}\right) \quad \forall A \in \mathfrak{S}^{(p)} \quad \forall k \in A,\ l \in \{-n, \ldots, n\} \setminus A.$

For $k \in A \in \mathfrak{S}^{(p)}$, we take

$$w_k^{(p)}(t) = \begin{cases} w_k^{(p-1)}(t), & 0 \le t \le \tau^{(p)}, \\ w_j^{(p-1)}(\tau^{(p)})\left(1 - \dfrac{\sqrt{m_1}}{\sqrt{m_2}}\right) + \dfrac{\sqrt{m_1}}{\sqrt{m_2}}\, w_j^{(p-1)}(t), & \tau^{(p)} < t \le 1, \end{cases}$$

where $j \in A$ and

$$|j| = \begin{cases} \min\{|i|;\ i \in A\}, & A \cap \{n_i,\ n_i + 1;\ i \in \mathbb{Z}\} = \varnothing, \\ \min\{|n_i|,\ |n_i + 1|;\ n_i,\ n_i + 1 \in A\}, & A \cap \{n_i,\ n_i + 1;\ i \in \mathbb{Z}\} \ne \varnothing, \end{cases}$$

$m_2 = \sum_{i \in A} a_i$, $m_1 = \sum_{i \in B} a_i$, and $B \in \mathfrak{S}^{(p-1)}$, where $j \in B$.

We set $x_n(k, \cdot) = w_k^{(2n)}$.

Remark 1. By $F_n^{n_j}$ we denote the rule according to which the system $\{w_k;\ k = \overline{-n, n}\}$ is associated with a system $\{x_n(k, \cdot);\ k = \overline{-n, n}\}$. Then $F_n^{n_j}$ is a measurable mapping of the space $(C_n, \mathcal{B}_n)$ into $(C_n, \mathcal{B}_n)$, where

$$C_n = \left\{f \in C\left([0,1],\, \mathbb{R}^{2n+1}\right):\ f(0) = (x_{-n}, \ldots, x_n)\right\} \quad \text{and} \quad \mathcal{B}_n = \mathcal{B}\left(C\left([0,1],\, \mathbb{R}^{2n+1}\right)\right) \cap C_n .$$

We now establish a property of the map $F_n^{n_j}$, $n \in \mathbb{N}$, guaranteeing the existence of the limit of the sequence $\{x_n(k, \cdot)\}_{n \ge k}$, $k \in \mathbb{Z}$, as $n \to +\infty$.

Lemma 1. *Let* $\{f_k;\ k \in \mathbb{Z}\} \subset C[0, 1]$ *and let* $f_k(0) = x_k$. *Denote*

$$\left(g_{-n}^{(n)}, \ldots, g_n^{(n)}\right) = F_n^{n_j}\left(f_{-n}, \ldots, f_n\right), \quad n \in \mathbb{N} .$$

1. If, for some $m \in \mathbb{N}$*, there exist* $C > 0$ *and* $\delta > 0$ *such that*

$$\max\left\{\max_{t\in[0,1]} f_k(t);\ k \in \left\{n_i,\ n_j + 1;\ i = \overline{0,\ m},\ j = \overline{0,\ m-1}\right\}\right\} < C,$$

$$\min_{t\in[0,1]} f_{n_m+1}(t) > C + \delta,$$

then, for any $l > n_m$ *and* $k = \overline{-l,\ n_m}$,

$$\max_{t\in[0,1]} g_k^{(l)}(t) \le \max_{t\in[0,1]} g_{n_m}^{(l)}(t) < C \quad \text{and} \quad \min_{t\in[0,1]} g_{n_m+1}^{(l)}(t) > C + \delta.$$

2. If, for some $-m \in \mathbb{N}$*, there exist* $C < 0$ *and* $\delta < 0$ *such that*

$$\min\left\{\min_{t\in[0,1]} f_k(t);\ k \in \left\{n_i,\ n_j + 1;\ i = \overline{m+1,\ 0},\ j = \overline{m,\ 0}\right\}\right\} > C,$$

$$\max_{t\in[0,1]} f_{n_m}(t) < C + \delta,$$

then, for any $l > -n_m$ *and* $k = \overline{n_m + 1,\ l}$,

$$\min_{t\in[0,1]} g_k^{(l)}(t) \ge \min_{t\in[0,1]} g_{n_m+1}^{(l)}(t) > C \quad \text{and} \quad \max_{t\in[0,1]} g_{n_m}^{(l)}(t) < C + \delta.$$

Proof. The proof of Lemma 1 directly follows from the construction of the maps $F_n^{n_j}$, $n \in \mathbb{N}$.

