# Spectral gap estimates for Brownian motion on domains with sticky-reflecting boundary diffusion

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**Abstract**

Introducing an interpolation method we estimate the spectral gap for Brownian motion on general domains with sticky-reflecting boundary diffusion associated to the first nontrivial eigenvalue for the Laplace operator with corresponding Wentzell-type boundary condition. In the manifold case our proofs involve novel applications of the celebrated Reilly formula.

## 1 Introduction and statement of main results

Brownian motion on smooth domains with sticky-reflecting diffusion along the boundary has a long history, dating back at least to Wentzell [34]. As a prototype consider a diffusion on the closure $\overline{\Omega}$ of a smooth domain $\Omega$ with Feller generator $(\mathcal{D}(A), A)$

$$\begin{aligned} \mathcal{D}(A) &= \{f \in C_0(\overline{\Omega}) \,|\, Af \in C_0(\overline{\Omega})\} \\ Af &= \Delta f \mathbb{I}_\Omega + (\beta \Delta^\tau f - \gamma \frac{\partial f}{\partial \nu}) \mathbb{I}_{\partial\Omega} \end{aligned} \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $\Delta^\tau$ is the Laplace-Beltrami operator on the boundary $\partial\Omega$ and $\beta > 0, \gamma \in \mathbb{R}$. The case of pure sticky reflection but no diffusion along the boundary corresponds to the regime $\beta = 0$; models with $\beta > 0$ have appeared recently in interacting particle systems with singular boundary or zero-range pair interaction [1, 7, 13, 19, 27]. The first rigorous process constructions on special domains $\Omega$ were given in [16, 33, 37] and were later extended to jump-diffusion processes on general domains [6] cf. [32]. An efficient construction in symmetric cases was given by Grothaus and Voßhall via Dirichlet forms in [15]. Qualitative regularity properties of the associated semigroups were studied e.g. in [14]. In this note we address the problem of estimating the spectral gap for such processes, which is a natural question also in algorithmic applications. To our knowledge this question has been considered only for $\beta = 0$ by Kennedy [17] and Shouman [30]. However, for $\beta > 0$ the properties of the process change

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significantly, which is indicated by the fact that the energy form of $A$ now also contains a boundary part and which also constitutes the main difference to the closely related work [18].

In the sequel we treat the case when $\gamma > 0$ which corresponds to an inward sticky reflection at $\partial\Omega$. Our ansatz to estimate the spectral gap is based on a simple interpolation idea. To this aim assume that $\Omega$ and $\partial\Omega$ have finite (Hausdorff) measure so that we may choose $\alpha \in (0,1)$ for which

$$\frac{\alpha}{1-\alpha}\frac{|\partial\Omega|}{|\Omega|} = \gamma.$$

Introducing $\lambda_\Omega$ and $\lambda_\partial$ as normalized volume and Hausdorff measures on $\Omega$ and $\partial\Omega$ and setting

$$\lambda_\alpha = \alpha\lambda_\Omega + (1-\alpha)\lambda_\partial,$$

we find that $-A$ is $\lambda_\alpha$-symmetric with first nonzero eigenvalue/spectral gap characterized by the Rayleigh quotient

$$\sigma_{\alpha,\beta} = \inf_{\substack{f\in C^1(\overline{\Omega})\\ \mathrm{Var}_{\lambda_\alpha}(f)>0}} \frac{\mathcal{E}_{\alpha,\beta}(f)}{\mathrm{Var}_{\lambda_\alpha} f},$$

where

$$\mathrm{Var}_{\lambda_\alpha} f = \int_\Omega f^2 d\lambda_\alpha - \left(\int_\Omega f d\lambda_\alpha\right)^2$$

and

$$\mathcal{E}_{\alpha,\beta}(f) = \alpha\int_\Omega \|\nabla f\|^2 d\lambda_\Omega + (1-\alpha)\int_{\partial\Omega} \beta\|\nabla^\tau f\|^2 d\lambda_\partial,$$

and $\nabla^\tau$ denotes the tangential derivative operator on $\partial\Omega$.

This representation of $\sigma_{\alpha,\beta}$ formally interpolates between the two extremal cases of the spectral gap for reflecting Brownian motion on $\Omega$ when $\alpha = 1$ and for Brownian motion on the surface $\partial\Omega$ when $\alpha = 0$. As our main result, in Proposition 2.1 we propose a simple method to estimate $\sigma_{\alpha,\beta}$ from below using only $\sigma_0$ and $\sigma_1$ and estimates for certain bulk-boundary interaction terms which are independent of $\alpha$. The method can lead to quite good results which is illustrated by the example when $\Omega = B_1 \subset \mathbb{R}^d$ is a $d$-dimensional unit ball. When $d = 2$ and $\beta = 1$, for instance, it yields the estimate

$$\sigma_\alpha \geq \frac{8(1+\alpha)\sigma_\Omega}{8(1-\alpha)\sigma_\Omega + 16\alpha + 3\alpha(1-\alpha)\sigma_\Omega} \text{ with } \alpha = \frac{\gamma}{2+\gamma},$$

where $\sigma_0 \approx 3.39$ is the spectral gap for the Neumann Laplacian on the 2-dimensional unit ball, c.f. Section 3.1. – In case when $\Omega$ is a $d$-dimensional manifold with Ricci curvature bounded from below by $k_R > 0$ and with boundary $\partial\Omega$ whose second fundamental form $\mathrm{II}_{\partial\Omega}$ is bounded from below by $k_2 > 0$ we obtain (again with $\beta = 1$, for simplicity) that

$$\sigma_\alpha \geq \min\left(\frac{dk_R}{C_\Omega dk_R + (1-\alpha)(d-1)}, \frac{dk_R}{C_{\partial\Omega}}\frac{2(1-\alpha)+\alpha k_2 C_{\partial\Omega}}{2(1-\alpha)dk_R + \alpha dk_2k_RC_\Omega + \alpha(1-\alpha)(d-1)k_2}\right),$$

where $C_\Omega$ and $C_{\partial\Omega}$ are the usual (Neumann) Poincaré constants of $\Omega$ and $\partial\Omega$ respectively. To derive this result we combine Escobar's lower bound [9] on the first Steklov eigenvalue [12, 20] of $\Omega$ with a novel estimate on the optimal zero mean trace Poincaré constant of $\Omega$ [22, 26], for which we obtain that

$$\int_\Omega f^2 dx \leq \frac{d-1}{dk_R}\int_\Omega |\nabla f|^2,$$

for all $f \in C^1(\Omega)$ with $\int_{\partial\Omega} f dS = 0$, and which is of independent interest. The proof is based on a novel application of Reilly's formula [28] which is also used for a complementary lower bound of $\sigma$ independent of the interpolation approach stating that

$$\sigma_\alpha \geq \min\left(\frac{dk_2}{3d-1}\frac{\alpha}{1-\alpha}\frac{|\partial\Omega|}{|\Omega|}, \frac{d}{d-1}k_R\right),$$

but which is generally weaker for small values of $\alpha$, c.f. Section 3.2.

