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STICKY-REFLECTED STOCHASTIC HEAT EQUATION DRIVEN BY COLORED NOISE

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We prove the existence of a sticky-reflected solution to the heat equation on the space interval $[0,1]$ driven by colored noise. The process can be interpreted as an infinite-dimensional analog of the sticky-reflected Brownian motion on the real line but, in this case, the solution obeys the ordinary stochastic heat equation, except the points where it reaches zero. The solution has no noise at zero and a drift pushes it to stay positive. The proof is based on a new approach that can also be applied to some other types of SPDEs with discontinuous coefficients.

1. Introduction

In the present paper, we study the existence of a continuous function $X\colon\ [0,1]\times[0,\infty)\to[0,\infty)$ that is a weak solution to the SPDE

$$\frac{\partial X_t}{\partial t}=\frac{1}{2}\frac{\partial^2 X_t}{\partial u^2}+\lambda\mathbb{I}_{\{X_t=0\}}+f(X_t)+\mathbb{I}_{\{X_t>0\}}Q\dot{W}_t \tag{1.1}$$

with Neumann

$$X_t'(0)=X_t'(1)=0,\quad t\ge 0, \tag{1.2}$$

or Dirichlet

$$X_t(0)=X_t(1)=0,\quad t\ge 0, \tag{1.3}$$

boundary conditions and the initial condition

$$X_0(u)=g(u),\quad u\in[0,1], \tag{1.4}$$

where $\dot{W}$ is a space-time white noise, the functions $g\in \mathrm{C}[0,1]$ and $\lambda\in \mathrm{L}_2:=\mathrm{L}_2[0,1]$ are nonnegative, f is a continuous function from $[0,\infty)$ to $[0,\infty)$, which has a linear growth and $f(0)=0$, and Q is a nonnegative definite self-adjoint Hilbert–Schmidt operator on L_2. We also assume that, in the case of Dirichlet boundary conditions,

$$g(0)=g(1)=0.$$

The analyzed equation appears as a sticky-reflected counterpart of the reflected SPDE introduced in [14, 22]. We assume that a solution of the stochastic heat equation is strictly positive but reaching zero; moreover, its diffusion vanishes and an additional drift at zero pushes the process to be positive. The form of equation (1.1) is similar

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to the form of the SDE for a sticky-reflected Brownian motion on the real line

$$dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} \sigma dw(t), \tag{1.5}$$

and we expect that the local behavior of X at zero is very similar to the behavior of the sticky-reflected Brownian motion x. Note that the SDE (1.5) admits only a weak unique solution due to its discontinuous coefficients, see, e.g., [4, 7, 19]. The approaches applicable to sticky processes in finite-dimensional spaces cannot be used for the solution of the SPDE (1.1). Thus, Engelbert and Peskir [7] showed that equation (1.5) admits a weak (unique) solution. Their approach was based on the time change for a reflected Brownian motion. This method is very restrictive and can be applied only for the sticky dynamics in a one-dimensional space of states. An equation for sticky-reflected dynamics in higher (finite) dimensions was considered by Grothaus with coauthors in [8, 12, 13], where they used the Dirichlet-form approach [10, 21]. This approach was based on the *a priori* knowledge of the invariant measure. Since, in our case, the space is infinite-dimensional, the problem of finding of the invariant measure seems to be very complicated (see, e.g., [9, 24] for the form of invariant measure for the reflected stochastic heat equation driven by the white noise).

In the present paper, we propose a new method aimed at proving the existence of weak solutions to equations that describe sticky-reflected behaviors. This approach is a modification of the method proposed by the author in [18]. It is based on the property of quadratic variation of semimartingales.

The paper leaves a couple of important open problems. The first problem is the uniqueness of solution to the SPDE (1.1)–(1.4). Similarly to the one-dimensional SDE for sticky-reflected Brownian motion (1.5), where strong uniqueness fails [4, 7], we do not expect the existence of strong uniqueness for the SPDE considered in what follows. However, we believe that weak uniqueness takes place.

Another interesting question is the existence of solutions to a similar sticky-reflected heat equation driven by the white-noise. It seems likely that the method proposed in the present work can be adapted to the case of SPDE of this kind. For this purpose, we need a statement similar to Theorem 3.1, which remains an open problem.

1.1. Definition of the Solution and Main Result. For the sake of convenience of notation, we introduce a parameter α_0 equal to 1 in the case of Neumann boundary conditions (1.2) and to 0 in the case of Dirichlet boundary conditions (1.3). For $k \geq 1$, we also introduce the space $\mathrm{C}^k[0,1]$ of k-times continuously differentiable functions on $(0,1)$, which (together with their derivatives up to the order k) can be extended to continuous functions on $[0,1]$. We write $\varphi \in \mathrm{C}^k_{\alpha_0}[0,1]$ if, in addition,

$$\varphi^{(\alpha_0)}(0) = \varphi^{(\alpha_0)}(1) = 0,$$

where $\varphi^{(0)} = \varphi$ and $\varphi^{(1)} = \varphi'$.

Denote the inner product in the space L_2 by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\| \cdot \|$. We now present a definition of weak solution to the SPDE (1.1).

Definition 1.1. *We say that a continuous function* $X\colon [0,1] \times [0,\infty) \to [0,\infty)$ *is a (weak) solution to the SPDE (1.1)–(1.4) if* $X_0 = g$ *and, for every* $\varphi \in \mathrm{C}^2_{\alpha_0}[0,1]$, *the process*

$$\mathcal{M}_t^{\varphi} = \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \frac{1}{2} \int_0^t \langle X_s, \varphi'' \rangle ds$$

$$- \int_0^t \langle \lambda \mathbb{I}_{\{X_s=0\}}, \varphi \rangle ds - \int_0^t \langle f(X_s), \varphi \rangle ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)$*-martingale with quadratic variation*

$$[M^{\varphi}]_t = \int\limits_0^t \left\| Q(\mathbb{I}_{\{X_s>0\}}\varphi) \right\|^2 ds, \quad t \geq 0.$$

In what follows, by $\{e_k,\ k \geq 1\}$ we denote the basis in L_2 formed by the eigenvectors of the nonnegative definite self-adjoint operator Q. Let $\{\mu_k,\ k \geq 1\}$ be the corresponding family of eigenvalues of Q. Note that

$$\sum_{k=1}^{\infty} \mu_k^2 < \infty,$$

since Q is a Hilbert–Schmidt operator. We introduce a function

$$\chi^2 := \sum_{k=1}^{\infty} \mu_k^2 e_k^2, \tag{1.6}$$

where the series trivially converges in $\mathrm{L}^1[0,1]$ and a.e. The main result of the present paper is the following theorem:

Theorem 1.1 (existence of solutions)**.** *If*

$$\lambda \mathbb{I}_{\{\chi>0\}} = \lambda \quad \textit{a.e.}, \tag{1.7}$$

then the SPDE (1.1)–(1.4) admits a weak solution.

Remark 1.1. Condition (1.7) means that the drift λ must be equal to zero for those u at which the noise vanishes.

Remark 1.2. The equation may have a solution even if condition (1.7) does not hold. The reason is that the existence can be violated if $X_t(u) = 0$ for $u \in [0,1]$ such that $\lambda(u) > 0$ and $\chi(u) = 0$ due to the term $\lambda \mathbb{I}_{\{X_t=0\}}$ and the absence of noise for these u. However, if the initial condition is strictly positive for these u, then the solution can always stay strictly positive for these u by virtue of the comparison principle for the classical heat equation. Therefore, the solution exists. We take, e.g, $Q = 0$ and $f = 0$. Then a weak solution to the heat equation

$$\frac{\partial X_t}{\partial t} = \frac{1}{2} \frac{\partial^2 X_t}{\partial u^2}$$

satisfying the corresponding boundary and initial conditions is a solution to the SPDE (1.1)–(1.4) if $X_t(u) > 0$, $t > 0$, and $u \in (0,1)$. However, the strong positivity of X is attained, e.g., under the assumption of strong positivity of the initial condition. Hence, the SPDE (1.1)–(1.4) has a weak solution even if $\lambda > 0$ for, e.g., $Q = 0$, $f = 0$, and $g > 0$.

We construct a solution to equation (1.1) as the limit of polygonal approximations by analogy with the approach realized in [11]. In tis case, the main difficulty is that the coefficients are discontinuous. Hence, we cannot pass to the limit directly. In the next section, we explain the key idea that allows us to overcome this difficulty.

1.2. Key Idea of Passing to the Limit. We demonstrate our idea of passing to the limit in the case of discontinuous coefficients by using the equation for a sticky-reflected Brownian motion in $\mathbb{R}$

$$\begin{aligned} &dx(t) = \lambda \mathbb{I}_{\{x(t)=0\}} dt + \mathbb{I}_{\{x(t)>0\}} \sigma dw(t), \quad t \geq 0, \\ &x(0) = x^0, \end{aligned} \tag{1.8}$$

where w is a standard Brownian motion in $\mathbb{R}$ and $\lambda,\ \sigma,$ and x^0 are positive constants. It is known that this equation has solely a unique weak solution (see, e.g., [7]).

We now show that a solution to the SDE (1.8) can be constructed as the weak limit of solutions to equations with "good" coefficients. The first three steps proposed in what follows are rather standard and the last step shows how one can overcome the problem of discontinuity of the coefficients.

Step I. Approximating Sequence. Consider a nondecreasing continuously differentiable function $\kappa\colon \mathbb{R} \to \mathbb{R}$ such that $\kappa(s) = 0,\ s \leq 0,$ and $\kappa(s) = 1,\ s \geq 1.$ Denote

$$\kappa_\varepsilon(s) := \kappa\left(\frac{s}{\varepsilon}\right), \quad s \in \mathbb{R},$$

and consider the SDE

$$\begin{aligned} &dx_\varepsilon(t) = \lambda \left(1 - \kappa_\varepsilon^2(x_\varepsilon(t))\right) dt + \kappa_\varepsilon(x_\varepsilon(t)) \sigma dw(t), \quad t \geq 0, \\ &x_\varepsilon(0) = x^0. \end{aligned} \tag{1.9}$$

This SDE has a unique strong solution for every $\varepsilon > 0$.

Step II. Tightness in an Appropriate Space. Consider the processes

$$a_\varepsilon(t) := \lambda \int_0^t \left(1 - \kappa_\varepsilon^2(x_\varepsilon(s))\right) ds, \quad t \geq 0,$$

$$\eta_\varepsilon(t) := \int_0^t \sigma \kappa_\varepsilon(x_\varepsilon(s))\, dw(s), \quad t \geq 0,$$

and

$$[\eta_\varepsilon]_t = \int_0^t \sigma^2 \kappa_\varepsilon^2(x_\varepsilon(s))\, ds, \quad t \geq 0,$$

where $[\eta_\varepsilon]$ is the quadratic variation of the martingale η_ε.

In view of the uniform boundedness of the coefficients of SDE (1.9), we can show that the family

$$\{(x_\varepsilon, a_\varepsilon, \eta_\varepsilon, [\eta_\varepsilon]),\ \varepsilon > 0\}$$

is tight in $(\mathrm{C}[0,\infty))^4$. By Prokhorov's theorem, we can choose a subsequence

$$(x_m, a_m, \eta_m, [\eta_m]) := (x_{\varepsilon_m}, a_{\varepsilon_m}, \eta_{\varepsilon_m}, [\eta_{\varepsilon_m}]), \qquad m \geq 1$$

that converges to (x, a, η, ρ) in $(\mathrm{C}[0,\infty))^4$ in distribution as $m \to \infty$. By the Skorokhod representation theorem, we can assume that

$$(x_m, a_m, \eta_m, [\eta_m]) \to (x, a, \eta, \rho) \quad \text{a.s.} \quad \text{as} \quad m \to \infty.$$

Step III. Properties of the Limit Process. It is easy to see that, for every $T > 0$, there exist a random element $\dot{\rho}$ in $\mathrm{L}_2[0,T]$ and a subsequence N such that

$$\sigma^2 \kappa_m^2(x_m) \to \dot{\rho} \quad \text{in the weak topology of} \quad \mathrm{L}_2[0,T] \tag{1.10}$$

along N and, for every $t \in [0,T]$,

$$x(t) = x^0 + a(t) + \eta(t), \quad \rho(t) = \int_0^t \dot{\rho}(s)\, ds, \quad a(t) = \lambda\left(t - \frac{1}{\sigma^2}\rho(t)\right), \tag{1.11}$$

and η is a continuous square-integrable martingale with quadratic variation ρ. It is possible to assume that $N = \mathbb{N}$.

Step IV. Identification of the Quadratic Variation and Drift. In view of the discontinuity of the coefficients of equation (1.8), we cannot make a direct conclusion that

$$\rho(t) = \int_0^t \sigma^2 \mathbb{I}_{\{x(s)>0\}} ds \quad \text{and} \quad a(t) = \lambda \int_0^t \mathbb{I}_{\{x(s)=0\}} ds, \quad t \in [0,T],$$

which would imply that x is a weak solution to the SDE (1.8). To overcome this problem we propose to use the following facts:

(a) if $x(t),\ t \geq 0$, is a continuous nonnegative semimartingale with quadratic variation

$$[x]_t = \int_0^t \sigma^2(s)\, ds, \quad t \geq 0,$$

then,[1] a.s.,

$$[x]_t = \int_0^t \sigma^2(s) \mathbb{I}_{\{x(s)>0\}} ds, \quad t \geq 0;$$

(b) if $s_m \to s$ in $\mathbb{R}$, then

$$\kappa_m^2(s_m)\mathbb{I}_{(0,+\infty)}(s) \to \mathbb{I}_{(0,+\infty)}(s) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad m \to \infty.$$

[1] See also Lemma A.1.

Hence, by using (1.10), (a), (b), and the dominated convergence theorem, we get (a.s.)

$$\rho(t) = \int_0^t \dot{\rho}(s)\,ds = \int_0^t \dot{\rho}(s)\mathbb{I}_{\{x(s)>0\}}ds$$

$$= \lim_{m\to\infty} \int_0^t \sigma^2\kappa_m^2(x_m(s))\mathbb{I}_{\{x(s)>0\}}ds = \int_0^t \sigma^2\mathbb{I}_{\{x(s)>0\}}ds, \quad t\in[0,T].$$

Thus, (1.11) implies that

$$a(t) = \lambda\left(t - \frac{1}{\sigma^2}\rho(t)\right) = \lambda\int_0^t \mathbb{I}_{\{x(s)=0\}}ds, \quad t\in[0,T].$$

Consequently,

$$x(t) = x^0 + \lambda\int_0^t \mathbb{I}_{\{x(s)=0\}}ds + \eta(t), \quad t\geq 0,$$

where η is a continuous square-integrable martingale with quadratic variation

$$[\eta]_t = \int_0^t \sigma^2\mathbb{I}_{\{x(s)>0\}}ds, \quad t\geq 0,$$

which means that x is a weak solution to (1.8).

Contents of the Paper. To show the existence of a weak solution to the SPDE (1.1)–(1.4), we follow the scheme presented above. Step I is realized in Subsection 2.1. More precisely, we construct a family of processes approximating a solution to the SPDE (1.1)–(1.4). The approximating sequence is similar to the sequence considered in [11]. Subsection 2.2 is devoted to the property of tightness, i.e., to Step II of the proposed scheme. Step III is done in Subsection 3.1, where we show that the limit process satisfies equalities similar to (1.11) (see Proposition 3.1). An analog of property (a) presented above is formulated for some infinite-dimensional semimartingales in Theorem 3.1 from Subsection 3.2. The proof of the existence theorem is given in Subsection 3.3, where we use the approach described in Step IV. Some auxiliary statements are proved in the appendix.

1.3. Preliminaries. We denote the inner product and the corresponding norm in a Hilbert space H by $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$, respectively.

For an essentially bounded function $\psi\in L_\infty$, we define a multiplication operator $[\psi\cdot]$ on L_2 as follows:

$$([\psi\cdot]\,h)\,(u) = \psi(u)h(u), \quad u\in[0,1], \quad h\in\mathrm{L}_2.$$

Let A be an operator on L_2 and let $\varphi_1,\ldots,\varphi_n\in\mathrm{L}_2$ be such that the product $\varphi_1\ldots\varphi_n$ belongs to L_2. To simplify notation, we always write $A\varphi_1\ldots\varphi_n$ for $A\,(\varphi_1\ldots\varphi_n)$.

Denote the space of Hilbert–Schmidt operators on L_2 by $\mathcal{L}_2$. Note that $\mathcal{L}_2$ equipped with the inner product

$$\langle A, B\rangle_{\mathcal{L}_2} = \sum_{k=1}^{\infty} \langle Ae_k, Be_k\rangle, \quad A, B \in \mathcal{L}_2,$$

is a Hilbert space whose norm does not depend on the choice of basis in the space L_2. The family of operators $\{e_k \odot e_l,\ k, l \geq 1\}$ form a basis in $\mathcal{L}_2$. Here, for any $\varphi, \psi \in \mathrm{L}_2$, $\varphi \odot \psi$ denotes the operator on L_2 defined as follows:

$$(\varphi \odot \psi)g = \langle g, \psi\rangle \varphi, \quad g \in \mathrm{L}_2.$$

We consider the set $\mathbb{R}^{n\times n}$ of all $(n\times n)$-matrices with real entries as a Hilbert space with the Hilbert–Schmidt inner product

$$\langle A, B\rangle_{\mathbb{R}^{n\times n}} = \sum_{k,l=1}^{n} A_{k,l} B_{k,l}.$$

The indicator function is defined as usually:

$$\mathbb{I}_S(x) = \begin{cases} 1 & \text{for} \quad x \in S, \\ 0 & \text{for} \quad x \notin S. \end{cases}$$

If $\phi : E_1 \to E_2$ is a function and S is a subset of E_2, then $\mathbb{I}_{\{\phi\in S\}}$ denotes a function $x \mapsto \mathbb{I}_S(\phi(x))$ from E_1 to E_2.