For all $k \in \mathbb{Z}$, we show that the limit of the sequence $\{x_n(k,\cdot)\}_{n \ge k}$ exists and take this limit as $x(k,\cdot)$. To this end, we formulate the following auxiliary lemma:

Lemma 2. *Let* $\{w_k;\ k \in \mathbb{N}\}$ *be a family of independent standard Wiener processes and let the sequences of real numbers* $\{y_k;\ k \in \mathbb{N}\}$ *and* $\{b_k;\ k \in \mathbb{N}\}$ *be such that*

(i) $y_k < y_{k+1}$ $\forall\, k \in \mathbb{N}$;

(ii) there exists $\delta > 0$ *such that* $b_k \ge \delta$ *and* $y_{k+1} - y_k \ge \delta$ *for any* $k \in \mathbb{N}$.

Denote

$$\xi_k = \max_{t\in[0,1]}\left\{y_k + \frac{1}{\sqrt{b_k}}\, w_k(t)\right\},$$

$$\eta_k = \min_{t \in [0,1]} \left\{ y_k + \frac{1}{\sqrt{b_k}} w_k(t) \right\}.$$

Then, for any $\delta_1 \in \left(0, \frac{\delta}{2}\right)$,

$$\mathbb{P}\left\{ \limsup_{n \to \infty} \left\{ \max_{k = \overline{1, n}} \xi_k \le y_n + \frac{\delta}{2},\ \eta_{n+1} > y_n + \frac{\delta}{2} + \delta_1 \right\} \right\} = 1.$$

Lemmas 1 and 2 imply that, for any $k \in \mathbb{Z}$,

$$\mathbb{P}\{\exists\, N \in \mathbb{N}\ \forall\, n \ge N: x_n(k, \cdot) = x_N(k, \cdot)\} = 1,$$

i.e., for any integer k, the sequence $\{x_n(k, \cdot)\}_{n \ge k}$ is stabilized with probability 1. As $x(k, \cdot)$, we take the limit of $\{x_n(k, \cdot)\}_{n \ge k}$.

Further, it follows from the construction of $x_n(k, \cdot)$ that

$$x_n(k, t) = x_k + \int_0^t \frac{d\tilde{w}_k^{(n)}(s)}{\sqrt{m_n(k, s)}},$$

where $\left\{\tilde{w}_k^{(n)};\ k = \overline{-n, n}\right\}$ is the family of standard Wiener processes such that

$$\left\langle \tilde{w}_l^{(n)}, \tilde{w}_k^{(n)} \right\rangle_t = \mathbb{I}_{\left\{t \ge \tau_{l,k}^{(n)}\right\}} \left(t - \tau_{l,k}^{(n)}\right),$$

$$\tau_{l,k}^{(n)} = \inf\{t : x_n(l, t) = x_n(k, t)\}.$$

As for the sequence $\{x_n(k, \cdot)\}_{n \ge k}$, we arrive at the equality

$$\mathbb{P}\left\{\exists\, N \in \mathbb{N}\ \forall\, n \ge N: \tilde{w}_k^{(n)} = \tilde{w}_k^{(N)}\right\} = 1.$$

As $\tilde{w}$, we take the limit of the sequence $\left\{\tilde{w}_k^{(n)}\right\}_{n \ge k}$.

It is clear that $\{\tilde{w}_k;\ k \in \mathbb{Z}\}$ is the family of standard Wiener processes such that

$$\langle \tilde{w}_l, \tilde{w}_k \rangle_t = \mathbb{I}_{\{t \ge \tau_{l,k}\}} \left(t - \tau_{l,k}\right).$$

Moreover,

$$x(k, t) = x_k + \int_0^t \frac{d\tilde{w}_k(s)}{\sqrt{m(k, s)}} .$$

This means that the system $\{x(k, t); k \in \mathbb{Z}, t \in [0, 1]\}$ satisfies conditions (i)–(v).