The interpolation approach also yields a sufficient condition for the continuity of $\sigma_\alpha$ at $\alpha \in \{0, 1\}$, which in general may fail. In Section 2.2 we present sufficient conditions for continuity and discontinuity of $\sigma_\alpha$ at $\{0, 1\}$ which hints towards a phase transition in the associated family of variational problems.

We conclude with the discussion of two applications of the method in non-standard or singular situations, c.f. Sections 3.3 and 3.4.

# 2 An interpolation approach

## 2.1 Generalized framework

It will be convenient to work with a slight generalisation of the setup above. To this aim let $\Omega$ be an open domain in $\mathbb{R}^d$ or a Riemannian manifold with a piecewise smooth boundary $\partial\Omega$. Let $\Sigma$ be a smooth compact and connected subset of $\partial\Omega$. We denote by $\partial\Sigma$ the boundary of $\Sigma$ in the space $\partial\Omega$, *i.e.* $\partial\Sigma = \Sigma \cap \overline{\partial\Omega\backslash\Sigma}$. We consider two probability measures $\lambda_\Omega$ and $\lambda_\Sigma$ with support $\Omega$ and $\Sigma$, which are absolutely continuous with respect to the Lebesgue and the Hausdorff measure on $\Omega$ and $\Sigma$, respectively.

Let $D : C^1(\Omega) \mapsto \Gamma^0(\Omega)$ and $D^\tau : C^1(\partial\Omega) \mapsto \Gamma^0(\partial\Omega)$ denote given first order gradient operators mapping differentiable functions into (tangential) vector fields on $\Omega$ and on $\partial\Omega$, respectively, and for $\alpha \in [0, 1]$ let

$$\lambda_\alpha := \alpha\lambda_\Omega + (1-\alpha)\lambda_\Sigma,$$
$$\mathcal{E}_\alpha(f) := \alpha\int_\Omega \| D f\|^2 d\lambda_\Omega + (1-\alpha)\int_\Sigma \|D^\tau f\|^2 d\lambda_\Sigma, \quad f \in \mathcal{D}_0,$$

where $\mathcal{D}_0 \subset \mathcal{C}^1(\overline{\Omega})$ is dense in $C_0(\Omega)$. We assume that for $\alpha \in [0, 1]$ the quadratic form $(\mathcal{E}_\alpha, \mathcal{D}_0)$ is a pre-Dirichlet form on $L^2(\overline{\Omega}, \lambda_\alpha)$ whose closure we shall denote by $(\mathcal{E}_\alpha, \mathcal{D})$, c.f. [15] for details. We wish to estimate from above $\sigma_\alpha^{-1} = C_\alpha$, where $C_\alpha$ is the optimal Poincaré constant given by

$$C_\alpha := \sup_{\substack{f\in\mathcal{D}_0 \\ \mathcal{E}_\alpha(f)>0}} \frac{\mathrm{Var}_{\lambda_\alpha} f}{\mathcal{E}_\alpha(f)}. \tag{2.1}$$

In the interpolation method presented below it is assumed that $C_\alpha$ are known or can be estimated at the two extremals $\alpha \in \{0, 1\}$. For instance, when $D = \nabla$, $D^\tau = \nabla^\tau$ are the standard gradient resp. tangential gradient operators and $\lambda_\Omega$ and $\lambda_\Sigma$ are normalized Lebesgue resp. Hausdorff measures on $\Omega$ and $\Sigma \subset \partial\Omega$, $C_\Omega := C_1$ is the optimal Poincaré constant associated to the Laplace operator on $\Omega$ with Neumann boundary conditions, whereas $C_\Sigma := C_0$ is the optimal Poincaré constant associated to the Laplace-Beltrami operator on $\Sigma$ with Neumann boundary conditions on $\partial\Sigma$.

The following proposition establishes an estimate of $C_\alpha$ in terms of $C_\Omega$ and $C_\Sigma$.

**Proposition 2.1.** *Assume there exists constants $K_{\Sigma,\Omega}$, $K_1, K_2$ such that for any $f \in \mathcal{D}_0$*

$$\operatorname{Var}_{\lambda_\Sigma} f \leq K_{\Sigma,\Omega} \int_\Omega \| D f\|^2 d\lambda_\Omega, \tag{2.2}$$

*and*

$$\left( \int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma \right)^2 \leq K_1 \int_\Omega \| D f\|^2 d\lambda_\Omega + K_2 \int_\Sigma \|D^\tau f\|^2 d\lambda_\Sigma, \tag{2.3}$$

*then it holds for any $\alpha \in (0,1)$,*

$$C_\alpha \leq \max\left( C_\Omega + (1-\alpha)K_1, \alpha K_2, \frac{(1-\alpha)K_{\Sigma,\Omega}C_\Sigma + \alpha C_\Omega C_\Sigma + \alpha(1-\alpha)(K_{\Sigma,\Omega}K_2 + C_\Sigma K_1)}{(1-\alpha)K_{\Sigma,\Omega} + \alpha C_\Sigma} \right). \tag{2.4}$$