Given a Hilbert space H, we write $H^T := \mathrm{L}_2([0, T], H)$ for the class of all Bochner integrable functions $\Phi\colon [0, T] \to H$ with

$$\|\Phi\|_{H,T} = \left(\int_0^T \|\Phi_s\|_H^2 ds\right)^{\frac{1}{2}} < \infty.$$

One can show that the space H^T equipped with the inner product

$$\langle \Phi, \Psi\rangle_{H,T} = \int_0^T \langle \Phi_s, \Psi_s\rangle_H ds, \quad \Phi, \Psi \in H^T,$$

is a Hilbert space.

Consider a sequence $\{Z^n\}_{n\geq 1}$ in H^T. We say that $Z^n \to Z$ a.e. as $n \to \infty$ if

$$\mathrm{Leb}_T\{t \in [0, T]\colon\ Z_t^n \not\to Z_t \text{ in } H,\ n \to \infty\} = 0,$$

where Leb_T denotes the Lebesgue measure on $[0, T]$.

Let $L \in \mathcal{L}_2^T$, let $Z \in \mathrm{L}_2^T$, and let S be a Borel measurable subset of $\mathbb{R}$. It is easy to see that $L_t\left[\mathbb{I}_{\{Z_t\in S\}}\cdot\right]$, $t \in [0, T]$, where $L_t\left[\mathbb{I}_{\{Z_t\in S\}}\cdot\right]$ is the composition of two operators, is well-defined and belongs to $L \in \mathcal{L}_2^T$. We denote this function shortly by $L_\cdot\left[\mathbb{I}_{\{Z_\cdot\in S\}}\cdot\right]$.

Let I be equal to $[0,T],\ [0,\infty),$ or $[0,T]\times[0,1]$. The space of all continuous functions from I to a Polish space E with the topology of uniform convergence on compact sets is denoted by $\mathrm{C}(I,E)$. If $I=[0,T]$ or $[0,\infty)$ and $E=\mathbb{R}$, then we simply write $\mathrm{C}[0,T]$ or $\mathrm{C}[0,\infty)$ instead of $\mathrm{C}\,(I,\mathbb{R})$.

We denote the right continuous complete filtration generated by continuous processes $\xi_1(t),\ t\in I,\dots,\xi_n(t),$ $t\in I$, by $(\mathcal{F}_t^{\xi_1,\dots,\xi_n})_{t\in I}$. Note that this filtration exists by Lemma 7.8 [17].

2. Finite Sticky Reflected Particle System

In this section, we construct a sequence of random processes used for the approximation of a solution to the SPDE (1.1)–(1.4).

Let $n\geq 1$ be fixed. We set

$$\pi_k^n=\mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)},\quad k\in[n]:=\{1,\dots,n\}.$$

Let $W_t,\ t\geq 0$, be a cylindrical Wiener process in L_2. We define the Wiener processes on $\mathbb{R}$ as follows:

$$w_k^n(t):=\sqrt{n}\int_0^t\left\langle\pi_k^n,QdW_s\right\rangle,\quad t\geq 0,\quad k\in[n],$$

and note that their joint quadratic variation is

$$[w_k^n,w_l^n]_t=n\langle Q\pi_k^n,Q\pi_l^n\rangle t=:q_{k,l}^n t,\quad t\geq 0.$$

Also let[2]

$$\lambda_k^n:=n\langle\lambda,\pi_k^n\rangle\mathbb{I}_{\left\{q_{k,k}^n>0\right\}}\quad\text{and}\quad g_k^n:=n\langle g,\pi_k^n\rangle,\quad k\in[n].$$

Consider the SDE

$$dx_k^n(t)=\frac{1}{2}\Delta^n x_k^n(t)dt+\lambda_k^n\mathbb{I}_{\left\{x_k^n(t)=0\right\}}dt$$
$$+f(x_k^n(t))dt+\sqrt{n}\mathbb{I}_{\left\{x_k^n(t)>0\right\}}dw_k^n(t),\quad k\in[n],\tag{2.1}$$

satisfying the initial condition

$$x_k^n(0)=g_k^n,\quad k\in[n],\tag{2.2}$$

where

$$\Delta^n x_k^n=(\Delta^n x^n)_k=n^2\left(x_{k+1}^n+x_{k-1}^n-2x_k^n\right)$$

and

$$x_0^n(t)=\alpha_0 x_1^n(t),\quad x_{n+1}^n(t)=\alpha_0 x_n^n(t),\quad t\geq 0.\tag{2.3}$$

[2] We add the indicator $\mathbb{I}_{\left\{q_{k,k}^n>0\right\}}$ to the definition of λ_k^n because we need the additional condition that $\lambda_k^n=0$ if $q_{k,k}^n=0$ for the existence of solution to the SDE (2.1).

We construct a solution of the SPDE (1.1)–(1.4) as a weak limit in $\mathrm{C}([0,\infty),\mathrm{C}[0,1])$ of the processes

$$\tilde{X}^n_t(u) = (un-k+1)x^n_k(t) + (k-nu)x^n_{k-1}(t), \quad t\in[0,T], \quad u\in\pi^n_k, \quad k\in[n]. \tag{2.4}$$

Note that equation (2.1) has discontinuous coefficients. Hence, the classical theory of SDE cannot be applied in our case. The existence of the solution follows from Theorem 2.1, which is formulated in what follows.

2.1. SDE for Sticky-Reflected Particle System. The aim of the present section is to prove the existence of solutions to (2.1) and (2.2). We formulate the problem in a slightly more general form. Hence, let $n\in\mathbb{N}$ and let $g_k,\ \lambda_k,\ k\in[n]$, be nonnegative numbers. We also consider a family of Brownian motions $w_k(t),\ t\ge 0$, $k\in[n]$ (with respect to the same filtration) with joint quadratic variation

$$[w_k, w_l]_t = q_{k,l}t, \quad t\ge 0.$$

As earlier, let $f\colon [0,\infty)\to[0,\infty)$ be a continuous function with linear growth such that $f(0)=0$. Consider the SDE

$$\begin{aligned} dy_k(t) &= \frac{1}{2}\Delta^n y_k(t)dt + \lambda_k \mathbb{I}_{\{y_k(t)=0\}}dt \\ &\quad + f(y_k(t))dt + \mathbb{I}_{\{y_k(t)>0\}}dw_k(t), \quad k\in[n], \end{aligned} \tag{2.5}$$

with the initial condition

$$y_k(0) = g_k, \quad k\in[n], \tag{2.6}$$

and the following boundary conditions:

$$y_0(t) = \alpha_0 y_1(t), \quad y_{n+1}(t) = \alpha_0 y_n(t), \quad t\ge 0. \tag{2.7}$$

Theorem 2.1. *Let $q_{k,k}=0$ imply that $\lambda_k=0$ for every $k\in[n]$. Then there exists a family of nonnegative (real-valued) continuous processes $y_k(t),\ t\ge 0,\ k\in[n]$, in $\mathbb{R}$, which is a weak (martingale) solution to (2.1), (2.2), i.e., $y_k(0)=g_k$ for any $k\in[n]$,*

$$\mathcal{N}_k(t) := y_k(t) - g_k - \frac{1}{2}\int_0^t \Delta^n y_k(s)\,ds - \lambda_k\int_0^t \mathbb{I}_{\{y_k(s)=0\}}ds - \int_0^t f(y_k(s))\,ds, \quad t\ge 0,$$

is an $(\mathcal{F}^y_t)$-martingale, and the joint quadratic variation of $\mathcal{N}_k$ and $\mathcal{N}_l,\ k,l\in[n]$, is equal to

$$[\mathcal{N}_k,\mathcal{N}_l]_t = q_{k,l}\int_0^t \mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds, \quad t\ge 0.$$

We are now going to construct a solution to the SDE by approximating the coefficients by Lipschitz continuous functions and using the method described in Subsection 1.2.

We take a nondecreasing function $\kappa \in \mathrm{C}^1(\mathbb{R})$ such that $\kappa(x) = 0$ for $x \leq 0$ and $\kappa(x) = 1$ for $x \geq 1$. Also let $\theta \in \mathrm{C}^1(\mathbb{R})$ be a nonnegative function with $\operatorname{supp}\theta \in [-1, 1]$ and $\int_{-\infty}^{+\infty} \theta(x)dx = 1$. For every $\varepsilon > 0$, we introduce the functions

$$\kappa_\varepsilon(x) = \kappa\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}, \qquad \text{and} \qquad \theta_\varepsilon(x) = \frac{1}{\varepsilon}\theta\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}.$$

Setting

$$f_\varepsilon(x) = \int_0^{+\infty} \theta_\varepsilon(x-y)f(y)\,dy, \quad x \in \mathbb{R},$$

we consider the SDE

$$dy_k^\varepsilon(t) = \frac{1}{2}\Delta^n y_k^\varepsilon(t)dt + \lambda_k\left(1 - \kappa_\varepsilon^2(y_k^\varepsilon(t))\right)dt + f_\varepsilon(y_k^\varepsilon(t))dt + \kappa_\varepsilon(y_k^\varepsilon(t))dw_k(t), \tag{2.8}$$

$$y_k^\varepsilon(0) = g_k, \quad k \in [n].$$

Since equation (2.8) has locally Lipschitz continuous coefficients with linear growth, it has a unique strong solution.

Our aim is to show that the sequence $\{y^\varepsilon = (y_k^\varepsilon)_{k=1}^n\}_{\varepsilon>0}$ has a subsequence that converges in distribution to a weak solution of Eq. (2.1). For every $k \in [n]$, we denote

$$a_k^\varepsilon(t) = \lambda_k \int_0^t \left(1 - \kappa_\varepsilon^2(y_k^\varepsilon(s))\right)ds, \quad t \geq 0,$$

and

$$\eta_k^\varepsilon(t) = \int_0^t \kappa_\varepsilon(y_k^\varepsilon(s))dw_k(s), \quad t \geq 0.$$

We set

$$a^\varepsilon = (a_k^\varepsilon)_{k=1}^n \quad \text{and} \quad \eta^\varepsilon = (\eta_k^\varepsilon)_{k=1}^n.$$

The quadratic variation $[\eta^\varepsilon]_t$, $t \geq 0$, of the $\mathbb{R}^n$-valued martingale η^ε takes values in the space of nonnegative definite $(n \times n)$-matrices with entries

$$[\eta_k^\varepsilon, \eta_l^\varepsilon]_t = \int_0^t \sigma_{k,l}^\varepsilon(s)\,ds,$$

where $\sigma_{k,l}^\varepsilon(s) = \kappa_\varepsilon(y_k^\varepsilon(s))\kappa_\varepsilon(y_l^\varepsilon(s))q_{k,l}$.

Remark 2.1. According to the choice of the approximating sequence for a, the equality

$$a_k^\varepsilon(t) = \lambda_k\left(t - \frac{1}{q_{k,k}}[\eta_k^\varepsilon]_t\right), \quad t \geq 0,$$

holds for every $k \in [n]$ satisfying $q_{k,k} > 0$.

Consider the following metric space:

$$\mathcal{W}_{\mathbb{R}^n} := (\mathrm{C}([0,\infty),\mathbb{R}^n))^3 \times \mathrm{C}\big([0,\infty),\mathbb{R}^{n\times n}\big).$$

Lemma 2.1. *The family* $\{(y^{\varepsilon_m}, a^{\varepsilon_m}, \eta^{\varepsilon_m}, [\eta^{\varepsilon_m}]),\ m \geq 1\}$ *is tight in* $\mathcal{W}_{\mathbb{R}^n}$, *where* $\varepsilon_m,\ m \geq 1,$ *is any sequence convergent to zero.*

Proof. In order to prove the statement, it is sufficient to show that each family of coordinate processes of $(y^{\varepsilon_m}, a^{\varepsilon_m}, \eta^{\varepsilon_m}, [\eta^{\varepsilon_m}]),\ m \geq 1,$ is tight in the corresponding space. We only prove the property of tightness for $\{y^{\varepsilon_m},\ m \geq 1\}$. The tightness for the other families can be established similarly.

According to the Aldous tightness criterion [2] (Theorem 1), it is sufficient to show that, for every $T > 0,$ any family of stopping times $\tau_m,\ m \geq 1,$ bounded by $T,$ and any sequence δ_m decreasing to zero,

$$y^{\varepsilon_m}(\tau_m + \delta_m) - y^{\varepsilon_m}(\tau_m) \to 0 \quad \text{in probability} \quad \text{as} \quad m \to \infty$$

and $\{y^{\varepsilon_m}(t),\ m \geq 1\}$ is tight in $\mathbb{R}^n$ for each $t \in [0,T]$.

The conditions presented above trivially follow from the convergence

$$\mathbb{E}\Big[\big\|y^{\varepsilon_m}(\tau_m + \delta_m) - y^{\varepsilon_m}(\tau_m)\big\|^2_{\mathbb{R}^n}\Big] \to 0 \quad \text{as} \quad m \to \infty$$

and the uniform boundedness of $\mathbb{E}\Big[\|y^{\varepsilon_m}(t)\|^2_{\mathbb{R}^n}\Big]$ in $m \geq 1$ for every $t \in [0,T]$.

By using the fact that there exists a constant $C > 0$ such that

$$|f_{\varepsilon_m}(x)| \leq C(1 + |x|), \quad x \in \mathbb{R}, \quad m \geq 1,$$

the inequality

$$\big\langle y^{\varepsilon_m}(t), \Delta^n y^{\varepsilon_m}(t)\big\rangle = -\sum_{k=1}^{n-1}\big(y^{\varepsilon_m}_{k+1}(t) - y^{\varepsilon_m}_k(t)\big)^2 - \alpha_0(y^{\varepsilon_m}_1(t) + y^{\varepsilon_m}_n(t)) \leq 2\varepsilon_m\alpha_0$$

valid for all $t \in [0,T],$ the Itô formula, and the Gronwall lemma, we can show that, for every $p \geq 1,$ there exists a constant $C_{p,T,n}$ depending on $p,\ T,$ and $n,$ such that

$$\mathbb{E}\Big[\|y^{\varepsilon_m}(t)\|^{2p}_{\mathbb{R}^n}\Big] \leq C_{p,T,n}, \quad t \in [0,T]. \tag{2.9}$$

Further, by the Itô formula and the optional sampling Theorem 7.12 [17], we get

$$\mathbb{E}\big[\|y^{\varepsilon_m}(\tau_m + \delta_m) - y^{\varepsilon_m}(\tau_m)\|^2_{\mathbb{R}^n}\big] \leq \mathbb{E}\left[\int_{\tau_m}^{\tau_m+\delta_m} \langle y^{\varepsilon_m}(r), \Delta^n y^{\varepsilon_m}(r)\rangle_{\mathbb{R}^n} dr\right]$$

$$+ 2\mathbb{E}\left[\int_{\tau_m}^{\tau_m+\delta_m} \langle y(r), \lambda\left(1 - \kappa^2_{\varepsilon_m}(y.(r))\right)\rangle_{\mathbb{R}^n} dr\right]$$

$$+ 2\mathbb{E}\left[\int_{\tau_m}^{\tau_m+\delta_m} \langle y^{\varepsilon_m}(r), f_{\varepsilon_m}(y^{\varepsilon_m}_\cdot(r))\rangle_{\mathbb{R}^n} dr\right]$$

$$+ \mathbb{E}\left[\int_{\tau_m}^{\tau_m+\delta_m} \sum_{k=1}^{n} \kappa^2_{\varepsilon_m}(y_k^{\varepsilon_m}(r)) q_{k,k} dr\right] \quad \text{for all} \quad m \geq 1. \tag{2.10}$$

By virtue of Hölder's inequality and estimate (2.9), we can conclude that

$$\mathbb{E}\Big[\|y^{\varepsilon_m}(\tau_m+\delta_m) - y^{\varepsilon_m}(\tau_m)\|^2_{\mathbb{R}^n}\Big] \to 0 \quad \text{as} \quad m \to \infty.$$

Lemma 2.1 is proved.