By $\{x^T(k, t); k \in \mathbb{Z}, t \in [0, T]\}$ we denote a collection of processes constructed in the same way as the system $\{x(k, t); k \in \mathbb{Z}, t \in [0, 1]\}$. Note that

$$x^{T_1}(k, t) = x^{T_2}(k, t) \quad \forall k \in \mathbb{Z}, \quad t \le T_1 \wedge T_2 .$$

This enables us to conclude that $x(k, \cdot)$ can be extended from $[0, 1]$ onto $\mathbb{R}^+$.

We now prove the second part of Theorem 2. Assume that $\{y(k, t); k \in \mathbb{Z}, t \ge 0\}$ satisfies conditions (i)–(v). Hence, by the Doob theorem [6], there exist a family of Wiener processes $\{w'_k; k \in \mathbb{Z}\}$ such that

$$y(k, t) = x_k + \int_0^t \frac{dw'_k(s)}{\sqrt{m(k, s)}}$$

and

$$\langle w'_l, w'_k \rangle_t = \mathbb{I}_{\{t \ge \tau_{l, k}\}} \left(t - \tau_{l, k}\right).$$

It is clear that

$$\left(x(l, t) - x(k, t)\right) \mathbb{I}_{\{t \ge \tau_{l, t}\}} = 0 .$$

Further, by using the system $\{w'_k; k \in \mathbb{Z}\}$, we can construct a family of standard independent Wiener processes $\{w_k; k \in \mathbb{Z}\}$ such that

$$y(k, \cdot) = \lim_{n \to \infty} y_n(k, \cdot),$$

where

$$\left(y_n(-n, \cdot), \ldots, y_n(n, \cdot)\right) = F_n^{\{n_j\}}\left(w_{-n}, \ldots, w_n\right).$$

The second part of the theorem is proved.

2. Stationary Point Measures

In the present section, we study some properties of stationary point measures. For the detailed analysis of these measures, see, e.g., [7].

Definition 1. *A measure* $\mu = \sum_{k \in I} a_k \delta_{x_k}$, *where* $a_k > 0$, $x_k \in \mathbb{R}$, $x_l \neq x_k$ *for* $l \neq k$, *and* $I \subseteq \mathbb{Z}$, *is called a point measure on* $\mathbb{R}$.

Parallel with μ, we consider the measure $\mu^* = \sum_{k \in I} \delta_{x_k}$.

Let $\mathcal{N}$ be a set of point measures μ on $\mathbb{R}$ such that $\mu^*(\mathcal{B}) < \infty$ for any bounded set $B \in \mathcal{B}(\mathbb{R})$.

Definition 2. *A mapping*

$$\mu : \mathcal{B}(\mathbb{R}) \times \Omega \to \mathbb{R}^+ \cup \{\infty\}$$

is called a stationary point measure μ *on* $\mathbb{R}$ *if it has the following properties:*

(i) for any $B \in \mathcal{B}(\mathbb{R})$, $\mu(B, \cdot)$ *is a random variable;*

(ii) $\mu(\cdot, \omega) \in \mathcal{N} \quad \forall\, \omega \in \Omega$;

(iii) for any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ *and* $h \in \mathbb{R}$,

$$\big(\mu(B_1), \ldots, \mu(B_n)\big) \overset{\mathrm{d}}{=} \big(\mu(B_1 + h), \ldots, \mu(B_n + h)\big)$$

Remark 2. In Definition 2, condition (iii) is equivalent to the condition

(3′) for any disjoint semiopen intervals $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and any $h \in \mathbb{R}$,

$$\big(\mu(B_1), \ldots, \mu(B_n)\big) \overset{\mathrm{d}}{=} \big(\mu(B_1 + h), \ldots, \mu(B_n + h)\big)$$

We now present some examples of stationary measures:

1. Let $\{x(u, \cdot);\, u \in \mathbb{R}\}$ be an Arratia flow. We take $\mu_t(B) = \lambda(B_t)$, where $B_t = \{u : x(u, t) \in B\}$ and λ is the Lebesgue measure on $\mathbb{R}$. Then μ_t is a stationary point measure.

For the proof of the fact that μ_t satisfies conditions (i) and (ii) of Definition 2, see, e.g., [8]. We now check condition (iii).