*Proof.* By definition of $C_\Sigma$ and by (2.2), for any $f \in \mathcal{D}_0$

$$\operatorname{Var}_{\lambda_\Sigma} f \leq t K_{\Sigma,\Omega} \int_\Omega \| D f\|^2 d\lambda_\Omega + (1-t) C_\Sigma \int_\Sigma \|D^\tau f\|^2 d\lambda_\Sigma,$$

for any $t \in [0,1]$. Let $\alpha \in (0,1)$. For any $f \in \mathcal{D}_0$ and any $t \in [0,1]$

$$\begin{aligned}
\operatorname{Var}_{\lambda_\alpha} f &= \alpha \operatorname{Var}_{\lambda_\Omega} f + (1-\alpha) \operatorname{Var}_{\lambda_\Sigma} f + \alpha(1-\alpha) \left( \int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma \right)^2 \\
&\leq \left( C_\Omega + \frac{(1-\alpha)t}{\alpha} K_{\Sigma,\Omega} + (1-\alpha)K_1 \right) \alpha \int_\Omega \| D f\|^2 d\lambda_\Omega \\
&\quad + \left( (1-t)C_\Sigma + \alpha K_2 \right)(1-\alpha) \int_\Sigma \|D^\tau f\|^2 d\lambda_\Sigma.
\end{aligned}$$

Therefore,

$$C_\alpha \leq \inf_{t\in[0,1]} \max\left( C_\Omega + \frac{(1-\alpha)t}{\alpha} K_{\Sigma,\Omega} + (1-\alpha)K_1, (1-t)C_\Sigma + \alpha K_2 \right).$$

For any positive constants $a, b, c, d$, we have

$$\inf_{t\in[0,1]} \max(a + bt, c - dt) = \begin{cases} a & \text{if } c - a < 0, \\ c - d & \text{if } c - a > b + d, \\ \frac{bc+ad}{b+d} & \text{if } 0 \leq c - a \leq b + d. \end{cases}$$

Therefore

$$C_\alpha \leq \begin{cases} C_\Omega + (1-\alpha)K_1 & \text{if } \alpha K_2 - (1-\alpha)K_1 + C_\Sigma - C_\Omega < 0, \\ \alpha K_2 & \text{if } \alpha K_2 - (1-\alpha)K_1 - C_\Omega > \frac{1-\alpha}{\alpha} K_{\Sigma,\Omega}, \\ \frac{(1-\alpha)K_{\Sigma,\Omega}C_\Sigma + \alpha C_\Omega C_\Sigma + \alpha(1-\alpha)(K_{\Sigma,\Omega}K_2 + C_\Sigma K_1)}{(1-\alpha)K_{\Sigma,\Omega} + \alpha C_\Sigma} & \text{if } 0 \leq \alpha K_2 - (1-\alpha)K_1 + C_\Sigma - C_\Omega \\ & \qquad \leq C_\Sigma + \frac{1-\alpha}{\alpha} K_{\Sigma,\Omega}. \end{cases}$$

The last term is equivalent to the announced result. □

## 2.2 Continuity of $C_\alpha$

In general, the function $\alpha \mapsto C_\alpha$ might have discontinuities at $\alpha \in \{0,1\}$ in which cases an upper bound for $C_\alpha$ which interpolates continuously between $C_0$ and $C_1$ cannot exist. For example, when $\Omega = (0,b) \times (0,1) \subset \mathbb{R}^2$ and $\Sigma = [0,b] \times \{0\}$, straightforward computations yield

$$\lim_{\alpha \to 0} C_\alpha = \max \left\{ C_\Sigma, \frac{4}{\pi^2} \right\},$$

where $C_\Sigma = \frac{b^2}{\pi^2}$. Hence $\alpha \mapsto C_\alpha$ is discontinuous at $\alpha = 0$ if and only if $b < 2$. – To generalize this to the framework of Section 2.1 let $\mathcal{C}^1_0(\overline{\Omega}) = \{ f \in \mathcal{C}^1(\overline{\Omega}) : f = 0 \text{ on } \Sigma \}$ and

$$\tilde{C}_0 := \sup_{\substack{f \in \mathcal{C}^1_0(\overline{\Omega}) \\ f \text{ non constant}}} \frac{\int_\Omega f^2 d\lambda_\Omega}{\int_\Omega \| D\, f \|^2 d\lambda_\Omega}.$$

(If $D = \nabla$, $\tilde{C}_0$ is the inverse of the spectral gap for Brownian motion on $\Omega$ with killing on $\Sigma$ and normal reflection at $\partial\Omega \setminus \Sigma$. ) We can then record the following statement as a partial corollary to Proposition 2.1.

**Proposition 2.2.** *In the setting of proposition 2.1 it holds that*

$$\underline{\lim}_{\alpha \to 0} C_\alpha \geq \tilde{C}_0.$$

*In particular, if $C_\Sigma < \tilde{C}_0$, then $\alpha \mapsto C_\alpha$ is discontinuous at $\alpha = 0$. Conversely, if $C_\Sigma \geq C_\Omega + K_1$ then $\alpha \mapsto C_\alpha$ is continuous at 0. If $C_\Omega \geq K_2$ continuity at 1 holds.*

*Proof.* To prove the second statement, take a non constant function $g \in \mathcal{C}^1_0(\overline{\Omega})$ and estimate

$$\begin{aligned} \underline{\lim}_{\alpha \to 0} C_\alpha &= \underline{\lim}_{\alpha \to 0} \sup_{\substack{f \in \mathcal{C}^1(\overline{\Omega}) \\ f \text{ non constant}}} \frac{\operatorname{Var}_{\lambda_\alpha} f}{\mathcal{E}_\alpha(f)} \geq \underline{\lim}_{\alpha \to 0} \frac{\operatorname{Var}_{\lambda_\alpha} g}{\mathcal{E}_\alpha(g)} \\ &= \underline{\lim}_{\alpha \to 0} \frac{\alpha \operatorname{Var}_{\lambda_\Omega} g + (1-\alpha) \operatorname{Var}_{\lambda_\Sigma} g + \alpha(1-\alpha) \left( \int_\Omega g d\lambda_\Omega - \int_\Sigma g d\lambda_\Sigma \right)^2}{\alpha \int_\Omega \| D\, g \|^2 d\lambda_\Omega + (1-\alpha) \int_\Sigma \| D^\tau g \|^2 d\lambda_\Sigma}. \end{aligned}$$

Since $g = 0$ on $\Sigma$, we obtain

$$\underline{\lim}_{\alpha \to 0} C_\alpha \geq \underline{\lim}_{\alpha \to 0} \frac{\alpha \operatorname{Var}_{\lambda_\Omega} g + \alpha(1-\alpha) \left( \int_\Omega g d\lambda_\Omega \right)^2}{\alpha \int_\Omega \| D\, g \|^2 d\lambda_\Omega} = \frac{\int_\Omega g^2 d\lambda_\Omega}{\int_\Omega \| D\, g \|^2 d\lambda_\Omega}.$$

Taking the supremum over $g \in \mathcal{C}^1_0(\overline{\Omega})$ yields the first statement.