By Lemma 2.1 and the Prokhorov theorem, there exists a sequence $\{\varepsilon_m\}_{m\geq 1}$ convergent to zero such that the sequence $\mathrm{y}^{\varepsilon_m} := (y^{\varepsilon_m}, a^{\varepsilon_m}, \eta^{\varepsilon_m}, [\eta^{\varepsilon_m}]),\ m \geq 1,$ converges to a random element $\mathrm{y} := (y, a, \eta, \rho)$ in $\mathcal{W}_{\mathbb{R}^n}$ in distribution. By the Skorokhod representation Theorem 3.1.8 [6], we can choose a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and determine (in this space) a family of random elements $\tilde{\mathrm{y}},\ \tilde{\mathrm{y}}^{\varepsilon_m},\ m \geq 1,$ taking values in $\mathcal{W}_{\mathbb{R}^n}$ and such that $\operatorname{Law}\tilde{\mathrm{y}} = \operatorname{Law}\mathrm{y},\ \operatorname{Law}\tilde{\mathrm{y}}^{\varepsilon_m} = \operatorname{Law}\mathrm{y}^{\varepsilon_m},\ m \geq 1,$ and $\tilde{\mathrm{y}}^{\varepsilon_m} \to \mathrm{y}$ in $\mathcal{W}_{\mathbb{R}^n}$ a.s. Hence, without loss of generality, we can assume that

$$\mathrm{y}^{\varepsilon_m} \to \mathrm{y} \quad \text{in} \quad \mathcal{W}_{\mathbb{R}^n} \quad \text{a.s.} \quad \text{as} \quad m \to \infty.$$

Since the sequence $\{\varepsilon_m\}_{m\geq 1}$ is fixed at the end of this section, we can write m instead of ε_m in order to simplify the notation.

Let $y = (y_k)_{k=1}^n,\ a = (a_k)_{k=1}^n,\ \eta = (\eta_k)_{k=1}^n,$ and $\rho = (\rho_{k,l})_{k,l=1}^n$.

Lemma 2.2.

(i) The coordinate processes $y_k(t),\ t \geq 0,\ k \in [n],$ *of* y *are nonnegative and*

$$y_k(t) = g_k + \frac{1}{2}\int_0^t \Delta^n y_k(s)\, ds + a_k(t) + \int_0^t f(y_k(s))\, ds + \eta_k(t), \quad t \geq 0, \quad k \in [n].$$

(ii) For every $k \in [n]$ *such that* $q_{k,k} > 0$, *the following equality is true:*

$$a_k = \lambda_k\bigg(t - \frac{1}{q_{k,k}}\rho_{k,k}\bigg).$$

(iii) For every $k \in [n]$ *and* $T > 0$, *there exists a random element* $\dot{a}_k$ *in* $\mathrm{L}_2([0,T],\mathbb{R})$ *such that (a.s.)*

$$a_k(t) = \int_0^t \dot{a}_k(s)\, ds, \quad t \in [0,T].$$

(iv) For every $k, l \in [n]$ *and* $T > 0$ *there exists a random element* $\dot{\rho}_{k,l}$ *in* $\mathrm{L}_2([0,T],\mathbb{R})$ *such that (a.s.)*

$$\rho_{k,l}(t) = \int_0^t \dot{\rho}_{k,l}(s)\, ds, \quad t \in [0,T].$$

(v) For every $k \in [n]$, the process $\eta_k(t)$, $t \geq 0$, is a continuous square-integrable $(\mathcal{F}_t^{\eta})$-martingale and the joint quadratic variation of η_k and η_l, $k, l \in [n]$, is equal to

$$[\eta_k, \eta_l]_t = \rho_{k,l}(t), \quad t \geq 0.$$

Proof. Note that, for every $k \in [n]$,

$$\mathbb{P}\big[\forall t \geq 0 \;\; f_m(y_k^m(t)) \to f(y_k(t)) \;\text{ as }\; m \to \infty\big] = 1$$

and, for every $m \geq 1$ and $k \in [n]$, a.s.,

$$y_k^m(t) = g_k + \frac{1}{2}\int_0^t \Delta^n y_k^m(s)\, ds + a_k^m(t) + \int_0^t f_m(y_k^m(s))\, ds + \eta_k^m(t), \quad t \geq 0.$$

Passing to the limit and using the dominated convergence theorem, we arrive at equality (i).

The equality in (ii) follows from Remark 2.1 and the convergence (in distribution) of $\left(a_k^m, [\eta_k^m]\right)$ to $(a_k, \rho_{k,k})$ in $(\mathrm{C}([0, +\infty), \mathbb{R}))^2$.

Further, we prove (iii). Let $T > 0$ be fixed. By B_r^T we denote the ball in $\mathrm{L}_2([0,T], \mathbb{R})$ with center 0 and radius $r > 0$ and equip it with the weak topology of the space $\mathrm{L}_2([0,T], \mathbb{R})$, i.e., a sequence $\{h_m\}_{m \geq 1}$ converges to h in B_r^T if $\langle h_m, b\rangle_{\mathbb{R},T} \to \langle h, b\rangle_{\mathbb{R},T}$ for all $b \in B_r^T$. By the Alaoglu Theorem V.4.2 [5] and Theorem V.5.1 [5], B_r^T is a compact metric space.

We fix $k \in [n]$ and take $r := \lambda_k \sqrt{T}$,

$$\dot{a}_k^m(t) := \lambda_k\big(1 - \kappa_m^2(y_k^m(t))\big), \quad t \in [0,T].$$

Then $\dot{a}_k^m$ is a random element in B_r^T for every $m \geq 1$. By the compactness of B_r^T, the family $\{\dot{a}_k^m,\ m \geq 1\}$ is tight in B_r^T. Consequently, the Prokhorov theorem implies the existence of a subsequence $N \subset \mathbb{N}$ such that

$$\dot{a}_k^m \to \tilde{a}_k \quad \text{(in distribution)} \quad \text{in} \quad B_r^T \quad \text{along} \quad N.$$

In particular, for every family $t_1, \ldots, t_l \in [0,T]$ and numbers $c_1, \ldots, c_l \in \mathbb{R}$, we get

$$\sum_{i=1}^{l} c_i \int_0^{t_i} \dot{a}_k^m(s)\, ds = \int_0^T \left(\sum_{i=1}^{l} c_i \mathbb{I}_{[0,t_i]}(s)\right) \dot{a}_k^m(s)\, ds$$

$$\to \int_0^T \left(\sum_{i=1}^{l} c_i \mathbb{I}_{[0,t_i]}(s)\right) \tilde{a}_k(s)\, ds = \sum_{i=1}^{l} c_i \int_0^{t_i} \tilde{a}_k(s)\, ds \quad \text{(in distribution)}$$

in $\mathbb{R}$ along N. Since the family of functions

$$\left\{ x \mapsto h\left(\sum_{l=1}^{l} c_i x_i\right),\;\; x = (x_i)_{i=1}^{l} \in \mathbb{R}^l :\;\; h \text{ is continuous and bounded on } \mathbb{R} \right\}$$

strongly separates points,[3] Theorem 3.4.5 [6] implies that

$$\left(\int_0^{t_1} \dot{a}_k^m(s)\,ds, \ldots, \int_0^{t_l} \dot{a}_k^m(s)\,ds\right) \to \left(\int_0^{t_1} \tilde{a}_k(s)\,ds, \ldots, \int_0^{t_l} \tilde{a}_k(s)\,ds\right) \quad \text{(in distribution)}$$

in $\mathbb{R}^l$ along N. On the other side,

$$\left(\int_0^{t_1} \dot{a}_k^m(s)\,ds, \ldots, \int_0^{t_l} \dot{a}_k^m(s)\,ds\right) \to (a_k(t_1), \ldots, a_k(t_l)) \quad \text{a.s.}$$

in $\mathbb{R}^l$ along N. This implies that

$$\operatorname{Law} a_k = \operatorname{Law} \int_0^{\cdot} \tilde{a}_k(s)\,ds. \tag{2.11}$$

We now show that there exists a random element $\dot{a}_k$ in $\mathrm{L}_2([0,T],\mathbb{R})$ such that, a.s.,

$$a_k = \int_0^{\cdot} \dot{a}_k(s)\,ds.$$

We now define a map $\Phi\colon \mathrm{L}_2([0,T],\mathbb{R}) \to \mathrm{C}[0,T]$ as follows:

$$\Phi(h)(t) = \int_0^t h(s)\,ds, \quad t \in [0,T].$$

Note that Φ is a bijective map from $\mathrm{L}_2([0,T],\mathbb{R})$ to its image

$$\operatorname{Im}\Phi = \{\Phi(h)\colon\, h \in \mathrm{L}_2([0,T],\mathbb{R})\}.$$

By the Kuratowski Theorem A.10.5 [6], the set $\operatorname{Im}\Phi$ is Borel measurable in $\mathrm{C}[0,T]$ and the map Φ^{-1} is Borel measurable. By (2.11), $a_k \in \operatorname{Im}\Phi$ a.s. Thus, we can define $\dot{a}_k = \Phi^{-1}(a_k)$. This completes the proof of (iii).

Similarly, we can prove (iv).

Statement (v) follows from the fact that the limit of local martingales is a local martingale and the uniform boundedness of $\mathbb{E}\left[(\eta_k^m(t))^2\right]$ in m. Indeed, for every $k,l \in [n]$, the processes η_k^m and $\eta_k^m\eta_l^m - [\eta_k^m,\eta_l^m]$ are $(\mathcal{F}_t^{\eta^m})$-martingales for all $m \geq 1$ and

$$(\eta^m, [\eta^m], \eta_k^m\eta_l^m - [\eta_k^m,\eta_l^m]) \to (\eta, \rho, \eta_k\eta_l - \rho_{k,l}) \quad \text{a.s.} \quad \text{as} \quad m \to \infty.$$

Proposition IX.1.17 [16] implies that η_k and $\eta_k\eta_l - \rho_{k,l}$ are $(\mathcal{F}_t^{(\eta,\rho)})$-local martingales. Note that, by the Fisk approximation Theorem 17.17 [17], $\mathcal{F}_t^{(\eta,\rho)} = \mathcal{F}_t^{\eta}$, $t \geq 0$. By the uniform boundedness of $\mathbb{E}\left[(\eta_k^m(t))^2\right]$ in m and the Fatou lemma, we conclude that η_k is a square-integrable $(\mathcal{F}_t^{\eta})$-martingale.

Lemma 2.2 is proved.

[3]See the definition in [6, p. 113].

Proposition 2.1. *Let* $\mathrm{y}(t) = (y(t), a(t), \eta(t), \rho(t)),\ t \geq 0,$ *be as in Lemma 2.2. In addition, let* $\lambda_k = 0$ *for* $q_{k,k} = 0,\ k \in [n]$*. Then*

(i) for every $k, l \in [n]$*, a.s.,*

$$\rho_{k,l}(t) = q_{k,l} \int_0^t \mathbb{I}_{\{y_k(s)>0\}} \mathbb{I}_{\{y_l(s)>0\}} ds, \quad t \geq 0;$$

(ii) for every $k \in [n]$*, a.s.,*

$$a_k(t) = \lambda_k \int_0^t \mathbb{I}_{\{y_k(s)=0\}} ds, \quad t \geq 0.$$

Proof. We take a sequence $\{y_n^m\}_{n\geq 1}$ as in the proof of Lemma 2.2. Again, without loss of generality, we can assume that it converges to y a.s. We first show that (a.s.)

$$\rho_{k,l}(t) = q_{k,l} \int_0^t \mathbb{I}_{\{y_k(s)>0\}} \mathbb{I}_{\{y_l(s)>0\}} ds, \quad t \geq 0.$$

Recall that (a.s.)

$$[\eta_k^m, \eta_l^m]_t = \int_0^t \sigma_{k,l}^m(s)\, ds, \quad t \geq 0,$$

where

$$\sigma_{k,l}^m(s) = q_{k,l} \kappa_m(y_k^m(s)) \kappa_m(y_l^m(s))$$

and, for each $T > 0,\ k, l \in [n]$, there exist random elements $\dot{\rho}_{k,l}$ in $\mathrm{L}_2([0,T], \mathbb{R})$ such that (a.s.)

$$\rho_{k,l}(t) = \int_0^t \dot{\rho}_{k,l}(s)\, ds, \quad t \in [0,T],$$

by Lemma 2.2.

Let $T > 0,\ k, l \in [n]$ be fixed. By the convergence of the sequence $[\eta_k^m, \eta_l^m],\ m \geq 1,$ to $\rho_{k,l}$ in $\mathrm{C}[0,T]$ a.s., the uniform boundedness of $\sigma_{k,l}^m$, and the fact that $\operatorname{span}\{\mathbb{I}_{[0,t]},\ t \in [0,T]\}$ is dense in $\mathrm{L}_2([0,T], \mathbb{R})$, we conclude that

$$\mathbb{P}\big[\sigma_{k,l}^m \to \dot{\rho}_{k,l} \text{ in the weak topology of } \mathrm{L}_2([0,T], \mathbb{R}) \text{ as } m \to \infty\big] = 1. \tag{2.12}$$

By Lemma 2.2, y_k and y_l are nonnegative continuous semimartingales with quadratic variation

$$[y_k, y_l]_t = \rho_{k,l}(t) = \int_0^t \dot{\rho}_{k,l}(s)\, ds, \quad t \in [0,T].$$

Thus, Lemma A.1 implies that (a.s.)

$$\int_0^t \dot{\rho}_{k,l}(s)\,ds = \int_0^t \dot{\rho}_{k,l}(s)\mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds, \quad t \in [0,T].$$

This equality and (2.12) imply that, for every $t \in [0,T]$, we can write (a.s.)

$$\begin{aligned}\rho_{k,l}(t) = \int_0^t \dot{\rho}_{k,l}(s)\,ds &= \int_0^t \dot{\rho}_{k,l}(s)\mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds \\ &= \lim_{m\to\infty}\int_0^t \sigma_{k,l}^m \mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds \\ &= \lim_{m\to\infty}\int_0^t q_{k,l}\kappa_m(y_k^m(s))\kappa_m(y_l^m(s))\mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds \\ &= \int_0^t q_{k,l}\mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds,\end{aligned}$$

where we have used the convergence $\kappa_m(x_m)\mathbb{I}_{(0,+\infty)}(x) \to \mathbb{I}_{(0,+\infty)}(x)$ as $x_m \to x$ in $\mathbb{R}$ and the dominated convergence theorem. Hence, a.s.,

$$\rho_{k,l}(t) = \int_0^t q_{k,l}\mathbb{I}_{\{y_k(s)>0\}}\mathbb{I}_{\{y_l(s)>0\}}ds, \quad t \geq 0,$$

and, consequently, according to Lemma 2.2(ii), a.s.,

$$a_k(t) = \lambda_k\left(1 - \frac{1}{q_{k,k}}\rho_{k,k}(t)\right) = \lambda_k\int_0^t \mathbb{I}_{\{y_k(s)=0\}}ds, \quad t \geq 0,$$

for all $k \in [n]$ such that $q_{k,k} \neq 0$. If $q_{k,k} = 0$, then $\lambda_k = 0$ by the assumption of Proposition 2.1. Therefore, $a_k^m = 0$ implies that $a_k = 0$.

Proposition 2.1 is proved.

Proof of Theorem 2.1. The statement of the theorem directly follows from Lemma 2.2 and Proposition 2.1.

2.2. Tightness. Let a family of nonnegative continuous processes $\{x_k^n(t),\ t \geq 0,\ k \in [n]\}$ be a weak solution to the SDE (2.1)–(2.3), which exists according to Theorem 2.1. Also let the continuous process $\tilde{X}_t^n,\ t \geq 0$, taking values in $\mathrm{C}[0,1]$ be defined by (2.4). We note that $\tilde{X}_t^n(u) \geq 0$ for all $u \in [0,1],\ t \geq 0$, and $n \geq 1$.

The aim of the present section is to prove the tightness of the family $\left\{\tilde{X}^n, n \geq 1\right\}$ in $\mathrm{C}\left([0,\infty), \mathrm{C}[0,1]\right)$. A similar problem was considered in [11] (Section 2), where the author studied the existence of solutions of an SPDE with Lipschitz continuous coefficients. In the cited work, the tightness argument was based on the properties of the fundamental solution to the discrete analog of the heat equation and the fact that coefficients of the equation have at most linear growth. The Lipschitz continuity was not needed for the proof of tightness. Since, in our case, the proof repeats the proof from [11], we only point out its main steps. The main statement of the section can be formulated as follows:

Proposition 2.2. *The family of processes* $\left\{\tilde{X}^n, n \geq 1\right\}$ *is tight in* $\mathrm{C}\left([0,\infty), \mathrm{C}[0,1]\right)$.

For the proof of this proposition, it is sufficient to show that the family $\left\{\tilde{X}^n, n \geq 1\right\}$ is tight in

$$\mathrm{C}\left([0,T], \mathrm{C}[0,1]\right) = \mathrm{C}\left([0,T] \times [0,1], \mathbb{R}\right)$$

for every $T > 0$. Hence, we fix $T > 0$ and apply Corollary 16.9 [17] which yields the property of tightness if $\left\{\tilde{X}^n, n \geq 0\right\}$ satisfies the following conditions:

(i) $\left\{\tilde{X}^n_0(0),\ n \geq 1\right\}$ is tight in $\mathbb{R}$;

(ii) there exist constants $\alpha, \beta, C > 0$ such that

$$\mathbb{E}\left[|\tilde{X}^n_t(u) - \tilde{X}^n_s(v)|^\alpha\right] \leq C\left(|t-s|^{2+\beta} + |u-v|^{2+\beta}\right)$$

for all $t, s \in [0,T],\ u, v \in [0,1]$, and $n \geq 1$.