Let $\{x(u, t);\, u \in \mathbb{R},\, t \geq 0\}$ be an Arratia flow. Then, for any h, $\{y(u, t);\, u \in \mathbb{R},\, t \geq 0\}$ is also an Arratia flow, where $y(u, t) = x(u - h, t) + h$. We take $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and consider

$$\begin{aligned}
\big(\mu_t(B_1), \ldots, \mu_t(B_n)\big) &= \big(\lambda\{u : x(u, t) \in B_1\}, \ldots, \lambda\{u : x(u, t) \in B_n\}\big) \\
&= \big(\lambda\{u : x(u, t) + h \in B_1 + h\}, \ldots, \lambda\{u : x(u, t) + h \in B_n + h\}\big) \\
&= \big(\lambda\{u : y(u + h, t) \in B_1 + h\}, \ldots, \lambda\{u : y(u + h, t) \in B_n + h\}\big)
\end{aligned}$$

$$= \left(\lambda\{u - h : y(u,t) \in B_1 + h\}, \ldots, \lambda\{u - h : y(u,t) \in B_n + h\}\right)$$

$$= \left(\lambda\{u : y(u,t) \in B_1 + h\}, \ldots, \lambda\{u : y(u,t) \in B_n + h\}\right)$$

$$\overset{\mathrm{d}}{=} \left(\mu_t(B_1 + h), \ldots, \mu_t(B_n + h)\right).$$

2. Let μ be a measure satisfying the conditions:

(i) $\mu(B)$ is a Poisson random variable with intensity $\lambda(B)$;

(ii) the random variables $\mu(B_1), \ldots, \mu(B_n)$ are jointly independent for any disjoint system of sets $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$.

Then μ is a stationary measure. (The existence of a measure with the indicated properties is established, e.g., in [9].)

Further, we prove the following lemma according to which there exists a collection of particles of variable mass with sticking such that the initial distribution of masses of these particles is a stationary measure:

Lemma 3. *Let* $\mu = \sum_{k \in I} a_k \delta_{x_k}$ *be a stationary point measure such that* $\mu \neq 0$ *almost surely and, for any* $l, k \in I$, *the facts that* $l < k$ *and* $l \le i \le k$ *imply that* $x_l < x_k$ *and* $i \in I$. *Then, with probability 1,* $I = \mathbb{Z}$ *and the sequences* $\{x_k;\ k \in \mathbb{Z}\}$ *and* $\{a_k;\ k \in \mathbb{Z}\}$ *satisfy conditions (1°)* and *(2°) of Theorem 2.*

Proof. Let

$$C_n^{(m)} = [mn,\ m(n+1)).$$

We consider

$$X_N^{(m)}(n) = \mu\left(C_n^{(m)}\right) \prod_{k : x_k \in C_n^{(m)}} \mathbb{I}_{\left\{a_k \ge \frac{1}{N}\right\}} \mathbb{I}_{\left\{\mu^*\left(C_n^{(m)}\right) > 1\right\}}.$$

The quantity $X_N^{(m)}(n)$ admits a representation

$$X_N^{(m)}(n) = \lim_{k \to \infty} \mathbb{I}_{\left\{\sum_{G \in \mathcal{A}_k\left(C_n^{(m)}\right)} \mathbb{I}_{\{\mu(G) > 0\}} > 1\right\}} \left(\sum_{G \in \mathcal{A}_k\left(C_n^{(m)}\right)} \mu(G) \right) \prod_{G \in \mathcal{A}_k\left(C_n^{(m)}\right)} \left(\mathbb{I}_{\left\{\mu(G) \ge \frac{1}{N}\right\}} + \mathbb{I}_{\{\mu(G) = 0\}} \right),$$

where $\mathcal{A}_k([a,b))$ is a finite imbedded division of $[a,b)$ by half intervals open form the right and such that

$$\max_{A \in \mathcal{A}_k([a,b))} \operatorname{diam} A \to 0 \quad \text{as} \quad k \to +\infty.$$

It is easy to see that $X_N^{(m)}$ is a stationary process in the restricted sense. From $X_N^{(m)}$, we form a new stationary process

$$Y_N^{(m)}(n) = X_N^{(m)}(n)\,\mathbb{I}_{\left\{X_N^{(m)}(n) \le N\right\}}.$$

Further, since $\mathbf{E}Y_N^{(m)}(n) < \infty$, by the Birkhoff–Khinchin theorem [10], we get

$$\lim_{K-M\to\infty} \frac{1}{K-M} \sum_{n=M+1}^{K} Y_N^{(m)}(n) = \mathbf{E}\left(Y_N^{(m)}(n) \,\middle|\, \mathcal{S}_{Y_N^{(m)}}\right),$$

where

$$\mathcal{S}_{Y_N^{(m)}} = \left(Y_N^{(m)}\right)^{-1}(\mathcal{S}), \quad \mathcal{S} = \left\{B \in \mathcal{B}(\mathbb{R}^\infty) : T^{-1}(B) = B\right\},$$

and T is the operator of shift in $\mathbb{R}^\infty$.