To prove the second assertion note that $\alpha \mapsto C_\alpha$ is the pointwise supremum of a family of continuous functions and therefore lower semi continuous. Thus $C_\Sigma = C_0 \leq \underline{\lim}_{\alpha \to 0} C_\alpha$. If $C_\Sigma \geq C_\Omega + K_1$, the r.h.s. of inequality (2.4) converges to $C_\Sigma$ as $\alpha$ goes to 0, which implies that $\overline{\lim}_{\alpha \to 0} C_\alpha \leq C_\Sigma$. Similarly, if $C_\Omega \geq K_2$, the r.h.s. of (2.4) converges to $C_\Omega$ as $\alpha$ goes □

**Remark 2.3.** For smooth enough boundary the constant $K_2$ can always be taken equal to zero, hence by proposition 2.2 continuity at $\alpha = 1$ holds. An example where a phase transition appears at $\alpha = 0$ is given in section 3.3. In section 3.4 we present an example where $C_\Omega < K_2$ but continuity of at $\alpha = 1$ can be established via Mosco-convergence [23] of the associated Dirichlet forms, see also [24].

# 3 Examples

## 3.1 Brownian motion on balls with sticky boundary diffusion

As our first example let $\Omega := B_1$ be the unit ball in $\mathbb{R}^d$, $\Sigma = \partial\Omega$ and $D = \nabla$ and $D^\tau = \sqrt{\beta}\,\nabla^\tau$ with $\mathcal{D}_0 = C^1(\overline{\Omega})$.

**Proposition 3.1.** *In the case when $\Omega = B_1 \subset \mathbb{R}^d$ the optimal Poincaré constant of the generator (1.1) is bounded from above by*

$$C_\alpha \leq \max\left(C_\Omega + (1-\alpha)\frac{d+1}{4d^2}, \frac{4(1-\alpha)d + 4\alpha d^2 C_\Omega + \alpha(1-\alpha)(d+1)}{4d(\alpha d + (1-\alpha)\beta(d-1))}\right), \tag{3.1}$$

*where $\alpha = \frac{\gamma}{d+\gamma}$ and $C_\Omega$ is the optimal Poincaré constant for reflecting Brownian motion on $B_1 \subset \mathbb{R}^d$.*

*Proof.* In order to apply Proposition 2.1, it is sufficient to compute the constants $C_\Sigma$, $K_{\Sigma,\Omega}$, $K_1$ and $K_2$. We claim that inequalities (2.2) and (2.3) holds with

$$C_\Sigma = \frac{1}{\beta(d-1)}, \quad K_{\Sigma,\Omega} = \frac{1}{d}, \quad K_1 = \frac{d+1}{4d^2}, \quad K_2 = 0.$$

First, according to [31, Theorem 22.1], the first eigenvalue of the Laplace-Beltrami operator on the unit sphere of dimension $d-1$ is equal to $d-1$, thus $C_\Sigma = \frac{1}{\beta(d-1)}$.

Moreover, according to [3, Theorem 4], for every $f \in \mathcal{C}^1(\partial\Omega)$ one has

$$\left(\int_{\partial\Omega} |f|^q d\lambda_\Sigma\right)^{\frac{2}{q}} \leq \frac{q-2}{d}\int_\Omega \|\nabla u\|^2 d\lambda_\Omega + \int_{\partial\Omega} f^2 d\lambda_\Sigma,$$

for $2 \leq q < \infty$ if $d = 2$ and $2 \leq q < \frac{2d-2}{d-2}$ if $d \geq 3$, where $u$ is the harmonic extension of $f$ to the unit ball $\Omega$. It implies the logarithmic Sobolev inequality $\mathrm{Ent}_{\lambda_\Sigma}(f^2) \leq \frac{2}{d}\int_\Omega \|\nabla u\|^2 d\lambda_\Omega$. Repeating the proof of Proposition 5.1.3 in [2], we get $\mathrm{Var}_{\lambda_\Sigma} f \leq \frac{1}{d}\int_\Omega \|\nabla u\|^2 d\lambda_\Omega$. Moreover, since the harmonic extension of $f$ is minimizing the energy functional $\mathcal{E}_1$ under any function with boundary condition $f$, the last inequality implies for any $f \in \mathcal{C}^1(\overline{\Omega})$

$$\mathrm{Var}_{\lambda_\Sigma} f \leq \frac{1}{d}\int_\Omega \|\nabla f\|^2 d\lambda_\Omega, \tag{3.2}$$

which implies $K_{\Sigma,\Omega} = \frac{1}{d}$.

Furthermore, note that $\int_{\partial\Omega} f(y)\lambda_\Sigma(dy) = \int_\Omega f(\pi_x)\lambda_\Omega(dx)$, where $\pi_x = \frac{x}{\|x\|}$, $x \neq 0$. Hence, using Jensen's inequality and polar coordinates

$$\begin{aligned}\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_\Sigma\right)^2 &\leq \int_\Omega (f(x) - f(\pi_x))^2 \lambda_\Omega(dx) \\ &= \frac{1}{|\Omega|}\int_{\partial\Omega}\int_0^1 (f(ry) - f(y))^2 r^{d-1} dr dy \\ &= \frac{1}{|\Omega|}\int_{\partial\Omega}\int_0^1 \left(\int_r^1 \frac{d}{ds} f(sy) ds\right)^2 r^{d-1} dr dy\end{aligned}$$

$$\le \frac{1}{|\Omega|}\int_{\partial\Omega}\int_0^1 (1-r)\left(\int_r^1 \left(\frac{d}{ds}f(sy)\right)^2 ds\right) r^{d-1}drdy$$
$$= \frac{1}{|\Omega|}\int_{\partial\Omega}\int_0^1 \left[\int_0^s (1-r)r^{d-1}dr\right]\left(\frac{d}{ds}f(sy)\right)^2 dsdy.$$