The family $\left\{\tilde{X}^n, n \geq 1\right\}$ trivially satisfies the first condition because $\tilde{X}^n_0(0) = g^n_1$ is uniformly bounded in $n \geq 1$. In order to check the second condition, we first write equation (2.1) in the integral form. Let $\{p^n_{k,l}(t),$ $t \geq 0,\ k, l \in [n]\}$ be the fundamental solution of the system of ordinary differential equations[4]

$$\frac{d}{dt} p^n_{k,l}(t) = \frac{1}{2}\Delta^n_{(k)} p^n_{k,l}(t), \quad t > 0, \quad k, l \in [n],$$

with the initial condition

$$p^n_{k,l}(0) = n\mathbb{I}_{\{k=l\}}, \quad k, l \in [n],$$

and the following boundary conditions:

$$p^n_{0,l}(t) = \alpha_0 p^n_{1,l}(t), \quad p^n_{n+1,l}(t) = \alpha_0 p^n_{n,l}(t), \quad t \geq 0, \quad l \in [n],$$

where the operator $\Delta^n_{(k)} = \Delta^n$ is applied to the vector $(p_{k,l}(t))^n_{k=1}$ for every $l \in [n]$. Note that $\left\{\left\langle W_t, \sqrt{n}\pi^n_k\right\rangle,\right.$ $\left. t \geq 0,\ k \in [n]\right\}$ is a family of standard Brownian motions. Thus, it is easy to see that $\tilde{X}^n$ has the same distribution as the solution to the integral equation

$$\tilde{X}^n_t(u) = \int_0^1 p^n(t,u,v) g(v) dv + \int_0^t \int_0^1 p^n(t-s,u,v)\tilde{\lambda}^n(v)\mathbb{I}_{\left\{\tilde{X}^n_s(\lceil v \rceil)=0\right\}} ds dv$$

[4] For more details about the properties of the fundamental solution to the discrete analog of the heat equation, see, e.g., [11] (Appendix II).

$$+\int_0^t \int_0^1 p^n(t-s,u,v) f\left(\tilde{X}_s^n(\lceil v \rceil)\right) dsdv$$

$$+\int_0^t \int_0^1 p^n(t-s,u,v) \mathbb{I}_{\left\{\tilde{X}_s^n(\lceil v \rceil)>0\right\}} Q dW_s du, \quad t \geq 0, \quad u \in [0,1], \tag{2.13}$$

where

$$p^n(t,u,v) = (1-n(\lceil u \rceil - u)) p^n_{k,n\lceil v \rceil}(t) + (\lceil u \rceil - u) p^n_{k,n\lceil v \rceil - 1}(t), \quad t \geq 0,$$

$$\tilde{\lambda}^n(v) = \lambda(v) \mathbb{I}_{\left\{q^n_{n\lceil v \rceil, n\lceil v \rceil}>0\right\}}, \quad v \in [0,1],$$

and

$$\lceil v \rceil = \lceil v \rceil^n := \frac{l}{n} \quad \text{for} \quad v \in \pi_l^n \quad \text{and} \quad l \in [n].$$

We denote by $\tilde{X}_t^{n,i}(u)$ the ith term on the right-hand side of equation (2.13).

Lemma 2.3. *For every $\gamma > 0$ and $T > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left(\tilde{X}_t^n(u)\right)^\gamma\right] \leq C$$

for all $t \in [0,T],\ u \in [0,1]$, and $n \geq 1$.

Lemma 2.4. *For each $\gamma \in \mathbb{N}$ and $T > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left|\tilde{X}_{t_2}^{n,i}(u_2) - \tilde{X}_{t_1}^{n,i}(u_1)\right|^{2\gamma}\right] \leq C\left(|t_2 - t_1|^{\frac{\gamma}{2}} + |u_2 - u_1|^{\frac{\gamma}{2}}\right)$$

for every $t_1, t_2 \in [0,T],\ u_1, u_2 \in [0,1],\ n \geq 1$, and $i \in [4]$.

To prove Lemmas 2.3 and 2.4, it is necessary to repeat the proofs of Lemmas 2.1 and 2.2 from [11] based on the properties of the fundamental solution $p^n(t,u,v),\ t \in [0,T],\ u, v \in [0,T]$, and the fact that the coefficients of the equation have at most linear growth. Here, we do not present the proofs of these lemmas.

Proposition 2.2 follows from Lemma 2.4.

Remark 2.2. Let $\tilde{X}_t,\ t \geq 0$, be a limit point of the sequence $\left\{\tilde{X}^n, n \geq 1\right\}$ in $\mathrm{C}([0,\infty), \mathrm{C}[0,1])$, i.e., $\tilde{X}$ is the limit (in distribution) of a subsequence of $\left\{\tilde{X}^n, n \geq 1\right\}$. Then the map $(t,u) \mapsto \tilde{X}_t(u)$ is a.s. locally Hölder continuous with exponent $\alpha \in (0, 1/4)$ according to Lemma 2.4 and Corollary 16.9 [17]. Moreover, Lemma 2.3 and Lemma 4.11 [17] imply that, for every $\gamma > 0$ and $T > 0$, there exists a constant $C = C(T,\gamma)$ such that

$$\mathbb{E}\left[\left(\tilde{X}_t(u)\right)^\gamma\right] \leq C, \quad t \in [0,T], \quad u \in [0,1].$$

3. Passing to the Limit

In the present section, we show that there exists a solution to the SPDE (1.1)–(1.4). The solution is constructed as a limit point of the family of processes $\left\{\tilde{X}^n, n \geq 1\right\}$ from Proposition 2.2, which exists by the Prokhorov theorem. Since the coefficients of the equation are discontinuous, we cannot pass to the limit directly. In the next section, we show that there exists a subsequence of $\left\{\tilde{X}^n, n \geq 1\right\}$ whose weak limit in $\mathrm{C}([0,\infty), \mathrm{C}[0,1])$ is a heat semimartingale.[5] After this, we prove an analog of the Itô formula and state a property similar to the property of ordinary $\mathbb{R}$-valued semimartingales formulated in Lemma A.1 for the analyzed heat semimartingales. Thus, by using the reasoning described in Subsection 1.2, we show that $\tilde{X}$ solves equation (1.1)–(1.4). In this section, $T > 0$ is fixed.

3.1. Martingale Problem for the Limit Points of the Discrete Approximation. We first introduce a new metric space in which we study convergence. Thus, we denote

$$r_0 := \left(1 + \|\lambda\| + \|Q\|_{\mathcal{L}_2}\right)\sqrt{T},$$

and consider the following balls:

$$\mathrm{B}(\mathrm{L}_2) := \left\{f \in \mathrm{L}_2^T :\ \|f\|_{\mathrm{L}_2,T} \leq r_0\right\},$$

$$\mathrm{B}(\mathcal{L}_2) := \left\{L \in \mathcal{L}_2^T :\ \|L\|_{\mathcal{L}_2,T} \leq r_0\right\}$$

in the Hilbert spaces L_2^T and $\mathcal{L}_2^T$, respectively. We equip these sets with the induced weak topologies. By Theorem V.5.1 [5], the indicated topological spaces are metrizable. Moreover, by the Alaoglu Theorem V.4.2 [5], they are compact metric spaces.

For every $n \geq 1,$ we take the family of processes

$$\left\{x_k^n(t),\ t \in [0,T],\ k \in [n]\right\},$$

which is a solution to the SDE (2.1)–(2.3). Let $\tilde{X}_t^n,\ t \in [0,T],$ be a continuous process in $\mathrm{C}[0,1]$ defined by (2.4), i.e.,

$$\tilde{X}_t^n(u) = (un - k + 1)x_k^n(t) + (k - nu)x_{k-1}^n(t), \quad u \in [0,1], \quad t \in [0,T],$$

and let $k \in [n]$ be such that

$$\frac{k-1}{n} \leq u < \frac{k}{n}.$$

We also introduce the process

$$X_t^n := \sum_{k=1}^n x_k^n(t)\pi_k^n, \quad t \in [0,T],$$

where $\pi_k^n = \mathbb{I}_{\left\{\left[\frac{k-1}{n}, \frac{k}{n}\right)\right\}}$. We set

$$\lambda^n := \sum_{k=1}^n n\langle\lambda, \pi_k^n\rangle \mathbb{I}_{\left\{q_{k,k}^n > 0\right\}}\pi_k^n \in \mathrm{L}_2,$$

[5] We call continuous processes in L_2 satisfying Eq. (3.9) in what follows *heat semimartingales*.

$$L_t^n := Q\left[\mathbb{I}_{\{X_t^n>0\}}\cdot\right]\mathrm{pr}^n, \quad t\in[0,T],$$

and

$$\Gamma_t^n := (L_t^n)^*\, L_t^n = \mathrm{pr}^n\left[\mathbb{I}_{\{X_t^n>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{X_t^n>0\}}\cdot\right]\mathrm{pr}^n, \quad t\in[0,T].$$

We can trivially estimate $\|\lambda^n\|\leq\|\lambda\|$ and

$$\|\Gamma_t^n\|_{\mathcal{L}_2}\leq\left\|Q\left[\mathbb{I}_{\{X_t^n>0\}}\cdot\right]\mathrm{pr}^n\right\|_{\mathcal{L}_2}^2\leq\|Q\|_{\mathcal{L}_2}^2, \quad t\in[0,T].$$

The last inequality follows from Lemma A.3. Hence, $\lambda^n\mathbb{I}_{\{X^n=0\}}$ and Γ^n are random elements in L_2^T and $\mathcal{L}_2^T$, respectively. We consider a random element

$$\mathrm{X}^n := \left(\tilde{X}^n, X^n, \lambda^n\mathbb{I}_{\{X^n=0\}}, \mathbb{I}_{\{X^n>0\}}, \Gamma^n\right), \quad n\geq 1, \tag{3.1}$$

in the complete separable metric space

$$\mathcal{W}_{\mathrm{L}_2} = \mathrm{C}\left([0,T],\mathrm{C}[0,1]\right)\times\mathrm{C}\left([0,T],\mathrm{L}_2\right)\times\mathrm{B}\left(\mathrm{L}_2\right)^2\times\mathrm{B}\left(\mathcal{L}_2\right).$$

The following statement is the main result of this section:

Proposition 3.1. *There exists a subsequence of* $\{\mathrm{X}^n, n\geq 1\}$ *that converges to* $\mathrm{X}=\left(\tilde{X},X,a,\sigma,\Gamma\right)$ *in* $\mathcal{W}_{\mathrm{L}_2}$ *in distribution. Moreover, the limit* X *satisfies the following properties:*

(i) $\tilde{X}_t = X_t$ *in* L_2 *for all* $t\in[0,T]$ *a.s. and* $a=\lambda(1-\sigma)$ *in* L_2^T *a.s.;*

(ii) there exists a random element L *in* $\mathcal{L}_2^T$ *such that*

$$\mathbb{P}\left[L^2=\Gamma \text{ and } L \text{ is self-adjoint a.e.}\right]=1$$

and

$$\mathbb{E}\left[\int_0^T\|L_t\|_{\mathcal{L}_2}^2\,dt\right]<+\infty; \tag{3.2}$$

(iii) there exists a continuous square-integrable $(\mathcal{F}_t^{X,M})$*-martingale* $M_t,\ t\in[0,T]$, *in* L_2 *such that, for every* $\varphi\in\mathrm{C}_{\alpha_0}^2[0,1]$,

$$\langle X_t,\varphi\rangle = \langle g,\varphi\rangle + \frac{1}{2}\int_0^t\left\langle X_s,\varphi''\right\rangle ds$$

$$+\int_0^t\langle a_s,\varphi\rangle ds + \int_0^t\langle f(X_s),\varphi\rangle ds + \langle M_t,\varphi\rangle, \quad t\in[0,T], \tag{3.3}$$

and

$$[\langle M_\cdot, \varphi \rangle]_t = \int_0^t \|L_s \varphi\|^2 ds, \quad t \in [0, T].$$

Remark 3.1. In view of equality (3.3) and Theorem 1.2 [23], the process

$$\int_0^t a_s ds, \quad t \in [0, T],$$

is $(\mathcal{F}_t^{X,M})$-adapted.

Proof. We first note that the families $\left\{ \lambda^n \mathbb{I}_{\{X^n=0\}},\ n \geq 1 \right\}$, $\left\{ \mathbb{I}_{\{X^n>0\}},\ n \geq 1 \right\}$, and $\{\Gamma^n, n \geq 1\}$ are tight due to the compactness of the spaces in which they are defined. Consequently, by Proposition 2.2 and Proposition 3.2.4 [6], the family $\left\{ \left(\tilde{X}^n, \lambda^n \mathbb{I}_{\{X^n=0\}}, \mathbb{I}_{\{X^n>0\}}, \Gamma^n \right),\ n \geq 1 \right\}$ is also tight. By the Prokhorov theorem, there exists a subsequence $N \subset \mathbb{N}$ such that

$$\left(\tilde{X}^n, \lambda^n \mathbb{I}_{\{X^n=0\}}, \mathbb{I}_{\{X^n>0\}}, \Gamma^n \right) \to (\tilde{X}, a, \sigma, \Gamma) \quad \text{in distribution}$$

along N. Without loss of generality, we can assume that $N = \mathbb{N}$.

Since

$$\max_{t\in[0,T]} \left\| \tilde{X}_t^n - X_t^n \right\|^2 = \max_{t\in[0,T]} \max_{k\in[n]} \left| x_k^n(t) - x_{k-1}^n(t) \right|^2$$

$$\leq \max_{t\in[0,T]} \sup_{0\leq\delta\leq\frac{1}{n}} \max_{|u-u'|\leq\delta} \left| \tilde{X}_t^n(u) - \tilde{X}_t^n(u') \right|^2,$$

it is easy to see that

$$\max_{t\in[0,T]} \left\| \tilde{X}_t^n - X_t^n \right\|^2 \xrightarrow{d} 0$$

by the Skorokhod representation Theorem 3.1.8 [6] and uniform convergence of $\tilde{X}^n$ to $\tilde{X}$. Hence,

$$\max_{t\in[0,T]} \left\| \tilde{X}_t^n - X_t^n \right\|^2 \to 0 \quad \text{in probability} \quad \text{as} \quad n \to \infty.$$

By using Corollary 3.3.3 [6] and the fact that $\tilde{X}^n,\ n \geq 1$, also converges to $\tilde{X}$ in $\mathrm{C}([0,T], \mathrm{L}_2)$ in distribution, we conclude that

$$X^n \xrightarrow{d} \tilde{X} =: X \quad \text{in} \quad \mathrm{C}([0,T], \mathrm{L}_2).$$

Further, we note that

$$\lambda^n \mathbb{I}_{\{X_t^n=0\}} = (\lambda^n - \lambda)\, \mathbb{I}_{\{X_t^n=0\}} + \lambda \left(1 - \mathbb{I}_{\{X_t^n>0\}} \right), \quad t \in [0, T]. \tag{3.4}$$

By Lemma A.2,

$$(\lambda^n - \lambda)\, \mathbb{I}_{\{X^n=0\}} \to 0 \quad \text{in} \quad \mathrm{B}(\mathrm{L}_2) \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Thus, relation (3.4) yields

$$\lambda^n \mathbb{I}_{\{X^n=0\}} \xrightarrow{d} \lambda\,(1-\sigma) \quad \text{in} \quad \mathrm{B}\,(\mathrm{L}_2) \quad \text{as} \quad n\to\infty.$$

This implies the equality $a=\lambda\,(1-\sigma)$ a.s.

The existence of a convergent subsequence of $\{\mathrm{X}^n\}_{n\geq 1}$ and statement (i) are proved.

The statement (ii) directly follows from Lemma A.5.