We now denote $\xi_N^{(m)} = \mathbf{E}\left(Y_N^{(m)}(n) \,\middle|\, \mathcal{S}_{Y_N^{(m)}}\right)$ and consider $B_N^{(m)} = \left\{\xi_N^{(m)} = 0\right\}$ and $A_N^{(m)} = \Omega \backslash B_N^{(m)}$.

Note that, on $A_N^{(m)}$, there exists a sequence of integers $\left\{n_j;\ j \in \mathbb{Z}\right\}$ such that

$$N \ge Y_N^{(m)}(n_j) \ge \frac{1}{N}$$

Thus, it follows from the structure of $Y_N^{(m)}$ that conditions *(1°)* and *(2°)* of Theorem 2 are satisfied on $A_N^{(m)}$.

We now show that

$$\mathbb{P}\left\{\bigcup_{m,\,N=1}^{\infty} A_N^{(m)}\right\} = 1,$$

which is equivalent to the fact that

$$\mathbb{P}\left\{\bigcap_{m,\,N=1}^{\infty} B_N^{(m)}\right\} = 0.$$

Assume that this is not true and consider

$$\int_{B_N^{(m)}} Y_N^{(m)}(n)\,\mathbb{P}(d\omega) = \int_{B_N^{(m)}} \xi_N^{(m)}(n)\,\mathbb{P}(d\omega) = 0.$$

Hence, $Y_N^{(n)}(n) = 0$ on $B_N^{(m)}$ and, therefore, $Y_N^{(m)} = 0$ for any $n, m,$ and N on $\bigcap_{m, N=1}^{\infty} B_N^{(m)}$. However, this means that, either on $\mathbb{R}^+$ or on $\mathbb{R}^-$, there exists exactly one point at which the measure μ is concentrated $\left(\text{on } \bigcap_{m, N=1}^{\infty} B_N^{(m)}\right)$..

We set

$$Y(n) = \mu^*\big([n, n+1)\big), \quad n \in \mathbb{Z}.$$

According to the remark presented above, one can find $\alpha > 0$ and $n \in \mathbb{Z}$ such that

$$\alpha = \mathbb{P}\{Y(n) = 1, Y(m) = 0, n \neq m\}.$$

Since μ is a stationary measure, μ^* is also stationary and, hence, Y is a stationary process in the restricted sense. This yields

$$\begin{aligned}
\alpha &= \mathbb{P}\{Y(n) = 1, Y(m) = 0, n \neq m\} \\
&= \lim_{m\to\infty} \mathbb{P}\{Y(-m) = 0, \ldots, Y(n) = 1, \ldots, Y(m) = 0\} \\
&= \lim_{m\to\infty} \mathbb{P}\{Y(-m+1) = 0, \ldots, Y(n+1) = 1, \ldots, Y(m+1) = 0\} \\
&= \mathbb{P}\{Y(n+1) = 1, Y(m+1) = 0, n \neq m\},
\end{aligned}$$

which is impossible. This contradiction proves Lemma 3.

3. Stationary Measure

Proof of Theorem 1. Let μ be a stationary point measure given in a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. In a different probability space $(\Omega'', \mathcal{F}'', \mathbb{P}'')$, we consider a countable collection of independent Wiener processes $\{w_k;\, k \in \mathbb{Z}\}$. For any fixed $\omega' \in \Omega'$, by virtue of Lemma 3, the sequences $\{x_k(\omega');\, k \in \mathbb{Z}\}$ and $\{\, a_k(\omega');\, k \in \mathbb{Z}\,\}$ satisfy conditions (1°) and (2°) of Theorem 2. By Theorem 2, there exists a collection of processes with sticking and the initial distribution $\mu(\omega') = \sum_{k\in\mathbb{Z}} a(\omega')\,\delta_{x_k(\omega')}$. We denote this collection by $X = \{x(k, t, \omega', \omega'');\, k \in \mathbb{Z},\, t \geq 0\}$. It is necessary to show that X is the required system of processes.