We separately estimate

$$\int_0^s (1-r)r^{d-1}dr = \left(\frac{s}{d}-\frac{s^2}{d+1}\right)s^{d-1} \le \frac{d+1}{4d^2}s^{d-1}.$$

for any $s \in [0,1]$. Hence,

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_\Sigma\right)^2 \le \frac{d+1}{4d^2|\Omega|}\int_{\partial\Omega}\int_0^1 (\nabla f(sy)\cdot y)^2 s^{d-1}ds$$
$$= \frac{d+1}{4d^2|\Omega|}\int_{\partial\Omega}\int_0^1 \|\nabla f(sy)\|^2 s^{d-1}dsdy$$
$$= \frac{d+1}{4d^2}\int_\Omega \|\nabla f(x)\|^2\lambda_\Omega(dx). \tag{3.3}$$

which implies $K_1 = \frac{d+1}{4d^2}$ and $K_2 = 0$. □

For illustration, in $d = 2$, we compare the bound from Proposition 3.1 for $\beta = 1, \gamma > 0$ to the optimal constant $C_\alpha$ which will be computed numerically. To evaluate the bound (3.1), note that in this case

$$C_\Omega = \frac{1}{\sigma_\Omega} \approx \frac{1}{3.39}, \tag{3.4}$$

where $\sigma_\Omega$ is the smallest positive eigenvalue of the Laplace operator with Neumann boundary condition on the circle. It is given as the minimal positive solution to the equation $J'_m(\sqrt{\gamma}) = 0$, $m \in \mathbb{N}_0$, where $J_m$ is the Bessel function of the first kind of parameter $m$, defined by $J_m(x) = \frac{1}{\pi}\int_0^\pi \cos(mt - x\sin t)dt$, $x \ge 0$. As a consequence, inequality (3.1) becomes

$$C_\alpha \le \frac{8(1-\alpha)\sigma_\Omega + 16\alpha + 3\alpha(1-\alpha)\sigma_\Omega}{8(1+\alpha)\sigma_\Omega}. \tag{3.5}$$

For the numerical computation of $C_\alpha$ one notes that the generator $A_\alpha$ associated with $\mathcal{E}_\alpha$ is defined on $D(A_\alpha) \subset \mathcal{C}^2(\overline{\Omega})$ as

$$A_\alpha f = \mathbb{I}_\Omega \Delta f + \mathbb{I}_{\partial\Omega}\left(\Delta^\tau f - \frac{2\alpha}{1-\alpha}\frac{\partial f}{\partial \nu}\right),$$

where $\Delta^\tau$ and $\frac{\partial}{\partial\nu}$ denote the Laplace-Beltrami operator and the outer normal derivative on the circle $\partial\Omega$. Hence, an eigenvector of $-A_\alpha$ for eigenvalue $\lambda \ge 0$ is a function $f \in D(A_\alpha)$ such that

$$A_\alpha f = -\lambda f \quad \text{in } \Omega.$$

This equation is equivalent to the system of partial differential equations

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega, \\ \Delta^\tau f - \frac{2\alpha}{1-\alpha}\frac{\partial f}{\partial\nu} = -\lambda f & \text{on } \partial\Omega, \end{cases}$$

which by the continuity of $f$ can be rewritten as

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega, \\ \Delta f = \Delta^\tau f - \frac{2\alpha}{1-\alpha}\frac{\partial f}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

Passing to polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta) \in \Omega$ in $d = 2$ and separating variables, we obtain the set of eigenfunctions $\{f^c_{m,l}, f^s_{m,l}\}_{m,l\in\mathbb{N}_0}$,

$$f^c_{m,l}(x_1, x_2) = J_m(\sqrt{\lambda_{m,l}}r)\cos(m\theta), \quad m, l \in \mathbb{N}_0,$$

$$f^s_{m,l}(x_1, x_2) = J_m(\sqrt{\lambda_{m,l}}r)\sin(m\theta), \quad m \in \mathbb{N},\ \ l \in \mathbb{N}_0,$$

where $\lambda_{m,l}$, $l \in \mathbb{N}_0$, are countable family of positive solutions to the equation

$$\sqrt{\lambda}J''_m(\sqrt{\lambda}) + \frac{1+\alpha}{1-\alpha}J'_m(\sqrt{\lambda}) = 0 \tag{3.6}$$

for every $m \in \mathbb{N}_0$. Since the family $\{f^c_{m,l},\ m, l \in \mathbb{N}_0\} \cup \{f^s_{m,l},\ m \in \mathbb{N}_0,\ l \in \mathbb{N}_0\}$ is dense in $L_2(\Omega, \lambda_\alpha)$ and the operator $A_\alpha$ is symmetric, the standard argument implies

$$C_\alpha = \frac{1}{\lambda_{\alpha,\star}}, \tag{3.7}$$

where $\lambda_{\alpha,\star} = \min\limits_{m,l\in\mathbb{N}_0} \lambda_{m,l}$. The resulting curves are plotted in Figure 1.

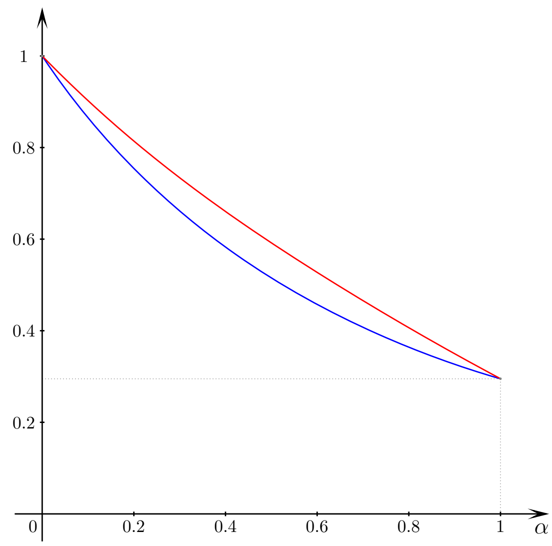



Figure 1: The blue curve represents $\alpha \mapsto C_\alpha$ the optimal Poincaré constant when $\Omega$ is the unit ball of $\mathbb{R}^2$ with full boundary diffusion. The red curve is the upper estimate given by (3.5).