In order to prove statement (iii) of the proposition, we first define the following L_2-valued martingale:

$$M_t^n := \sum_{k=1}^n \int_0^t \sqrt{n}\,\mathbb{I}_{\{x_k^n(s)>0\}} dw_k^n(s)\pi_k^n$$

$$= \int_0^t \mathrm{pr}^n\left[\mathbb{I}_{\{X_s^n>0\}}\cdot\right] Q dW_s \;= \int_0^t (L_s^n)^*\, dW_s, \quad t\in[0,T].$$

For $\varphi\in\mathrm{L}_2,$ we set

$$\tilde{\Delta}^n\varphi := n^3\sum_{k=1}^n \Delta^n\varphi_k^n\pi_k^n, \tag{3.5}$$

where

$$\varphi_k^n = \langle\varphi,\pi_k^n\rangle, \qquad \varphi_0^n=\alpha_0\varphi_1^n \qquad \text{and} \qquad \varphi_{n+1}^n=\alpha_0\varphi_n^n.$$

Since $X^n=\sum_{k=1}^n x_k^n\pi_k^n$ and the family $\{x_k^n,\ k\in[n]\}$ solves the SDE (2.1)–(2.3), for every $\varphi\in\mathrm{L}_2,$ we get

$$\langle M_t^n,\varphi\rangle = \langle M_t^n,\mathrm{pr}^n\,\varphi\rangle$$

$$= \langle X_t^n,\mathrm{pr}^n\,\varphi\rangle - \langle g^n,\mathrm{pr}^n\,\varphi\rangle - \frac{1}{2}\int_0^t \left\langle \tilde{\Delta}^n X_s^n,\mathrm{pr}^n\,\varphi\right\rangle ds$$

$$- \int_0^t \left\langle \lambda^n\mathbb{I}_{\{X_s^n=0\}},\mathrm{pr}^n\,\varphi\right\rangle ds - \int_0^t \langle f(X_s^n),\mathrm{pr}^n\,\varphi\rangle\, ds$$

$$= \langle X_t^n,\varphi\rangle - \langle g^n,\varphi\rangle - \frac{1}{2}\int_0^t \left\langle X_s^n,\tilde{\Delta}^n\varphi\right\rangle ds$$

$$- \int_0^t \left\langle \lambda^n\mathbb{I}_{\{X_s^n=0\}},\varphi\right\rangle ds - \int_0^t \langle f(X_s^n),\varphi\rangle\, ds, \quad t\in[0,T], \tag{3.6}$$

and the quadratic variation of the $(\mathcal{F}_t^{X^n})$-martingale $\langle M_\cdot^n,\varphi\rangle$ is equal to

$$[\langle M_\cdot^n,\varphi\rangle]_t = \int_0^t \left\|Q\mathbb{I}_{\{X_s^n>0\}}\,\mathrm{pr}^n\,\varphi\right\|^2 ds, \quad t\in[0,T].$$

Let

$$\tilde{e}_1(u) = 1, \ u \in [0,1], \quad \text{and} \quad \tilde{e}_k(u) = \sqrt{2}\cos\pi(k-1)u, \ u \in [0,1], \ k \ge 2, \quad \text{for} \quad \alpha_0 = 1$$

and let

$$\tilde{e}_k(u) = \sqrt{2}\sin\pi k u, \ u \in [0,1], \ k \ge 1, \quad \text{for} \quad \alpha_0 = 0.$$

Then $\tilde{e}_k \in \mathrm{C}^2_{\alpha_0}[0,1]$ for all $k \ge 1$ and $\{\tilde{e}_k, \ k \ge 1\}$ form an orthonormal basis in L_2. Since

$$\left\| Q\mathbb{I}_{\{X^n_t > 0\}} \operatorname{pr}^n \tilde{e}_k \right\|^2 \le \|Q\|^2, \quad t \in [0,T], \quad k \ge 1,$$

the families $\{\langle M^n_\cdot, \tilde{e}_k\rangle, \ n \ge 1\}$ and $\{[\langle M^n_\cdot, \tilde{e}_k\rangle], n \ge 1\}$ are tight in $\mathrm{C}[0,T]$ for every $k \ge 1$ by the Aldous tightness criterion. In view of the tightness of $\{\mathrm{X}^n, n \ge 1\}$, we can also conclude that $\{\langle X^n_\cdot, \tilde{e}_k\rangle, \ n \ge 1\}$ is tight in $\mathrm{C}[0,T]$ for each $k \ge 1$. By using Proposition 2.4 [6] and the Prokhorov theorem, we can choose a subsequence $N \subset \mathbb{N}$ such that

$$\Big(\langle X^n_\cdot, \tilde{e}_k\rangle, \langle M^n_\cdot, \tilde{e}_k\rangle, [\langle M^n_\cdot, \tilde{e}_k\rangle] \Big)_{k\ge 1} \to \big(\bar{X}_k, \bar{M}_k, \bar{V}_k\big)_{k \ge 1} \tag{3.7}$$

in $\big((\mathrm{C}[0,T])^3\big)^{\mathbb{N}}$ in distribution along N. In particular, we conclude that

$$\langle M^n_\cdot, \tilde{e}_k\rangle^2 - [\langle M^n_\cdot, \tilde{e}_k\rangle], \quad n \ge 1,$$

is a sequence of martingales that converges to $\bar{M}^2_k - \bar{V}_k$ in $\mathrm{C}[0,T]$ in distribution along N for all $k \ge 1$.

We fix $m \ge 1$. Let $(\bar{\mathcal{F}}^{\bar{X},\bar{M},\bar{V},m}_t)_{t\in[0,T]}$ be the complete right continuous filtration generated by $(\bar{X}_k, \bar{M}_k, \bar{V}_k)$, $k \in [m]$. By Proposition IX.1.17 [16], we conclude that $\bar{M}_k$ and $\bar{M}^2_k - \bar{V}_k$ are continuous local $(\bar{\mathcal{F}}^{\bar{X},\bar{M},\bar{V},m}_t)$-martingales for all $k \in [m]$. Since

$$\mathbb{E}\Big[\langle M^n_T, e_k\rangle^2\Big] = \int_0^T \mathbb{E}\Big[\big\|Q\mathbb{I}_{\{X^n_s > 0\}}\tilde{e}_k\big\|^2\Big]\, ds \le \|Q\|^2 T,$$

we get

$$\mathbb{E}\big[\bar{M}^2_k(T)\big] < +\infty$$

by Lemma 4.11 [17]. Hence, $\bar{M}^2_k$ is a continuous square-integrable $\big(\bar{\mathcal{F}}^{\bar{X},\bar{M},\bar{V},m}_t\big)$-martingale with quadratic variation $\big[\bar{M}_k\big] = \bar{V}, \ k \in [m]$. By using Theorem 17.17 [17], we conclude that

$$\mathcal{F}^{\bar{X},\bar{M},\bar{V},m}_t = \bar{\mathcal{F}}^{\bar{X},\bar{M},m}_t, \quad t \in [0,T],$$

where $\big(\bar{\mathcal{F}}^{\bar{X},\bar{M},m}_t\big)_{t\in[0,T]}$ is the complete right continuous filtration generated by $\big(\bar{X}_k, \bar{M}_k\big), \ k \in [m]$. Since, for every $t \in [0,T]$, the σ-algebra $\bar{\mathcal{F}}^{\bar{X},\bar{M},m}_t$ increases to $\bar{\mathcal{F}}^{\bar{X},\bar{M}}_t$ as $m \to \infty$, Theorem 1.6 [20] implies that $\bar{M}_k$ is a continuous square-integrable $\big(\bar{\mathcal{F}}^{\bar{X},\bar{M}}_t\big)$-martingale with quadratic variation $[\bar{M}_k] = \bar{V}_k$ for each $k \ge 1$.

Further, we recall that

$$\left(\tilde{X}^n, X^n, \lambda^n \mathbb{I}_{\{X^n=0\}}, \Gamma^n\right) \to (X, X, a, \Gamma)$$

in $\mathrm{C}([0,T], \mathrm{C}[0,1]) \times \mathrm{C}([0,T], \mathrm{L}_2) \times \mathrm{B}(\mathrm{L}_2) \times \mathrm{B}(\mathcal{L}_2)$ in distribution as $n \to \infty$. By the Skorokhod representation Theorem 3.1.8 [6], we can assume that this sequence converges a.s. Therefore, for every $t \in [0,T]$ and $k \geq 1$,

$$\langle X_t^n, \tilde{e}_k \rangle \to \langle X_t, \tilde{e}_k \rangle =: X_k(t),$$

$$\langle g^n, \tilde{e}_k \rangle \to \langle g, \tilde{e}_k \rangle,$$

$$\int_0^t \left\langle \lambda^n \mathbb{I}_{\{X_s^n=0\}}, \tilde{e}_k \right\rangle ds \to \int_0^t \langle a_s, \tilde{e}_k \rangle \, ds,$$

$$\int_0^t \langle f(X_s^n), \tilde{e}_k \rangle \, ds \to \int_0^t \langle f(X_s), \tilde{e}_k \rangle \, ds,$$

$$[\langle M_\cdot^n, \tilde{e}_k \rangle]_t = \int_0^t \|L_s^n \tilde{e}_k\|^2 \, ds \to \int_0^t \|L_s \tilde{e}_k\|^2 \, ds =: V_k(t)$$

a.s. as $n \to \infty$. By the Taylor formula and the fact that $\tilde{e}_k \in \mathrm{C}^3_{\alpha_0}[0,1],\ k \geq 1,$ it is easy to see that, for every $t \in [0,T]$ and $k \geq 1$,

$$\int_0^t \left\langle X_s^n, \tilde{\Delta}^n \tilde{e}_k \right\rangle ds \to \int_0^t \langle X_s, \tilde{e}_k'' \rangle \, ds \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Consequently, for every $t \in [0,T]$, the sequence $\langle M_t^n, \tilde{e}_k \rangle,\ n \geq 1,$ converges to

$$M_k(t) := \langle X_t, \tilde{e}_k \rangle - \langle g, \tilde{e}_k \rangle - \frac{1}{2} \int_0^t \langle X_s, \tilde{e}_k'' \rangle \, ds \; - \int_0^t \langle a_s, \tilde{e}_k \rangle ds \; - \int_0^t \langle f(X_s), \tilde{e}_k \rangle \, ds$$

a.s. as $n \to \infty$. Thus, for every $m \in \mathbb{N}$ and $t_i \in [0,T],\ i \in [m]$,

$$\left(\left(\langle X_{t_i}^n, \tilde{e}_k \rangle, \langle M_{t_i}^n, \tilde{e}_k \rangle, [\langle M_\cdot^n, \tilde{e}_k \rangle]_{t_i} \right)_{i \in [m]} \right)_{k \geq 1} \to \left((X_k(t_i), M_k(t_i), V_k(t_i))_{i \in [m]} \right)_{k \geq 1} \quad \text{a.s.}$$

in $\left(\mathbb{R}^{3m}\right)^{\mathbb{N}}$ as $n \to \infty$. This fact and convergence (3.7) imply that

$$\mathrm{Law}\left\{ (X_k, M_k, V_k)_{k \geq 1} \right\} = \mathrm{Law}\left\{ \left(\bar{X}_k, \bar{M}_k, \bar{V}_k\right)_{k \geq 1} \right\}$$

in $\left((\mathrm{C}[0,1])^3\right)^{\mathbb{N}}$. Hence, for every $k \geq 1$, the process M_k is a continuous square-integrable $\left(\bar{\mathcal{F}}_t^{X,M}\right)$-martingale with quadratic variation

$$[M_k]_t = V_k(t) = \int_0^t \|L_s\tilde{e}_k\|^2\, ds, \quad t \in [0,T],$$

where $\left(\bar{\mathcal{F}}_t^{X,M}\right)_{t\in[0,T]}$ is the complete right continuous filtration generated by X_k and $M_k,\ k \geq 1$.

We now introduce the following process in L_2:

$$M_t := \sum_{k=1}^{\infty} M_k(t)\tilde{e}_k, \quad t \in [0,T]. \tag{3.8}$$

Note that $M_t,\ t \in [0,T]$, is a well-defined continuous process in L_2. Indeed, by the Burkholder–Davis–Gundy inequality, Lemma A.4, inequality (3.2), and the dominated convergence theorem, for every $n, m \geq 1$, we get

$$\mathbb{E}\left[\max_{t\in[0,T]}\left\|\sum_{k=1}^{n} M_k(t)\tilde{e}_k - \sum_{k=1}^{n+m} M_k(t)\tilde{e}_k\right\|^2\right] = \mathbb{E}\left[\max_{t\in[0,T]}\sum_{k=n+1}^{n+m} M_k^2(t)\right]$$

$$\leq \int_0^T \mathbb{E}\left[\sum_{k,l=n+1}^{n+m}\langle L_t\tilde{e}_k, L_t\tilde{e}_l\rangle\right]dt$$

$$= \int_0^T \mathbb{E}\left[\sum_{k,l=1}^{\infty}\left\langle L_t\widetilde{\mathrm{pr}}^{n,n+m}\tilde{e}_k, L_t\widetilde{\mathrm{pr}}^{n,n+m}\tilde{e}_l\right\rangle\right]dt$$

$$= \int_0^T \mathbb{E}\left[\left\|L_t\widetilde{\mathrm{pr}}^{n,n+m}\right\|_{\mathcal{L}_2}^2\right]dt \to 0 \quad \text{as} \quad n, m \to \infty,$$

where $\widetilde{\mathrm{pr}}^{n,n+m}$ is the orthogonal projection in L_2 onto $\mathrm{span}\{\tilde{e}_k,\ k = n+1, \ldots, n+m\}$. This implies the convergence of series (3.8) and the continuity of $M_t,\ t \in [0,T]$, in L_2.

Since

$$\bar{\mathcal{F}}_t^{X,M} = \mathcal{F}_t^{X,M}, \quad t \in [0,T], \quad \text{and} \quad \langle M_t, \tilde{e}_k\rangle = M_k(t), \quad t \in [0,T], \quad \text{for all} \quad k \geq 1,$$

it is easy to see that M is a continuous square-integrable $(\mathcal{F}_t^{X,M})$-martingale in L_2 with quadratic variation

$$[M]_t = \int_0^t L_s^2 ds = \int_0^t \Gamma_s ds, \quad t \in [0,T].$$

This implies statement (iii).

Proposition 3.1 is proved.

3.2. Property of Quadratic Variation of Heat Semimartingales. In this section, we assume that

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \text{ is a filtered complete probability space,}$$

where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is complete and right continuous. Let $T > 0$ be fixed. Consider a continuous $(\mathcal{F}_t)$-adapted L_2-valued process $Z_t,\ t \in [0,T]$, for which there exist random elements a and L in L_2^T and $\mathcal{L}_2^T$, respectively, such that, for every $\varphi \in \mathrm{C}^2_{\alpha_0}[0,1]$, the processes $\int_0^t \langle a_s, \varphi\rangle ds,\ t \in [0,T]$, and $\int_0^t \|L_s\varphi\|^2 ds$, $t \in [0,T]$, are $(\mathcal{F}_t)$-adapted and

$$\mathcal{M}_Z^{\varphi}(t) := \langle Z_t, \varphi\rangle - \langle Z_0, \varphi\rangle - \frac{1}{2}\int_0^t \langle Z_s, \varphi''\rangle ds - \int_0^t \langle a_s, \varphi\rangle ds, \quad t \in [0,T], \tag{3.9}$$

is a local $(\mathcal{F}_t)$-martingale with quadratic variation

$$[\mathcal{M}_Z^{\varphi}]_t = \int_0^t \|L_s\varphi\|^2 ds, \quad t \in [0,T].$$

Note that the assumptions imposed on L imply that the continuous process $\int_0^t \|L_s\|^2_{\mathcal{L}_2} ds,\ t \in [0,T]$, is well defined and $(\mathcal{F}_t)$-adapted.

In what follows, we consider the case of the Neumann boundary conditions, where $\alpha_0 = 1$. All conclusions of this section have the same form for the Dirichlet boundary conditions with $\alpha_0 = 0$. Let $\{\tilde{e}_k,\ k \geq 1\}$ be the family of eigenfunctions of Δ on $[0,1]$ with Neumann boundary conditions. We recall that

$$\tilde{e}_1(u) = 1, \quad u \in [0,1], \qquad \text{and} \qquad \tilde{e}_k(u) = \sqrt{2}\cos\pi(k-1)u, \quad u \in [0,1], \quad k \geq 2.$$

Denote the orthogonal projection in L_2 onto $\mathrm{span}\{\tilde{e}_k, k \in [n]\}$ by $\widetilde{\mathrm{pr}}^n$.

Let

$$Z_t^n = \widetilde{\mathrm{pr}}^n Z_t, \quad t \geq 0, \qquad \text{and} \qquad a_t^n = \widetilde{\mathrm{pr}}^n a_t, \quad t \in [0,T].$$

We also introduce

$$\dot{Z}_t^n = \sum_{k=1}^n \langle Z_t, \tilde{e}_k\rangle \tilde{e}_k', \quad t \in [0,T], \quad n \geq 1,$$

and note that $\dot{Z}^n,\ n \geq 1$, is a sequence of random elements in L_2^T.

Lemma 3.1.

(i) The equality

$$\mathbb{P}\Big[\dot{Z}^n,\ n \geq 1,\ \textit{converges in}\ \mathrm{L}_2^T\ \textit{and a.e. as}\ n \to \infty\Big] = 1$$

holds.