As in Sec. 1, one can show that $x(k, \cdot)$ is a random process for any $k \in \mathbb{Z}$. It is necessary check that $x(k, \cdot) - x_k$ is a square integrable local martingale.

Let

$$\sigma_n^{(k)} = \inf\{t : | x(k, t) - x_k | \geq n\} \wedge n.$$

Since $x(k,\cdot) - x_k$ is a continuous process consistent with

$$(\mathcal{F}_t) = \big(\sigma(a_k, x_k, w_k(s); s \le t, k \in \mathbb{Z})\big),$$

$\sigma_n^{(k)}$ is a Markov moment with respect to $(\mathcal{F}_t)_{t \ge 0}$. By the theorem on transformation of the free choice [11], for any $\omega' \in \Omega'$, $x\left(k, \cdot \wedge \sigma_n^{(k)}, \omega'\right) - x_k(\omega')$ is a square integrable martingale with the following characteristic:

$$\langle x(k,\cdot,\omega') - x_k(\omega')\rangle_t = \int_0^{t \wedge \sigma_n^{(k)}(\omega')} \frac{ds}{m(k, s, \omega')}.$$

We now show that $x\left(k, \cdot \wedge \sigma_n^{(k)}\right) - x_k$ is a square integrable martingale on $\Omega' \times \Omega''$. To do this, it suffices to check that

$$\int_{A' \times A''} \left(x\left(k, t \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}' \otimes \mathbb{P}'' = \int_{A' \times A''} \left(x\left(k, s \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}' \otimes \mathbb{P}'',$$

where $s \le t$, $A' \in \sigma\{a_k, x_k; k \in \mathbb{Z}\}$, and $A'' \in \sigma\{w_k(r); r \le s\}$.

Thus,

$$\begin{aligned}\int_{A' \times A''} \left(x\left(k, t \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}' \otimes \mathbb{P}'' &= \int_{A'} d\mathbb{P}' \int_{A''} \left(x\left(k, t \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}'' \\ &= \int_{A'} d\mathbb{P}' \int_{A''} \left(x\left(k, s \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}'' \\ &= \int_{A' \times A''} \left(x\left(k, s \wedge \sigma_n^{(k)}\right) - x_k\right) d\mathbb{P}' \otimes \mathbb{P}''.\end{aligned}$$

Further, we consider

$$M_n(k, t) = \left(x\left(k, t \wedge \sigma_n^{(k)}\right) - x_k\right)^2 - \int_0^{t \wedge \sigma_n^{(k)}} \frac{ds}{m(k, s)}.$$

By using the same reasoning for M_n, we conclude that M_n is a square integrable martingale. By the Doob–Meyer theorem [6], this yields

$$\langle x(k,\cdot) - x_k\rangle_t = \int_0^{t \wedge \sigma_n^{(k)}} \frac{ds}{m(k, s)}.$$

It remains to show that

$$\left\langle x(l,\cdot) - x_l,\ x(k,\ \cdot) - x_k \right\rangle_t \mathbb{I}_{\{t < \tau_{l,k}\}} = 0.$$

Denote

$$\tau_n = \tau_{l,k} \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}.$$

We take

$$M_n(t) = \big(x(l,\ t \wedge \tau_n) - x_l\big)\big(x(k,\ t \wedge \tau_n) - x_k\big).$$

It is easy to see that M_n is a square integrable martingale. By the Doob–Meyer theorem,

$$\Big(x\Big(l,\ t \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_l\Big)\Big(x\Big(k,\ t \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_k\Big)$$

$$= \widetilde{M}_n(t) + \Big\langle x\Big(l,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_l,\ x\Big(k,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_k \Big\rangle_t.$$

In the last equality, we replace t by $t \wedge \tau_{l,k}$ and obtain

$$\big(x\big(l,\ t \wedge \tau_n\big) - x_l\big)\big(x\big(k,\ t \wedge \tau_n\big) - x_k\big)$$

$$= \widetilde{M}_n(t \wedge \tau_n) + \Big\langle x\Big(l,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_l,\ x\Big(k,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_k \Big\rangle_{t \wedge \tau_n}.$$

By the theorem on transformation of the free choice, $\widetilde{M}_n(\cdot \wedge \tau_n)$ is a martingale and, hence, $M_n = \widetilde{M}_n(\cdot \wedge \tau_n)$. This yields

$$\Big\langle x\Big(l,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_l,\ x\Big(k,\cdot \wedge \sigma_n^{(l)} \wedge \sigma_n^{(k)}\Big) - x_k \Big\rangle_{t \wedge \tau_n} = 0.$$

Therefore,

$$\left\langle x(l,\cdot) - x_l,\ x(k,\cdot) - x_k \right\rangle_{t \wedge \tau_n} = 0.$$

The theorem is proved.