## 3.2 Smooth manifold with boundary

Let $\Omega$ be a smooth compact Riemannian manifold of dimension $d$ with piecewise smooth boundary $\partial\Omega$. We denote by $\mathrm{Ric}$ the Ricci curvature of $\Omega$ and by $\mathrm{II}$ the second fundamental form on the boundary $\partial\Omega$.

Assume in this section that:

$$\text{Assumption (M)}: \qquad \exists k_r > 0, k_2 > 0, \quad \mathrm{Ric}\,|_\Omega \geq k_R\,\mathrm{id} \quad \text{and} \quad \mathrm{II}|_{\partial\Omega} \geq k_2\,\mathrm{id}\,.$$

As before we consider $\Sigma = \partial\Omega$, $D = \nabla$ and $D^\tau = \nabla^\tau$ with $\mathcal{D}_0 = C^1(\overline{\Omega})$.

**Proposition 3.2.** *Under assumption (M), it holds that*

$$C_\alpha \leq \max\left(C_\Omega + \frac{(1-\alpha)(d-1)}{dk_R}, \frac{C_\Sigma}{dk_R} \cdot \frac{2(1-\alpha)dk_R + \alpha dk_2 k_R C_\Omega + \alpha(1-\alpha)(d-1)k_2}{2(1-\alpha) + \alpha k_2 C_\Sigma}\right) =: M_1. \tag{3.8}$$

This statement is obtained via Proposition 2.1 and the two statements below.

**Proposition 3.3.** *Under assumption (M), inequality (2.3) is satisfied with $K_2 = 0$ and*

$$K_1 = \frac{d-1}{dk_R}.$$

*Proof.* Our goal is to obtain an lower bound of

$$\inf_{f \in C^1(\overline{\Omega})} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\left(\int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma\right)^2},$$

where we recall that $\Sigma = \partial\Omega$. We note that

$$\inf_{f \in C^1(\overline{\Omega})} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\left(\int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma\right)^2} = \inf_{\substack{f \in C^1(\overline{\Omega}) \\ \int_\Sigma f d\lambda_\Sigma = 0}} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\left(\int_\Omega f d\lambda_\Omega\right)^2} \geq \inf_{\substack{f \in C^1(\overline{\Omega}) \\ \int_\Sigma f d\lambda_\Sigma = 0}} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\int_\Omega f^2 d\lambda_\Omega} =: \sigma.$$

Let $f \in C^1(\overline{\Omega})$ be a minimizer for $\sigma$. Then $\int_\Sigma f d\lambda_\Sigma = 0$ and

$$\int_\Omega \nabla f \cdot \nabla \xi d\lambda_\Omega = \sigma \int_\Omega f \xi d\lambda_\Omega$$

for each $\xi \in C^1(\overline{\Omega})$ with $\int_\Sigma \xi d\lambda_\Sigma = 0$. By integration by parts, the latter equality is equivalent to

$$-\int_\Omega \Delta f \xi d\lambda_\Omega + \frac{|\Sigma|}{|\Omega|} \int_\Sigma \frac{\partial f}{\partial \nu} \xi d\lambda_\Sigma = \sigma \int_\Omega f \xi d\lambda_\Omega$$

for each $\xi \in C^1(\overline{\Omega})$ satisfying $\int_\Sigma \xi d\lambda_\Sigma = 0$. In particular, choosing $\xi \in C_0^\infty(\Omega)$ (which obviously satisfies $\int_\Sigma \xi d\lambda_\Sigma = 0$), we get that $f$ should satisfy $-\Delta f = \sigma f$ in $\Omega$. Hence $\int_\Sigma \frac{\partial f}{\partial \nu} \xi d\lambda_\Sigma = 0$ for each $\xi$ with zero mean, so it follows that $\int_\Sigma \frac{\partial f}{\partial \nu}\left(\xi - \int_\Sigma \xi d\lambda_\Sigma\right) d\lambda_\Sigma = 0$ for every $\xi \in C^1(\overline{\Omega})$, which is equivalent to

$$\int_\Sigma \left(\frac{\partial f}{\partial \nu} - \int_\Sigma \frac{\partial f}{\partial \nu} d\lambda_\Sigma\right) \xi d\lambda_\Sigma = 0$$

for every $\xi \in C^1(\overline{\Omega})$. It follows that $\frac{\partial f}{\partial \nu}$ is constant on $\Sigma$. Therefore, $f$ satisfies

$$\begin{cases} \Delta f = -\sigma f & \text{in } \Omega, \\ \frac{\partial f}{\partial \nu} \equiv c & \text{on } \partial\Omega, \\ \int_\Sigma f d\lambda_\Sigma = 0, \end{cases} \tag{3.9}$$

for some constant $c$.

Moreover, recall Reilly's formula (see [28])

$$\begin{aligned}\int_\Omega \left((\Delta f)^2 - \|\nabla^2 f\|^2\right) dx = \int_\Omega \mathrm{Ric}(\nabla f, \nabla f) dx \\ + \int_\Sigma \left( H(\frac{\partial f}{\partial \nu})^2 + \mathrm{II}(\nabla^\tau f, \nabla^\tau f) + 2\Delta^\tau f \frac{\partial f}{\partial \nu}\right) dS\end{aligned} \tag{3.10}$$

where $dx$ and $dS$ denote the Riemannian volume resp. surface measure on $\Omega$ and $\partial\Omega$, $\nabla^2 f$ is the Hessian of $f$ and $H$ is the mean curvature of $\Sigma$ (*i.e.* the trace of II). Since $f$ satisfies (3.9),

$$\begin{aligned}\int_\Omega (\Delta f)^2 dx = -\sigma \int_\Omega f \Delta f dx = \sigma \int_\Omega \|\nabla f\|^2 dx - \sigma \int_\Sigma \frac{\partial f}{\partial \nu} f dS \\ = \sigma \int_\Omega \|\nabla f\|^2 dx - \sigma c \int_\Sigma f dS = \sigma \int_\Omega \|\nabla f\|^2 dx,\end{aligned}$$