(ii) Let

$$\dot{Z} := \lim_{n\to\infty} \dot{Z}^n,$$

where the limit is taken a.e. Then $\dot{Z}$ *is a random element in* L_2^T *and, for every* $t \in [0,T]$,

$$\int_0^t \|\dot{Z}_s^n\|^2 ds \to \int_0^t \|\dot{Z}_s\|^2 ds \quad a.s. \quad as \quad n\to\infty.$$

Proof. We set $z_k(t) := \langle Z_t, \tilde{e}_k\rangle,\ t \in [0,T],\ k \geq 1$. Thus, by the definition of Z, for every $k \geq 1$, the process

$$\xi_k(t) := z_k(t) - z_k(0) + \frac{\pi^2(k-1)^2}{2} \int_0^t z_k(s)\, ds - \int_0^t a_k(s)\, ds, \quad t \in [0,T],$$

is a continuous local $(\mathcal{F}_t)$-martingale with quadratic variation

$$[\xi_k]_t = \int_0^t \|L_s \tilde{e}_k\|^2 ds, \quad t \in [0,T],$$

where $a_k(s) := \langle a_s, \tilde{e}_k\rangle$. Denote

$$\sigma_{k,l}^2(t) := \langle L_t \tilde{e}_k, L_t \tilde{e}_l\rangle, \quad t \in [0,T].$$

Note that

$$Z_t^n = \sum_{k=1}^n z_k(t)\tilde{e}_k \quad \text{and} \quad a_t^n = \sum_{k=1}^n a_k(t)\tilde{e}_k, \quad t\in[0,T], \quad n \geq 1.$$

By the Itô formula and the polarization equality, we get

$$\|Z_t^n\|^2 = \|Z_0^n\|^2 - \int_0^t \|\dot{Z}_s^n\|^2 ds$$
$$+ 2\int_0^t \langle a_s^n, Z_s^n\rangle\, ds + \int_0^t \|L_s \widetilde{\mathrm{pr}}^n\|_{\mathcal{L}_2}^2 ds + \mathcal{M}^n(t), \quad t \in [0,T], \tag{3.10}$$

where $\mathcal{M}^n(t),\ t\in[0,T]$, is a continuous local $(\mathcal{F}_t)$-martingale defined as

$$\mathcal{M}^n(t) = 2\sum_{k=1}^n \int_0^t z_k(s) d\xi_k(s), \quad t \in [0,T].$$

As a result of simple computations, we obtain

$$[\mathcal{M}^n]_t = 4\int_0^t \|L_s Z_s^n\|^2 ds, \quad t \in [0,T].$$

Trivially,

$$\|Z_t^n\|^2 \to \|Z_t\|^2 \quad \text{a.s.} \quad \text{as} \quad n \to \infty$$

for all $t \in [0, T]$. By using the dominated convergence theorem, we conclude that

$$\int_0^t \langle a_s^n, Z_s^n \rangle \, ds \to \int_0^t \langle a_s, Z_s \rangle \, ds \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Further, by Lemma A.4 and the dominated convergence theorem,

$$\int_0^t \|L_s \widetilde{\mathrm{pr}}^n\|_{\mathcal{L}_2}^2 ds \to \int_0^t \|L_s\|_{\mathcal{L}_2}^2 \, ds \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Further, we show that $\mathcal{M}^n(t)$ converges in probability. Since $\mathcal{M}^n$ is a local martingale, it is necessary to choose a localization sequence of $(\mathcal{F}_t)$-stopping times defined as follows:

$$\tau_k := \inf \left\{ t \in [0, T] : \int_0^t \|L_s\|_{\mathcal{L}_2}^2 \, ds \geq k \right\} \wedge T.$$

Thus, the processes $\mathcal{M}^n(t \wedge \tau_k), \ t \in [0, T], \ n \geq 1$, are square-integrable $(\mathcal{F}_t)$-martingales for every $k \geq 1$, and $\tau_k \uparrow T$ as $k \to \infty$. By the Burkholder–Davis–Gundy inequality (see, e.g., [15], Theorem III.3.1), for every $k, n, m \geq 1, \ n < m$, we find

$$\mathbb{E}\left[\max_{t \in [0,T]} \left| \mathcal{M}^n(t \wedge \tau_k) - \mathcal{M}^m(t \wedge \tau_k) \right|^2 \right] \leq 16 \mathbb{E}\left[\int_0^{\tau_k} \|L_s \widetilde{\mathrm{pr}}^{n,m} Z_s^m\|^2 \, ds \right],$$

where $\widetilde{\mathrm{pr}}^{n,m}$ is the orthogonal projection in L_2 onto $\mathrm{span}\{\tilde{e}_k, \ k = n+1, \dots, m\}$. Hence, by the dominated convergence theorem,

$$\mathbb{E}\left[\max_{t \in [0,T]} |\mathcal{M}^n(t \wedge \tau_k) - \mathcal{M}^m(t \wedge \tau_k)|^2 \right] \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that there exists a continuous square-integrable $(\mathcal{F}_t)$-martingale $\mathcal{M}_k(t), \ t \in [0, T]$, such that

$$\max_{t \in [0,T]} \left| \mathcal{M}^n(t \wedge \tau_k) - \mathcal{M}_k(t) \right| \to 0 \quad \text{in probability} \quad \text{as} \quad n \to \infty.$$

By Lemma B.11 [3],

$$[\mathcal{M}_k]_t = 4 \int_0^{t \wedge \tau_k} \|L_s Z_s\|^2 \, ds, \quad t \in [0, T].$$

Furthermore, for every $k \geq 1$, we have, a.s.,

$$\mathcal{M}_k = \mathcal{M}_{k+1}(\cdot \wedge \tau_k).$$

We define $\mathcal{M}(t) := \mathcal{M}_k(t)$ for $t \le \tau_k,\ k \ge 1$. Trivially, $\mathcal{M}$ is a continuous local $(\mathcal{F}_t)$-martingale with quadratic variation

$$[\mathcal{M}]_t = 4\int_0^t \|L_s Z_s\|^2\, ds, \quad t \in [0,T].$$

By using Lemma 4.2 [17], $\mathcal{M}^n(t) \to \mathcal{M}(t)$ in probability as $n \to \infty$ for every $t \in [0,T]$.

It has been shown that every term of equality (3.10), except $\int_0^t \|\dot{Z}^n_s\|^2 ds$, converges in probability. Hence, $\int_0^t \|\dot{Z}^n_s\|^2 ds$ also converges in probability. Moreover, this sequence is monotone. By Lemma 4.2 [17], it converges almost surely. By the Fatou lemma,

$$\int_0^T \lim_{n\to\infty} \|\dot{Z}^n_s\|^2 ds < \infty \quad \text{a.s.} \tag{3.11}$$

This yields the convergence of $\left\{\dot{Z}^n_s(\omega)\right\}_{n\ge 1}$ in L_2 for almost all s and ω. Hence, $\dot{Z}^n,\ n \ge 1$, converges to $\dot{Z}$ a.e. a.s. as $n \to \infty$. The equality in the second part of the lemma follows from the monotone convergence theorem and (3.11). In particular,

$$\|\dot{Z}^n\|_{\mathrm{L}_2,T} \to \|\dot{Z}\|_{\mathrm{L}_2,T}.$$

Thus, $\dot{Z}^n \to \dot{Z}$ in L_2^T a.s. according to Proposition 2.12 [17].

Proposition 3.2. *Assume that $F \in \mathrm{C}^2(\mathbb{R})$ has a bounded second derivative and $h \in \mathrm{C}^1[0,1]$. Then*

$$\langle F(Z_t), h\rangle = \langle F(Z_0), h\rangle - \frac{1}{2}\int_0^t \left\langle \left(F'(Z_s)h\right)', \dot{Z}_s\right\rangle ds$$
$$+ \int_0^t \langle F'(Z_s)h, a_s\rangle\, ds + \frac{1}{2}\int_0^t \left\langle L_s\left[F''(Z_s)h\cdot\right], L_s\right\rangle_{\mathcal{L}_2} ds + \mathcal{M}_{F,h}(t), \quad t \in [0,T], \tag{3.12}$$

where $\mathcal{M}_{F,h}(t),\ t \in [0,T]$, is a continuous local $(\mathcal{F}_t)$-martingale with quadratic variation

$$[\mathcal{M}_{F,h}]_t = \int_0^t \left\|L_s F'(Z_s)h\right\|^2 ds, \quad t \in [0,T],$$

and $(F'(Z_s)h)' := F''(Z_s)\dot{Z}_s h + F'(Z_s)h' \in \mathrm{L}_2$.

Proof. As in the proof of Lemma 3.1, we can compute, for every $n \ge 1$,

$$\langle F(Z^n_t), h\rangle = \langle F(Z^n_0), h\rangle - \sum_{k=1}^n \frac{\pi^2(k-1)^2}{2}\int_0^t \langle F'(Z^n_s)h, \tilde{e}_k\rangle\, z_k(s)\, ds$$

$$+\sum_{k=1}^{n}\int_0^t \left\langle F'(Z_s^n)h,\tilde{e}_k\right\rangle a_k(s)\,ds+\frac{1}{2}\sum_{k,l=1}^{n}\int_0^t \left\langle F''(Z_s^n)h\tilde{e}_k,\tilde{e}_l\right\rangle \sigma_{k,l}^2(s)\,ds$$

$$+\sum_{k=1}^{n}\int_0^t \left\langle F'(Z_s^n)h,\tilde{e}_k\right\rangle d\xi_k(s),\quad t\in[0,T].$$

Therefore,

$$\langle F(Z_t^n),h\rangle=\langle F(Z_0^n),h\rangle-\frac{1}{2}\int_0^t \left\langle \left(F'(Z_s^n)h\right)_n',\dot{Z}_s^n\right\rangle ds\;+\int_0^t \left\langle F'(Z_s^n)h,a_s^n\right\rangle ds$$

$$+\int_0^t \left\langle L_s\widetilde{\mathrm{pr}}^n\left[F''(Z_s^n)h\cdot\right],L_s\widetilde{\mathrm{pr}}^n\right\rangle_{\mathcal{L}_2} ds\;+\;\mathcal{M}_{F,h}^n(t),\quad t\in[0,T],$$

where

$$\mathcal{M}_{F,h}^n(t)=\sum_{k=1}^{n}\int_0^t \left\langle F'(Z_s^n)h,\tilde{e}_k\right\rangle d\xi_k(s),$$

and

$$\left(F'(Z_s^n)h\right)_n'=\sum_{k=1}^{n}\left\langle F'(Z_s^n)h,\tilde{e}_k\right\rangle \tilde{e}_k'.$$

The process $\mathcal{M}_{F,h}^n(t),\ t\in[0,T]$, is a continuous local $(\mathcal{F}_t)$-martingale with quadratic variation

$$\left[\mathcal{M}_{F,h}^n\right]_t=\sum_{k,l=1}^{n}\int_0^t \left\langle F'(Z_s^n)h,\tilde{e}_k\right\rangle\left\langle F'(Z_s^n)h,\tilde{e}_l\right\rangle \sigma_{k,l}^2 ds$$

$$=\int_0^t \left\|L_s\widetilde{\mathrm{pr}}^n F'(Z_s^n)h\right\|^2 ds,\quad t\in[0,T].$$

By the boundedness of the second derivative of F, we conclude that there exists a constant $C>0$ such that

$$|F'(x)|\le C(1+|x|)\quad\text{and}\quad |F(x)|\le C(1+|x|^2).$$

Therefore,

$$\langle F(Z_t^n),h\rangle\to\langle F(Z_t),h\rangle\quad\text{a.s.}\quad\text{as}\quad n\to\infty,$$

and

$$F'(Z_t^n)h\to F'(Z_t)h\quad\text{and}\quad F''(Z_t^n)h\to F''(Z_t)h\quad\text{in}\quad \mathrm{L}_2\quad\text{a.s.}\quad\text{as}\quad n\to\infty$$

for all $t\in[0,T]$.

By the dominated convergence theorem and Lemma A.4, we get

$$\int_0^t \left\langle F'(Z_s^n)h, a_s^n \right\rangle ds \to \int_0^t \left\langle F'(Z_s)h, a_s \right\rangle ds \quad \text{a.s.}$$

and

$$\int_0^t \left\langle L_s\, \widetilde{\mathrm{pr}}^n \big[F''(Z_s^n)h\cdot\big], L_s \widetilde{\mathrm{pr}}^n \right\rangle_{\mathcal{L}_2} ds \to \int_0^t \left\langle L_s \left[F''(Z_s)h\cdot\right], L_s \right\rangle_{\mathcal{L}_2} ds \quad \text{a.s.}$$

as $n \to \infty$. By using the localization sequence, one can show that, for every $t \in [0,T]$, $\mathcal{M}_{F,h}^n(t) \to \mathcal{M}_{F,h}(t)$ in probability as in the proof of the previous lemma.

In order to complete the proof of the proposition, we only need to show that $\int_0^t \left\langle \big(F'(Z_s^n)h\big)_n', \dot{Z}_s^n \right\rangle ds$ converges to the corresponding term. By Lemma 3.1, it is sufficient to show that

$$\big(F'(Z_\cdot^n)h\big)_n' \to \big(F'(Z_\cdot)h\big)' = F''(Z_\cdot)\dot{Z}_\cdot h + F'(Z_\cdot)h' \quad \text{a.e.} \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

However, this easily follows from the formula of integration by parts.

Theorem 3.1. *Let the process* $Z_t,\ t \in [0,T]$, *and the random element* $L \in \mathcal{L}_2^T$ *be as above. Assume that* $Z_t \geq 0$ *a.e.,* $t \in [0,T]$. *Then the equality*

$$L_\cdot \left[\mathbb{I}_{\{Z_\cdot \neq 0\}}\cdot\right] = L \quad \textit{in} \quad \mathcal{L}_2^T \quad \textit{a.s.}$$

holds.

Proof. In order to prove the theorem, we use Proposition 3.2. We fix a function $\psi \in \mathrm{C}(\mathbb{R})$ such that

$$\operatorname{supp}\psi \subset [-1,1], \quad 0 \leq \psi(x) \leq 1, \quad x \in \mathbb{R}, \quad \text{and} \quad \psi(0) = 1.$$

Define

$$\psi_\varepsilon(x) := \psi\Big(\frac{x}{\varepsilon}\Big), \quad x \in \mathbb{R}, \qquad \text{and} \qquad F_\varepsilon(x) := \int_{-\infty}^{x} \left(\int_{-\infty}^{y} \psi_\varepsilon(r) dr \right) dy, \quad x \in \mathbb{R}.$$

Then $0 \leq F_\varepsilon'(x) \leq 2\varepsilon,\ x \in \mathbb{R}$, and $F_\varepsilon''(x) \to \mathbb{I}_{\{0\}}(x)$ as $\varepsilon \to 0+$ for all $x \in \mathbb{R}$.

Let a nonnegative function $h \in \mathrm{C}^1[0,1]$ be fixed. By Proposition 3.2,

$$\langle F_\varepsilon(Z_t), h\rangle = \langle F_\varepsilon(Z_0), h\rangle - \frac{1}{2}\int_0^t \left\langle \big(F_\varepsilon'(Z_s)h\big)', \dot{Z}_s \right\rangle ds$$

$$+ \int_0^t \left\langle F_\varepsilon'(Z_s)h, a_s \right\rangle ds \ + \ \frac{1}{2}\int_0^t \left\langle L_s \left[F_\varepsilon''(Z_s)h\cdot\right], L_s \right\rangle_{\mathcal{L}_2} ds \ + \ \mathcal{M}_{F_\varepsilon,h}(t), \quad t \in [0,T],$$

and the quadratic variation of the local $(\mathcal{F}_t)$-martingale $\mathcal{M}_{F_\varepsilon,h}$ is equal to

$$[\mathcal{M}_{F_\varepsilon,h}]_t = \int_0^t \left\|L_s F'_\varepsilon(Z_s)h\right\|^2 ds, \quad t \in [0,T].$$

Letting $\varepsilon \to 0+$, we can immediately conclude that, for every $t \in [0,T]$,

$$|\langle F_\varepsilon(Z_t), h\rangle - \langle F_\varepsilon(Z_0), h\rangle| \le 2\varepsilon \|Z_t - Z_0\| \|h\| \to 0 \quad \text{a.s.}$$

and

$$\left|\int_0^t \langle F'_\varepsilon(Z_s)h, a_s\rangle ds\right| \le 2\varepsilon\|h\| \int_0^t \|a_s\| ds \to 0 \quad \text{a.s.}$$

As in the proof of Lemma 3.1, by using the localization sequence, we can show that

$$M_{F_\varepsilon,h}(t) \to 0 \quad \text{in probability.}$$

By the dominated convergence theorem and Lemma A.4, we get

$$\int_0^t \left\langle L_s\left[F''_\varepsilon(Z_s)h\cdot\right], L_s\right\rangle_{\mathcal{L}_2} ds \to \int_0^t \left\langle L_s\left[\mathbb{I}_{\{Z_s=0\}}h\cdot\right], L_s\right\rangle_{\mathcal{L}_2} ds \quad \text{a.s.}$$

Further, by the dominated convergence theorem and Lemma A.6, we find

$$\int_0^t \left\langle (F'_\varepsilon(Z_s)h)', \dot{Z}_s\right\rangle ds = \int_0^t \left\langle F''_\varepsilon(Z_s)\dot{Z}_s h, \dot{Z}_s\right\rangle ds$$

$$+ \int_0^t \left\langle F'(Z_s)h', \dot{Z}_s\right\rangle ds \ \to \int_0^t \left\|\mathbb{I}_{\{Z_s=0\}}\dot{Z}_s\sqrt{h}\right\|^2 ds = 0 \quad \text{a.s.}$$

For every $t \in [0,T]$, we have

$$\int_0^t \left\langle L_s\left[\mathbb{I}_{\{Z_s=0\}}h\cdot\right], L_s\right\rangle_{\mathcal{L}_2} ds = 0 \quad \text{a.s.}$$

Thus, taking $h = 1$ and applying Lemma A.3, it is easy to see that

$$\int_0^T \left\|L_s\left[\mathbb{I}_{\{Z_s=0\}}\cdot\right]\right\|^2_{\mathcal{L}_2} ds = 0.$$

Theorem 3.1 is proved.