Remark 3. By the second part of Theorem 2, the distribution of the constructed family of processes $(\ \ldots,\ x(-n,\cdot),\ \ldots,\ x(n,\cdot), \ldots\ )$ is independent of the choice of the system of Wiener processes $\{w_k;\ k \in \mathbb{Z}\}$.

We prove that the distribution of the processes $(\ \ldots,\ x(-n,\cdot),\ \ldots,\ x(n,\cdot), \ldots\ )$ depends solely on the distribution of the stationary measure μ.

Assume that the measures μ and ν satisfy the conditions of Theorem 1 and are such that $\mu \overset{d}{=} \nu$, i.e., for any collection of Borel sets $B_1, \ldots, B_n$, we have

$$\big(\mu(B_1), \ldots, \mu(B_n)\big) \overset{d}{=} \big(\nu(B_1), \ldots, \nu(B_n)\big).$$

As in the proof of Theorem 1, we construct the systems of processes $\{x(k, t);\ k \in \mathbb{Z},\ t \geq 0\}$ and $\{y(k, t);\ k \in \mathbb{Z},\ t \geq 0\}$ with initial distributions μ and ν, respectively. The following theorem is true:

Theorem 3. *The random elements* $\big(\ldots, x(-n,\cdot), \ldots, x(n,\cdot), \ldots\big)$ *and* $\big(\ldots, y(-n,\cdot), \ldots, y(n,\cdot), \ldots\big)$ *are identically distributed.*

Proof. We use the same notation as in the proof of Theorem 1. It suffices to show [12] that, for almost all fixed ω'', the distributions

$$\big(\ldots, x(-n,\cdot, \omega''), \ldots, x(n,\cdot, \omega''), \ldots\big) \quad \text{and} \quad \big(\ldots, y(-n,\cdot, \omega''), \ldots, y(n,\cdot, \omega''), \ldots\big)$$

coincide. To this end, it suffices to establish the existence of sequences of the collections of processes

$$\Big\{x_{m_j}(k, t);\ k \in \overline{-m_j, m_j},\ t \in [0, T]\Big\}_{j \geq 0} \quad \text{and} \quad \Big\{y_{m_j}(k, t);\ k \in \overline{-m_j, m_j},\ t \in [0, T]\Big\}_{j \geq 0}$$

satisfying the following conditions (for fixed ω''):

(i) $\big(x_{m_j}(-m_j,\cdot), \ldots, x_{m_j}(m_j,\cdot)\big)$ and $\big(y_{m_j}(-m_j,\cdot), \ldots, y_{m_j}(m_j,\cdot)\big)$ are identically distributed in the space $\Big(\big(C([0, T])\big)^{m_j},\ \mathcal{B}\Big(\big(C([0, T])\big)^{m_j}\Big)\Big)$;

(ii) for any $k_1, \ldots, k_m \in \mathbb{Z}$,

$$\big(x_{m_j}(k_1,\cdot), \ldots, x_{m_j}(k_m,\cdot)\big) \to \big(x(k_1,\cdot), \ldots, x(k_m,\cdot)\big),$$

$$\big(y_{m_j}(k_1,\cdot), \ldots, y_{m_j}(k_m,\cdot)\big) \to \big(y(k_1,\cdot), \ldots, y(k_m,\cdot)\big) \quad \text{as} \quad j \to \infty$$

in the uniform topology of the space $\big(C([0, T])\big)^m$.