because $\int_\Sigma f dS = |\Sigma| \int_\Sigma f d\lambda_\Sigma = 0$. Furthermore, note that $\|\nabla^2 f\|^2 = \sum_{i,j} (\partial^2_{ij} f)^2 \geq \sum_{i=1}^d (\partial^2_{ii} f)^2 \geq \frac{1}{d}(\sum_{i=1}^d \partial^2_{ii} f)^2 = \frac{1}{d}(\Delta f)^2$. Therefore, the l.h.s. of (3.10) is bounded by

$$\int_\Omega \left((\Delta f)^2 - \|\nabla^2 f\|^2\right) dx \leq \frac{d-1}{d} \int_\Omega (\Delta f)^2 dx \leq \frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 dx.$$

On the other hand, by assumption (M), $H \geq 0$, $\mathrm{II}(\nabla^\tau f, \nabla^\tau f) \geq 0$ and

$$\int_\Omega \mathrm{Ric}(\nabla f, \nabla f) dx \geq k_R \int_\Omega \|\nabla f\|^2 dx.$$

Since

$$\int_\Sigma \Delta^\tau f \frac{\partial f}{\partial \nu} dS = c \int_\Sigma \Delta^\tau f dS = 0$$

the r.h.s. of (3.10) is bounded from below by $k_R \int_\Omega \|\nabla f\|^2 dx$. It turns out that

$$\frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 dx \geq k_R \int_\Omega \|\nabla f\|^2 dx,$$

which implies that $\sigma \geq \frac{d}{d-1} k_R$. It follows that inequality (2.3) holds with $K_1 = \frac{d-1}{d k_R}$. □

**Remark 3.4.** Instead of using $K_1$ from Proposition 3.3 another admissible choice is

$$K_1' = \frac{|\Omega|}{|\partial\Omega|} B^2 (1 + C_\Omega) < \infty,$$

where $B$ is the optimal Sobolev trace constant of $\Omega$, i.e. the norm of the embedding $H^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$. $B^{-2}$ is the first nontrivial eigenvalue of a Steklov-type eigenvalue problem

$$\begin{cases} -\Delta f + f = 0 & \text{in } \Omega \\ \frac{\partial f}{\partial \nu} = \sigma f & \text{on } \partial\Omega, \end{cases}$$

for which however explicit lower bounds in terms of the geometry of $\Omega$ seem yet unknown [4, 5, 11, 21, 29].

**Proposition 3.5.** *Under assumption (M), inequality* (2.2) *holds with* $K_{\Sigma,\Omega} = \frac{2}{k_2}$.

*Proof.* The optimal choice for $K_{\Sigma,\Omega}$ is $\sigma^{-1}$, where $\sigma$ given by

$$\sigma = \inf_{\substack{f\in C^1(\overline{\Omega}) \\ \int_\Sigma f d\lambda_\Sigma = 0}} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\left(\int_\Sigma f^2 d\lambda_\Sigma\right)^2}$$

is the first nontrivial eigenvalue of the Steklov-problem c.f. [12]

$$\begin{cases} \Delta f = 0 & \text{in } \Omega, \\ \frac{\partial f}{\partial \nu} = \sigma f & \text{on } \partial\Omega. \end{cases}$$

Escobar [9] showed $\sigma \geq \frac{k_2}{2}$ in this case. □

Alternatively, we obtain another upper bound for $C_\alpha$ by a direct application of Reilly's formula.

**Proposition 3.6.** *Under assumption (M) it holds that*

$$C_\alpha \leq \max\left( \frac{(3d-1)(1-\alpha)}{d\alpha k_2} \frac{|\Omega|}{|\partial\Omega|}, \frac{d-1}{dk_R} \right) =: M_2. \tag{3.11}$$

*Proof.* We estimate equivalently from below the first nontrivial eigenvalue $\sigma = C_\alpha^{-1}$ for the problem

$$\begin{cases} \Delta f + \sigma f = 0 & \text{in } \Omega \\ \Delta^\tau f - \gamma \frac{\partial f}{\partial \nu} + \sigma f = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma = \frac{\alpha}{1-\alpha} \frac{|\partial\Omega|}{|\Omega|}$. As in the proof of Proposition 3.3 we apply Reilly's formula (3.10) to the corresponding eigenfunction $f$. In this case, for the l.h.s. we estimae

$$\begin{aligned} \int_\Omega \left((\Delta f)^2 - \|\nabla^2 f\|^2\right) dx &\leq \frac{d-1}{d} \int_\Omega (\Delta f)^2 dx = -\frac{d-1}{d}\sigma \int_\Omega f \Delta f dx \\ &= \frac{d-1}{d}\sigma \int_\Omega \|\nabla f\|^2 dx - \frac{d-1}{d}\sigma \int_\Sigma \frac{\partial f}{\partial \nu} f dS \\ &= \frac{d-1}{d}\sigma \int_\Omega \|\nabla f\|^2 dx - \frac{d-1}{d}\frac{\sigma}{\gamma} \int_\Sigma (\Delta^\tau f + \sigma f) f dS \\ &= \frac{d-1}{d}\sigma \int_\Omega \|\nabla f\|^2 dx + \frac{d-1}{d}\frac{\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS - \frac{d-1}{d}\frac{\sigma^2}{\gamma} \int_\Sigma f^2 dS \\ &\leq \frac{d-1}{d}\sigma \int_\Omega \|\nabla f\|^2 dx + \frac{d-1}{d}\frac{\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS. \end{aligned}$$