3.3. Proof of the Existence Theorem. In this section, we consider the random element X^n defined in Subsection 3.1. According to Proposition 3.1, there exists a subsequence $N \subset \mathbb{N}$ such that

$$\mathrm{X}^n = \left(\tilde{X}^n, X^n, \lambda^n \mathbb{I}_{\{X^n=0\}}, \mathbb{I}_{\{X^n>0\}}, \Gamma^n\right) \to \left(\tilde{X}, X, a, \sigma, \Gamma\right) \quad \text{in distribution}$$

in $\mathcal{W}_{\mathrm{L}_2}$ along N. As earlier, without loss of generality, we can assume that $N = \mathbb{N}$. Moreover, by the Skorokhod representation theorem, we can assume that this sequence converges a.s. Since

$$\tilde{X}^n \to \tilde{X} \quad \text{in} \quad \mathrm{C}\left([0,T], \mathrm{C}[0,1]\right) \quad \text{a.s.}$$

and, for all $t \in [0,T]$, the equality $\tilde{X}_t = X_t$ holds a.s. in L_2, the inequality

$$\max_{t\in[0,T]} \left\|\tilde{X}^n_t - X^n_t\right\| \le \max_{t\in[0,T]} \sup_{0\le\delta\le\frac{1}{n}} \max_{|u-u'|\le\delta} \left|\tilde{X}^n_t(u) - \tilde{X}^n_t(u')\right|$$

implies that

$$\mathbb{P}[\,\forall t \in [0,T], \quad X^n_t \to X_t \ \text{ a.e.}] = 1. \tag{3.13}$$

I. We first show that $\Gamma = \left[\mathbb{I}_{\{X_\cdot>0\}}\cdot\right] Q^2 \left[\mathbb{I}_{\{X_\cdot>0\}}\cdot\right]$ a.s.

By virtue of Proposition 3.1(ii), there exists a random element L in $\mathcal{L}_2^T$ such that $\Gamma = L^2$ a.s. Next, by Proposition 3.1 and Theorem 3.1, we have

$$L_\cdot \mathbb{I}_{\{X_\cdot>0\}} = L \quad \text{a.s.}$$

Therefore, in view of the convergence of $\Gamma^n = \mathrm{pr}^n\, \mathbb{I}_{\{X^n_\cdot>0\}} Q^2 \mathbb{I}_{\{X^n_\cdot>0\}}\, \mathrm{pr}^n$ to Γ in $\mathrm{B}\left(\mathcal{L}_2\right)$ a.s., we conclude that, for every $t \in [0,T] \cap \mathbb{Q}$ and $k, l \ge 1$, a.s.,

$$\begin{aligned}
\int_0^t \langle \Gamma_s, e_k \odot e_l\rangle_{\mathcal{L}_2} ds &= \int_0^t \langle \Gamma_s e_l, e_k\rangle\, ds = \int_0^t \langle L_s e_l, L_s e_k\rangle\, ds = \int_0^t \left\langle L_s \mathbb{I}_{\{X_s>0\}} e_l,\ L_s \mathbb{I}_{\{X_s>0\}} e_k\right\rangle ds \\
&= \int_0^t \left\langle \Gamma_s \mathbb{I}_{\{X_s>0\}} e_l,\ \mathbb{I}_{\{X_s>0\}} e_k\right\rangle ds = \lim_{n\to\infty} \int_0^t \left\langle \Gamma^n_s \mathbb{I}_{\{X_s>0\}} e_l,\ \mathbb{I}_{\{X_s>0\}} e_k\right\rangle ds \\
&= \lim_{n\to\infty} \int_0^t \left\langle Q \mathbb{I}_{\{X^n_s>0\}}\, \mathrm{pr}^n\left(\mathbb{I}_{\{X_s>0\}} e_l\right),\ Q \mathbb{I}_{\{X^n_s>0\}}\, \mathrm{pr}^n\left(\mathbb{I}_{\{X_s>0\}} e_k\right)\right\rangle ds \\
&= \lim_{n\to\infty} \int_0^t \left\langle Q\, \mathrm{pr}^n\left(\mathbb{I}_{\{X^n_s>0\}} \mathbb{I}_{\{X_s>0\}} e_l\right),\ Q\, \mathrm{pr}^n\left(\mathbb{I}_{\{X^n_s>0\}} \mathbb{I}_{\{X_s>0\}} e_k\right)\right\rangle ds \\
&= \int_0^t \left\langle Q \mathbb{I}_{\{X_s>0\}} e_l, Q \mathbb{I}_{\{X_s>0\}} e_k\right\rangle ds = \int_0^t \left\langle \mathbb{I}_{\{X_s>0\}} Q^2 \mathbb{I}_{\{X_s>0\}}, e_k \odot e_l\right\rangle_{\mathcal{L}_2} ds.
\end{aligned}$$

In the last equality, we have used the fact that

$$\mathbb{I}_{(0,+\infty)}(x_n)\mathbb{I}_{(0,+\infty)}(x) \to \mathbb{I}_{(0,+\infty)}(x) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad x_n \to x,$$

convergence (3.13), and the dominated convergence theorem. Since the family $\{\mathbb{I}_{[0,t]}e_k \odot e_l, t \in [0,T] \cap \mathbb{Q},\ k,l \geq 1\}$ is countable and its linear span is dense in $\mathcal{L}_2^T$, we immediately conclude that

$$\Gamma = \mathbb{I}_{\{X_\cdot>0\}}Q^2\mathbb{I}_{\{X_\cdot>0\}} \quad \text{a.s.} \tag{3.14}$$

II. Let χ^2 be defined by (1.6). We now want to show that

$$\mathbb{I}_{\{\chi>0\}}\sigma = \mathbb{I}_{\{\chi>0\}}\mathbb{I}_{\{X>0\}} \quad \text{in} \quad \mathrm{L}_2^T \quad \text{a.s.} \tag{3.15}$$

However, this directly follows from the following lemma:

Lemma 3.2. *Let $Z_t^n,\ t \in [0,T],\ n \geq 1,$ be a sequence of L_2-valued measurable functions such that $Z_t^n \geq 0$ for all $t \in [0,T]$ and $n \geq 1,$ and let*

$$\mathrm{Leb}_T \otimes \mathrm{Leb}_1\left\{(t,u) \in [0,T] \times [0,1] :\ Z_t^n(u) \not\to Z_t(u)\right\} = 0.$$

If

$$\mathrm{pr}^n\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right]\mathrm{pr}^n \to \left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right] \quad \textit{in} \quad \mathrm{B}\left(\mathcal{L}_2\right) \tag{3.16}$$

and

$$\mathbb{I}_{\{Z^n>0\}} \to \sigma \quad \textit{in} \quad \mathrm{B}\left(\mathrm{L}_2\right)$$

as $n \to \infty$, then

$$\mathbb{I}_{\{\chi>0\}}\sigma = \mathbb{I}_{\{\chi>0\}}\mathbb{I}_{\{Z>0\}}. \tag{3.17}$$

We postpone the proof of the lemma to the end of this section.

III. By using equality (3.15), Proposition 3.1 (i), and assumption (1.7) in Theorem 1.1, we get

$$\begin{aligned} a = \lambda(1-\sigma) &= \lambda\mathbb{I}_{\{\chi>0\}}(1-\sigma) = \lambda\mathbb{I}_{\{\chi>0\}}\left(1-\mathbb{I}_{\{X>0\}}\right) \\ &= \lambda\mathbb{I}_{\{\chi>0\}}\mathbb{I}_{\{X=0\}} = \lambda\mathbb{I}_{\{X=0\}}. \end{aligned} \tag{3.18}$$

Hence, by Proposition 3.1(iii) and equalities (3.14) and(3.18), we conclude that, for every $\varphi \in \mathrm{C}^2_{\alpha_0}[0,1]$, a.s.,

$$\begin{aligned} \langle X_t, \varphi\rangle = \langle g, \varphi\rangle + \frac{1}{2}\int_0^t \left\langle X_s, \varphi''\right\rangle ds + \int_0^t \left\langle \lambda\mathbb{I}_{\{X_s=0\}}, \varphi\right\rangle ds \\ + \int_0^t \langle f(X_s), \varphi\rangle\, ds + \langle M_t, \varphi\rangle, \quad t \in [0,T], \end{aligned} \tag{3.19}$$

and $\langle M_t, \varphi\rangle,\ t \in [0,T],$ is a continuous square-integrable $\left(\mathcal{F}_t^{X,M}\right)$-martingale with quadratic variation

$$[\langle M_\cdot, \varphi\rangle]_t = \int\limits_0^t \left\|Q\mathbb{I}_{\{X_s>0\}}\varphi\right\|^2 ds, \quad t \in [0,T].$$

In particular, (3.19) implies that $\mathcal{F}_t^{X,M} = \mathcal{F}_t^X,\ t \in [0,T]$.

Theorem 1.1 is proved.

Proof of Lemma 3.2. It is easy to see that convergence (3.16) is equivalent to the convergence

$$\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right] \to \left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right] \quad \text{in} \quad \mathrm{B}\left(\mathcal{L}_2\right) \quad \text{as} \quad n \to \infty.$$

Hence, for every $\varphi \in \mathrm{L}_2^T,$ we find

$$\int\limits_0^T \left\|Q\mathbb{I}_{\{Z_t^n>0\}}\varphi_t\right\|^2 dt = \left\langle\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot^n>0\}}\cdot\right], \varphi_\cdot \odot \varphi_\cdot\right\rangle_{\mathcal{L}_2,T}$$

$$\to \left\langle\left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right]Q^2\left[\mathbb{I}_{\{Z_\cdot>0\}}\cdot\right], \varphi_\cdot \odot \varphi_\cdot\right\rangle_{\mathcal{L}_2,T} = \int\limits_0^T \left\|Q\mathbb{I}_{\{Z_t>0\}}\varphi_t\right\|^2 dt \quad \text{as} \quad n \to \infty,$$

where $\varphi_\cdot \odot \varphi_\cdot$ is defined as $\varphi_t \odot \varphi_t,\ t \in [0,T]$. Replacing φ by $e_k\mathbb{I}_{\{Z=0\}}$ for every $k \geq 1,$ we obtain

$$\int\limits_0^T \left\|Q\mathbb{I}_{\{Z_t^n>0\}}\mathbb{I}_{\{Z_t=0\}}e_k\right\|^2 dt \ \to \int\limits_0^T \left\|Q\mathbb{I}_{\{Z_t>0\}}\mathbb{I}_{\{Z_t=0\}}e_k\right\|^2 dt = 0 \quad \text{as} \quad n \to \infty. \tag{3.20}$$

We set

$$\tilde{\mathbb{I}}_t^n := \mathbb{I}_{\{Z_t^n>0\}}\mathbb{I}_{\{Z_t=0\}}, \quad t \in [0,T].$$

Then (3.20) and the equality

$$\int\limits_0^T \left\|Q\tilde{\mathbb{I}}_t^n e_k\right\|^2 dt = \sum_{l=1}^\infty \int\limits_0^T \mu_l^2 \left\langle \tilde{\mathbb{I}}_t^n e_k, e_l\right\rangle^2 dt$$

imply that

$$\int\limits_0^T \left\langle \tilde{\mathbb{I}}_t^n e_k, e_k\right\rangle^2 dt \to 0, \quad n \to \infty,$$

for every $k \geq 1$ such that $\mu_k > 0$. Thus, by the Hölder inequality,

$$\left(\int\limits_0^T \int\limits_0^1 \tilde{\mathbb{I}}_t^n(u)e_k^2(u)dtdu\right)^2 \leq T\int\limits_0^T \left\langle \tilde{\mathbb{I}}_t^n e_k, e_k\right\rangle^2 dt \to 0, \quad n \to \infty.$$

Taking into account the equality $\tilde{\mathbb{I}}_t^n = \left(\tilde{\mathbb{I}}_t^n\right)^2$, $t \in [0,T]$, we can conclude that

$$\tilde{\mathbb{I}}^n e_k \to 0 \quad \text{in} \quad \mathrm{L}_2^T, \quad n \to \infty, \tag{3.21}$$

for every $k \geq 1$ such that $\mu_k > 0$.

We claim that $\chi\tilde{\mathbb{I}}^n$, $n \geq 1$, converges to 0 in L_2^T as $n \to \infty$. Indeed, by convergence (3.21) and the dominated convergence theorem,

$$\left\|\chi\tilde{\mathbb{I}}^n\right\|_{\mathrm{L}_2,T}^2 = \sum_{k=1}^{\infty} \mu_k^2 \int_0^T \int_0^1 \tilde{\mathbb{I}}_t^n(u) e_k^2(u)\,dt\,du \to 0, \quad n \to \infty. \tag{3.22}$$

Further, since $\mathbb{I}_{\{Z^n>0\}} \to \sigma$ in the weak topology of L_2^T as $n \to \infty$ and $\mathbb{I}_{\{Z>0\}}$ and $\mathbb{I}_{\{Z=0\}}$ are uniformly bounded, we trivially obtain

$$\begin{aligned} \mathbb{I}_{\{Z^n>0\}}\mathbb{I}_{\{Z>0\}} &\to \sigma\mathbb{I}_{\{Z>0\}}, \\ \tilde{\mathbb{I}}^n = \mathbb{I}_{\{Z^n>0\}}\mathbb{I}_{\{Z=0\}} &\to \sigma\mathbb{I}_{\{Z=0\}}, \end{aligned} \tag{3.23}$$

in the weak topology of L_2^T as $n \to \infty$. Using the fact that

$$\mathbb{I}_{(0,+\infty)}(x_n)\mathbb{I}_{(0,+\infty)}(x) \to \mathbb{I}_{(0,+\infty)}(x) \quad \text{as} \quad x_n \to x \quad \text{in} \quad \mathbb{R},$$

and the uniqueness of the weak limit, we get

$$\sigma\mathbb{I}_{\{Z>0\}} = \mathbb{I}_{\{Z>0\}}. \tag{3.24}$$

Since $\chi \in \mathrm{L}_2$, convergence (3.23) yields

$$\int_0^T \int_0^1 \chi(u)\tilde{\mathbb{I}}_t^n(u)\,dt\,du \to \int_0^T \int_0^1 \chi(u)\sigma_t(u)\mathbb{I}_{\{Z_t=0\}}(u)\,dt\,du, \quad n \to \infty.$$

On the other hand, $\chi\tilde{\mathbb{I}}^n \to 0$ in L_2^T by (3.22). Hence,

$$\chi\sigma\mathbb{I}_{\{Z=0\}} = 0.$$

The last equality and (3.24) yield

$$\chi\sigma = \chi\sigma\mathbb{I}_{\{Z>0\}} + \chi\sigma\mathbb{I}_{\{Z=0\}} = \chi\mathbb{I}_{\{Z>0\}} \quad \text{in} \quad \mathrm{L}_2^T,$$

which is equivalent to equality (3.17).

Lemma 3.2 is proved.

A. Auxiliary Statements

Lemma A.1. *Let* $\xi_k(t),\ t \geq 0,\ k \in [2]$, *be continuous real-valued semimartingales with respect to the same filtration. Also let the quadratic variations be equal to*

$$[\xi_k, \xi_l]_t = \int_0^t \sigma_{k,l}(s)\, ds, \qquad t \geq 0, \quad k, l \in [2].$$

Then, for all $k, l \in [2]$, *a.s.,*

$$[\xi_k, \xi_l]_t = \int_0^t \sigma_{k,l}(s) \mathbb{I}_{\{\xi_k(s) \neq 0\}} \mathbb{I}_{\{\xi_l(s) \neq 0\}} ds, \quad t \geq 0.$$

Proof. By Theorem 22.5 [17], for $k \in [2]$, we get, a.s.,

$$\begin{aligned} \int_0^t \sigma_{k,k}(s) \mathbb{I}_{\{\xi_k(s)=0\}} ds &= \int_0^t \mathbb{I}_{\{0\}}(\xi_k(s)) d[\xi_k]_s \\ &= \int_{-\infty}^{+\infty} \mathbb{I}_{\{0\}}(x) L_t^{k,x} dx = 0, \quad t \geq 0, \end{aligned}$$

where $L_t^{k,x},\ t \geq 0,\ x \in \mathbb{R}$, is the local time of ξ_k. Applying the Cauchy-type inequality [17] (Proposition 17.9), for every $t \geq 0$, we estimate, a.s.,

$$\int_0^t |\sigma_{1,2}(s)| \mathbb{I}_{\{\xi_1(s)=0\}} ds \leq \int_0^t \sigma_{1,1}(s) \mathbb{I}_{\{\xi_1(s)=0\}} ds \int_0^t \sigma_{2,2}(s)\, ds = 0.$$

Similarly, we get

$$\int_0^t |\sigma_{1,2}(s)| \mathbb{I}_{\{\xi_2(s)=0\}} ds = 0, \quad t \geq 0, \quad \text{a.s.}$$

These equalities immediately yield the statement of the lemma.

Lemma A.2. *Let* λ *be a nonnegative function from* L_2, *let* Q *be a nonnegative-definite self-adjoint Hilbert–Schmidt operator on* L_2, *let* χ^2 *be defined by (1.6), and let*

$$\lambda^n = \sum_{k=1}^n n \langle \lambda, \pi_k^n \rangle \mathbb{I}_{\{q_{k,k}^n > 0\}} \pi_k^n, \quad n \geq 1,$$

where $q_{k,k}^n = n \|Q \pi_k^n\|^2$. *If* $\lambda \mathbb{I}_{\{\chi > 0\}} = \lambda$ *a.e., then* $\lambda^n \to \lambda$ *in* L_2 *as* $n \to \infty$.