For any $n \in \mathbb{N}$, we construct a collection of processes $\Big\{x_{n2^n}(k, t);\ k = \overline{-n2^n, n2^n},\ t \in [0, T]\Big\}$ from the system of continuous functions $\{w_k(\omega'');\ k \in \mathbb{Z}\}$. Here, $\{w_k;\ k \in \mathbb{Z}\}$ is a family of independent standard Wiener processes used to construct $\{x(k, t);\ k \in \mathbb{Z},\ t \geq 0\}$ and $\{y(k, t);\ k \in \mathbb{Z},\ t \geq 0\}$. We introduce the notation

$$\Delta_k^n = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right),$$

$$\tilde{a}_k^{(n)} = \mu\left(\Delta_k^n\right),$$

$$\tilde{x}_k^{(n)} = \frac{k}{2^n}, \quad k = \overline{-n2^n, n2^n}.$$

Further, from the sets

$$\left\{\tilde{a}_k^{(n)}; k = \overline{-n2^n, n2^n}\right\} \quad \text{and} \quad \left\{\tilde{x}_k^{(n)}; k = \overline{-n2^n, n2^n}\right\},$$

we extract the numbers k for which $\tilde{a}_k^{(n)} = 0$. Moreover, if the number of elements remaining after this procedure is even, then we additionally extract the maximum number. Redenoting the remaining elements, we obtain the sequences $\left\{a_k^{(n)}; k = \overline{l, l}\right\}$ and $\left\{x_k^{(n)}; k = \overline{-l, l}\right\}$. For the sake of convenience, we set

$$a_k^{(n)} = 1, \quad k \in \left\{-n2^n, \ldots, n2^n\right\} \setminus \{-l, \ldots, l\},$$

$$x_i^{(n)} = x_{i-1}^{(n)} + 1, \quad i = \overline{l+1, n2^n},$$

$$x_j^{(n)} = x_{j+1}^{(n)} - 1, \quad j = \overline{-n2^n, -l-1}.$$

As in the proof of Theorem 2, from the sequences $\left\{a_k^{(n)}; k = \overline{-n2^n, n2^n}\right\}$, $\left\{x_k^{(n)}; k = \overline{-n2^n, n2^n}\right\}$ and $\left\{w_k(\omega''); k \in \mathbb{Z}\right\}$, we construct $\left\{x_{n2^n}(k, t); k = \overline{-n2^n, n2^n}, t \in [0, T]\right\}$, i.e., we set

$$\left(x_{n2^n}\left(-n2^n, \cdot\right), \ldots, x_{n2^n}\left(n2^n, \cdot\right)\right) = F_{n2^n}^{n_j}\left(w_{n2^n}(\omega''), \ldots, w_{n2^n}(\omega'')\right),$$

where $F_{n2^n}^{n_j}$ is the map constructed in Theorem 2.

Similarly, we construct a collection of processes $\left\{y_{n2^n}(k, t); k = \overline{-n2^n, n2^n}, t \in [0, T]\right\}$.

Since

$$\left(\mu\left(\Delta_{-n}^n\right), \ldots, \mu\left(\Delta_n^n\right)\right) \stackrel{\mathrm{d}}{=} \left(\nu\left(\Delta_{-n}^n\right), \ldots, \nu\left(\Delta_n^n\right)\right) \quad \forall n \in \mathbb{N},$$

condition (i) is satisfied.

Note that, by virtue of Lemmas 2 and 3, for sufficiently large numbers n, one can find N such that $\left(x_{n2^n}(k_1,\cdot),\ldots,x_{n2^n}(k_m,\cdot)\right)$ depends only on the collections $\left\{w_k(\omega''); k=\overline{-N,N}\right\}$, $\left\{a_k^{(n)}; k=\overline{-N,N}\right\}$, and $\left\{x_k^{(n)}; k=\overline{-N,N}\right\}$. Further, since the measure μ has finitely many atoms on each segment and Wiener paths were used for the construction of $\left\{x_{n2^n}(k,t); k=\overline{-n2^n,n2^n},\ t\in[0,T]\right\}$, the sequence $\left(x_{n2^n}(k_1,\cdot),\ldots,x_{n2^n}(k_m,\cdot)\right)$ converges to $\left(x(k_1,\cdot),\ldots,x(k_m,\cdot)\right)$ as $n\to\infty$.

The theorem is proved.

Corollary 1. *For any* $t\ge 0$, *the measure* $\mu_t=\sum_{k\in\mathbb{Z}}a_k\delta_{x(k,t)}$ *is stationary.*

It follows from the corollary that the probability distribution of the measure μ_t is independent of the shift by an arbitrary number $h\in\mathbb{R}$, i.e., $\mu_t\overset{\mathrm{d}}{=}\mu_t(\cdot+h)$.

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