Proof. Denote

$$\tilde{\lambda}^n := \mathrm{pr}^n \lambda = \sum_{k=1}^{n} n\langle \lambda, \pi_k^n \rangle \pi_k^n, \quad n \geq 1.$$

In this proof, the functions from L_2 are considered as random elements on the probability space

$$([0,1], \mathcal{B}([0,1]), \mathrm{Leb}_1),$$

where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$. We note that $\tilde{\lambda}^n$ is the conditional expectation $\mathbb{E}[\lambda|\mathcal{S}^n]$ determined on the probability space, where $\mathcal{S}^n = \sigma\left\{\pi_k^n,\ k \in [n]\right\}$. By Proposition 1 [1], $\tilde{\lambda}^n \to \lambda$ in L_2 as $n \to \infty$. In particular, $\tilde{\lambda}^n$ converges to λ in probability as $n \to \infty$.

Let

$$q^n := \sum_{k=1}^{n} n q_{k,k}^n \pi_k^n = \sum_{k=1}^{n} n^2 \|Q\pi_k^n\|^2 \pi_k^n = \sum_{k=1}^{n} \left(n^2 \sum_{l=1}^{\infty} \mu_l^2 \langle e_l, \pi_k^n \rangle^2 \right) \pi_k^n$$

$$= \sum_{l=1}^{\infty} \mu_l^2 \left(\sum_{k=1}^{n} n^2 \langle e_l, \pi_k^n \rangle^2 \pi_k^n \right) = \sum_{l=1}^{\infty} \mu_l^2 \left(\mathrm{pr}^n e_l\right)^2, \quad n \geq 1.$$

Note that, for all $l \geq 1$, $\mathrm{pr}^n e_l \to e_l$ in probability as $n \to \infty$.

We fix a subsequence $N \subset \mathbb{N}$. Then, by Lemma 4.2 [17], there exists a subsequence $N' \subset N$ such that $\tilde{\lambda}^n \to \lambda$ a.s. along N'. By using Lemma 4.2 [17] once again and the diagonalization argument, we can find a subsequence $N'' \subset N'$ such that $\mathrm{pr}^n e_l \to e_l$ a.s. along N'' for all $l \geq 1$. By the Fatou lemma,

$$\liminf_{N'' \ni n \to \infty} q^n \geq \sum_{l=1}^{\infty} \mu_l^2 e_l^2 = \chi^2 \quad \text{a.s.}$$

This inequality and the lower semicontinuity of the map $\mathbb{R} \ni x \mapsto \mathbb{I}_{(0,+\infty)}(x)$ imply that

$$\liminf_{N'' \ni n \to \infty} \mathbb{I}_{\{q^n > 0\}} \geq \mathbb{I}_{\{\chi^2 > 0\}} = \mathbb{I}_{\{\chi > 0\}} \quad \text{a.s.}$$

Hence, by virtue of the equality

$$\lambda^n = \sum_{k=1}^{n} n\langle \lambda, \pi_k^n \rangle \mathbb{I}_{\left\{q_{k,k}^n > 0\right\}} \pi_k^n = \tilde{\lambda}^n \mathbb{I}_{\{q^n > 0\}}, \tag{A.1}$$

and the convergence $\tilde{\lambda}^n \to \lambda$ a.s. along N'', we obtain

$$\liminf_{N'' \ni n \to \infty} \lambda^n = \liminf_{N'' \ni n \to \infty} \tilde{\lambda}^n \mathbb{I}_{\{q^n > 0\}} \geq \lambda \mathbb{I}_{\{\chi > 0\}} = \lambda \quad \text{a.s.}$$

By (A.1), we also have

$$\limsup_{N'' \ni n \to \infty} \lambda^n \leq \limsup_{N'' \ni n \to \infty} \tilde{\lambda}^n = \lambda \quad \text{a.s.}$$

This yields the convergence $\lambda^n \to \lambda$ a.s. along N'' and, hence,

$$\lambda^n \to \lambda \quad \text{in probability} \quad \text{as} \quad n \to \infty$$

by Lemma 4.2 [17]. We also remark that $\lambda^n \leq \tilde{\lambda}^n$, $n \geq 1$, and $\tilde{\lambda}^n \to \lambda$ in L_2. Hence, the dominated convergence Theorem 1.21 [17] implies that $\|\lambda^n\| \to \|\lambda\|$. By Proposition 4.12 [17], $\lambda^n \to \lambda$ in L_2 as $n \to \infty$.

Lemma A.3. *Let $A \in \mathcal{L}_2$ and B_i, $i = 1, 2$, be bounded operators on L_2. Then $AB_i \in \mathcal{L}_2$, $i = 1, 2$, and*

$$\langle AB_1, AB_2 \rangle_{\mathcal{L}_2} = \sum_{n=1}^{\infty} \nu_n^2 \langle B_1^* \varepsilon_n, B_2^* \varepsilon_n \rangle,$$

where $\{\varepsilon_n,\ n \geq 1\}$ and $\left\{\nu_n^2,\ n \geq 1\right\}$ are the eigenvectors and eigenvalues of A^*A, respectively.

Proof. We set

$$A^n := \sum_{l=1}^{n} \nu_l \varepsilon_l \odot \varepsilon_l, \quad n \geq 1.$$

Then it is easy to see that the sequence $\{A^n\}_{n\geq 1}$ converges to $\sqrt{A^*A} = \sum_{l=1}^{\infty} \nu_l \varepsilon_l \odot \varepsilon_l$ in $\mathcal{L}_2$. Hence,

$$\begin{aligned}
\langle AB_1, AB_2 \rangle_{\mathcal{L}_2} &= \sum_{k=1}^{\infty} \langle AB_1 \varepsilon_k, AB_2 \varepsilon_k \rangle = \sum_{k=1}^{\infty} \langle A^*AB_1 \varepsilon_k, B_2 \varepsilon_k \rangle \\
&= \left\langle \sqrt{A^*A} B_1, \sqrt{A^*A} B_2 \right\rangle_{\mathcal{L}_2} = \lim_{n\to\infty} \langle A_n B_1, A_n B_2 \rangle_{\mathcal{L}_2} \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} \langle A_n B_1 \varepsilon_k, A_n B_2 \varepsilon_k \rangle = \lim_{n\to\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{n} \nu_l^2 \langle B_1 \varepsilon_k, \varepsilon_l \rangle \langle B_2 \varepsilon_k, \varepsilon_l \rangle \\
&= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \nu_l^2 \langle \varepsilon_k, B_1^* \varepsilon_l \rangle \langle \varepsilon_k, B_2^* \varepsilon_l \rangle = \sum_{l=1}^{\infty} \nu_l^2 \langle B_1^* \varepsilon_l, B_2^* \varepsilon_l \rangle.
\end{aligned}$$

Lemma A.4. *Let $A \in \mathcal{L}_2$ and let a sequence of bounded operators B_n, $n \geq 1$, in L_2 be pointwise convergent to an operator B, i.e., for every $\varphi \in \mathrm{L}_2$, $B_n \varphi \to B\varphi$ in L_2 as $n \to \infty$. Then B is bounded and $AB_n^* \to AB^*$ in $\mathcal{L}_2$ as $n \to \infty$.*

Proof. We first note that the norms $\|B_n\|$, $n \geq 1$, are uniformly bounded by the Banach–Steinhaus theorem. Consequently, B is a bounded operator on L_2.

Further, we show that $\{AB_n^*\}_{n\geq 1}$ converges to AB^* in the weak topology of $\mathcal{L}_2$. Let $\{\varepsilon_n,\ n \geq 1\}$ and $\{\nu_n^2,\ n \geq 1\}$ be the eigenvectors and eigenvalues of A^*A, respectively. Then, for every $k, l \geq 1$,

$$\begin{aligned}
\langle AB_n^*, \varepsilon_k \odot \varepsilon_l \rangle_{\mathcal{L}_2} &= \langle AB_n^* \varepsilon_l, \varepsilon_k \rangle = \langle \varepsilon_l, B_n A^* \varepsilon_k \rangle \\
&\to \langle \varepsilon_l, BA^* \varepsilon_k \rangle = \langle AB^*, \varepsilon_k \odot \varepsilon_l \rangle_{\mathcal{L}_2} \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since $\mathrm{span}\{\varepsilon_k \odot \varepsilon_l,\ k,l \geq 1\}$ is dense in $\mathcal{L}_2$ and

$$\|AB_n^*\|_{\mathcal{L}_2} \leq \|A\|_{\mathcal{L}_2}\|B_n^*\|, \qquad n \geq 1,$$

is uniformly bounded, the sequence $\{AB_n^*\}_{n\geq 1}$ converges to AB^* in the weak topology of $\mathcal{L}_2$. By the dominated convergence theorem, the uniform boundedness of the norms of $\|B_n\|,\ n \geq 1,$ and Lemma A.3, we obtain

$$\|AB_n^*\|_{\mathcal{L}_2}^2 = \sum_{l=1}^{\infty} \nu_l^2 \|B_n\varepsilon_l\|^2 \to \sum_{l=1}^{\infty} \nu_l^2 \|B\varepsilon_l\|^2 = \|AB^*\|_{\mathcal{L}_2}^2 \quad \text{as} \quad n \to \infty.$$

This implies that $\{AB_n^*\}_{n\geq 1}$ converges to AB^* in the strong topology of $\mathcal{L}_2$.

Let $\mathcal{L}_2^{p,sa}$ be a closed subset of $\mathcal{L}_2$ consisting of nonnegative-definite self-adjoint operators. Consider

$$\mathrm{B}\left(\mathcal{L}_2^{p,sa}\right) := \{L \in \mathrm{B}\left(\mathcal{L}_2\right) : \ L \in \mathcal{L}_2^{p,sa} \ \text{ a.e.}\}.$$

Note that a nonnegative-definite self-adjoint operator A on L_2 has the square root, i.e., there exists a unique nonnegative definite self-adjoint operator $\sqrt{A}$ on L_2 such that $\left(\sqrt{A}\right)^2 = A$. This trivially follows from the spectral theorem.

Lemma A.5.

(i) *The set* $\mathrm{B}\left(\mathcal{L}_2^{p,sa}\right)$ *is closed in* $\mathrm{B}(\mathcal{L}_2)$.

(ii) *For every* $r > 0$, *the set*

$$S_r := \left\{ L \in \mathrm{B}\left(\mathcal{L}_2^{p,sa}\right) : \ \int_0^T \left\|\sqrt{L_t}\right\|_{\mathcal{L}_2}^2 dt \leq r \right\}$$

is closed in $\mathrm{B}(\mathcal{L}_2)$.

(iii) *For every* $r > 0$, *the map* $\Phi^r : \ S_r \to \mathrm{B}\left(\mathcal{L}_2^{p,sa}\right)$ *defined as*

$$\Phi_t^r(L) = \sqrt{L_t}, \qquad t \in [0,T], \quad L \in S_r,$$

is Borel measurable.

Proof. Let $L^n,\ n \geq 1,$ be a sequence from $\mathrm{B}(\mathcal{L}_2^{p,sa})$ convergent to L in $\mathrm{B}(\mathcal{L}_2)$. We take arbitrary $t \in [0,T]$ and $\varphi, \psi \in \mathrm{L}_2$ and consider

$$\int_0^t \langle L_s\varphi, \psi\rangle\, ds = \int_0^t \langle L_s, \psi \odot \varphi\rangle_{\mathcal{L}_2} ds = \lim_{n\to\infty} \int_0^t \langle L_s^n, \psi \odot \varphi\rangle_{\mathcal{L}_2} ds$$

$$= \lim_{n\to\infty} \int_0^t \langle L_s^n\varphi, \psi\rangle\, ds = \lim_{n\to\infty} \int_0^t \langle \varphi, L_s^n\psi\rangle\, ds = \int_0^t \langle \varphi, L_s\psi\rangle\, ds.$$

Due to the fact that the set $\operatorname{span}\left\{\mathbb{I}_{[0,t]}\varphi \odot \psi,\ t \in [0,T],\ \varphi,\psi \in \mathrm{L}_2\right\}$ is dense in $\mathcal{L}_2^T$, we conclude that L is self-adjoint a.e. Similarly, we can show that L is nonnegative definite. Hence, $\mathrm{B}(\mathcal{L}_2^{p,sa})$ is closed.

We now prove (ii). We take a sequence $L^n,\ n \geq 1$, from S_r that converges to L in $\mathrm{B}(\mathcal{L}_2)$ and remark that $L \in \mathrm{B}(\mathcal{L}_2^{p,sa})$ due to (i). Thus,

$$\int_0^T \left\|\sqrt{L_t}\right\|_{\mathcal{L}_2}^2 dt = \int_0^T \left[\sum_{k=1}^{\infty} \left\|\sqrt{L_t} e_k\right\|^2\right] dt = \sum_{k=1}^{\infty} \int_0^T \langle L_t e_k, e_k\rangle\, dt$$

$$\leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} \int_0^T \langle L_t^n e_k, e_k\rangle\, dt = \liminf_{n\to\infty} \int_0^T \left\|\sqrt{L_t^n}\right\|_{\mathcal{L}_2}^2 dt \leq r,$$

by the Fatou lemma and the fact that

$$\int_0^T \langle L_t^n e_k, e_k\rangle \to \int_0^T \langle L_t e_k, e_k\rangle, \qquad n \to \infty, \quad \text{for all} \quad k \geq 1.$$

Therefore, S_r is closed.

In order to check (iii), we first note that it is sufficient to show that, for every $t \in [0,T]$ and $\varphi,\psi \in \mathrm{L}_2$, the map

$$S_r \ni L \mapsto \int_0^T \left\langle \Phi_s^r(L), \mathbb{I}_{[0,t]}(s)\psi \odot \varphi\right\rangle_{\mathcal{L}_2} ds = \int_0^t \langle \Phi_s^r(L)\varphi, \psi\rangle\, ds \in \mathbb{R} \tag{A.2}$$

is Borel measurable. By Theorem 1.2 [23], the Borel σ-algebra on $\mathrm{B}(\mathcal{L}_2)$ coincides with the σ-algebra of all Borel measurable sets of $\mathcal{L}_2^T$ contained in the ball $\mathrm{B}(\mathcal{L}_2)$. Consequently, it is sufficient to show that map (A.2) is Borel measurable as a map from S_r to $\mathbb{R}$, where S_r is embedded with the strong topology of $\mathcal{L}_2^T$. However, in this case, the map (A.2)

$$S_r \ni L \mapsto \int_0^t \langle \Phi_s^r(L)\varphi, \psi\rangle = \int_0^t \langle L_s\varphi, L_s\psi\rangle\, ds$$

is continuous and, hence, Borel measurable.

Lemma A.5 is proved.

Assume that the basis $\{\tilde{e}_k,\ k \geq 1\}$ in L_2 is defined as in Subsection 3.2, i.e.,

$$\tilde{e}_1(u) = 1, \quad u \in [0,1], \quad \text{and} \quad \tilde{e}_k(u) = \sqrt{2}\cos \pi(k-1)u, \quad u \in [0,1], \quad k \geq 2.$$

For $h \in \mathrm{L}_2$, we define

$$\dot{h} = \sum_{n=1}^{\infty} \langle h, \tilde{e}_n\rangle \tilde{e}_n',$$

if the series is convergent in L_2. Note that $\langle \dot{h}, \varphi\rangle = -\langle h, \varphi'\rangle$ for every $\varphi \in \mathrm{C}^1[0,1]$ with $\varphi(0) = \varphi(1) = 0$.

Lemma A.6. *Let $h \in \mathrm{L}_2$ be nonnegative and let $\dot{h}$ exist. Then $\dot{h}\mathbb{I}_{\{0\}}(h) = 0$ a.e.*

Proof. For every $\varepsilon > 0,$ we consider the function

$$\psi_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2} - \varepsilon, \quad x \in \mathbb{R}.$$

Then ψ_ε is continuously differentiable, $\psi_\varepsilon(0) = 0,$ and $\psi_\varepsilon(x) \to |x|$ as $\varepsilon \to 0+$ for all $x \in \mathbb{R}.$ Moreover, $|\psi'_\varepsilon(x)| \leq 1$ and $\psi'_\varepsilon(x) \to \mathrm{sgn}(x)$ for all $x \in \mathbb{R}.$

Take any function $\varphi \in \mathrm{C}[0,1]$ satisfying $\varphi(0) = \varphi(1) = 0.$ By the dominated convergence theorem, it is easy to see that

$$\langle \psi_\varepsilon(h), \varphi' \rangle = -\langle \psi'_\varepsilon(h)\dot{h}, \varphi \rangle.$$

Letting $\varepsilon \to 0+$ and using the nonnegativity of $h,$ we get

$$-\langle \dot{h}, \varphi \rangle = \langle h, \varphi' \rangle = -\langle \mathbb{I}_{(0,+\infty)}(h)\dot{h}, \varphi \rangle.$$

Since φ is arbitrary, we conclude that $\dot{h} = \dot{h}\mathbb{I}_{(0,+\infty)}(h)$ a.e.

Lemma A.6 is proved.

Remark A.1. The same statement of Lemma A.6 remains true if the "cos" basis is replaced by the "sin" basis $\tilde{e}_k = \sqrt{2}\sin \pi k u,\ u \in [0,1],\ k \geq 1.$

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