Since

$$\begin{aligned} \int_\Sigma \frac{\partial f}{\partial \nu} \Delta^\tau f dS &= \frac{1}{\gamma} \int_\Sigma (\Delta^\tau f + \sigma f) \Delta^\tau f dS \\ &= \frac{1}{\gamma} \int_\Sigma (\Delta^\tau f)^2 dS - \frac{\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS \geq -\frac{\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS \end{aligned}$$

the r.h.s. of (3.10) is bounded from below by

$$\begin{aligned} k_R \int_\Omega \|\nabla f\|^2 dx - \frac{2\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS + \int_\Sigma h |\frac{\partial f}{\partial \nu}|^2 dS + k_2 \int_\Sigma \|\nabla^\tau f\|^2 dS \\ \geq k_R \int_\Omega \|\nabla f\|^2 dx - \frac{2\sigma}{\gamma} \int_\Sigma \|\nabla^\tau f\|^2 dS + k_2 \int_\Sigma \|\nabla^\tau f\|^2 dS. \end{aligned}$$

Combining the two bounds for (3.10) yields

$$\left(\frac{d-1}{d}\sigma - k_R\right)\int_\Omega \|\nabla f\|^2 dx \geq \left(k_2 - \frac{3d-1}{d}\frac{\sigma}{\gamma}\right)\int_\Sigma \|\nabla^\tau f\|^2 dS,$$

which implies that either

$$k_2 - \frac{3d-1}{d}\frac{\sigma}{\gamma} \leq 0, \quad \text{i.e.} \quad \sigma \geq \frac{dk_2\gamma}{3d-1}$$

or

$$\frac{d-1}{d}\sigma - k_R \geq 0, \quad \text{i.e.} \quad \sigma \geq \frac{d}{d-1}k_r.$$

Consequently,

$$\sigma \geq \min\left(\frac{dk_2\gamma}{3d-1}, \frac{d}{d-1}k_R\right).$$

□

**Corollary 3.7.** *Under assumption (M), it holds that*

$$C_\alpha \leq \min(M_1, M_2),$$

*where* $M_1 = M_1(\alpha)$ *and* $M_2 = M_2(\alpha)$ *are defined by* (3.8) *and* (3.11)*, respectively.*

When $\alpha$ goes to 0, $M_1$ tends to $\max(C_\Omega, \frac{d-1}{dk_R}, C_\Sigma)$ and $M_2$ tends to $+\infty$, so the estimation via the interpolation method is always stronger. When $\alpha$ goes to 1, $M_1$ tends to $C_\Omega$ and $M_2$ tends to $\frac{d-1}{dk_R}$, so the relative strength of each method depends on the values of $C_\Omega$, $d$ and $k_R$.

## 3.3 Brownian motion on balls with partial sticky reflecting boundary diffusion

As in Section 3.1, let $\Omega := B_1$ be the unit ball of $\mathbb{R}^2$. Now, define for a fixed $\delta \in (0,1)$

$$\Sigma = \{(\cos\theta, \sin\theta) \in \partial\Omega : -\delta\pi \leq \theta \leq \delta\pi\}, \qquad \Sigma_{\mathrm{N}} := \partial\Omega \backslash \Sigma.$$

**Proposition 3.8.** *It holds that*

$$C_\alpha \leq \max\left(C_\Omega + (1-\alpha)K_1(\delta), \frac{4(1-\alpha)\delta^2 + 8\alpha\delta^3 C_\Omega + 8\alpha(1-\alpha)\delta^3 K_1(\delta)}{(1-\alpha) + 8\alpha\delta^3}\right), \tag{3.12}$$

*where* $C_\Omega = \frac{1}{\sigma_\Omega} \approx \frac{1}{3.39}$ *and* $K_1(\delta) = \left(\sqrt{1-\delta}\pi + \frac{1}{4}\sqrt{\frac{3}{\delta}}\right)^2$.

As previously, we will start by computing the needed constants $C_\Omega$, $C_\Sigma$, $K_{\Sigma,\Omega}$, $K_1$ and $K_2$. The first constant, $C_\Omega = \frac{1}{\sigma_\Omega} \approx \frac{1}{3.39}$, remains unchanged.

**Lemma 3.9.** *The following inequalities hold true*

$$\mathrm{Var}_{\lambda_\Sigma} f \leq C_\Sigma \int_\Sigma \|\nabla^\tau f\|^2 d\lambda_\Sigma, \tag{3.13}$$

$$\mathrm{Var}_{\lambda_\Sigma} f \leq K_{\Sigma,\Omega} \int_\Omega \|\nabla f\|^2 d\lambda_\Omega, \tag{3.14}$$

*where* $C_\Sigma = 4\delta^2$ *and* $K_{\Sigma,\Omega} = \frac{1}{2\delta}$.

*Proof.* Inequality (3.13) corresponds to the Poincaré inequality of the Laplacian on the one-dimensional interval $[-\delta\pi, \delta\pi]$ with Neumann boundary conditions. It is well known (see [2, Prop. 4.5.5]) that the optimal Poincaré constant is given by $C_\Sigma = 4\delta^2$.

Moreover, let us decompose the normalized Hausdorff measure $\lambda_\partial$ on the sphere $\partial\Omega$ into the normalized Hausdorff measure $\lambda_\Sigma$ on $\Sigma$ and the normalized Hausdorff measure $\lambda_{\rm N}$ on $\Sigma_{\rm N}$: $\lambda_\partial = \delta\lambda_\Sigma + (1-\delta)\lambda_{\rm N}$. Therefore

$$\mathrm{Var}_{\lambda_\partial} f = \delta\,\mathrm{Var}_{\lambda_\Sigma} f + (1-\delta)\,\mathrm{Var}_{\lambda_{\rm N}} f + \delta(1-\delta)\left(\int_\Sigma f d\lambda_\Sigma - \int_{\Sigma_{\rm N}} f d\lambda_{\rm N}\right)^2 \geq \delta\,\mathrm{Var}_{\lambda_\Sigma} f,$$

Furthermore, recall that by inequality (3.2), for any $f \in \mathcal{C}^1(\overline{\Omega})$, $\mathrm{Var}_{\lambda_\partial} f \leq \frac{1}{2}\int_\Omega \|\nabla f\|^2 d\lambda_\Omega$. It implies (3.14). □

**Lemma 3.10.** *It holds that*

$$\left(\int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma\right)^2 \leq K_1(\delta)\int_\Omega \|\nabla f\|^2 d\lambda_\Omega$$

*with* $K_1(\delta) = \left(\sqrt{1-\delta}\pi + \frac{1}{4}\sqrt{\frac{3}{\delta}}\right)^2$.

*Proof.* For every $x \in \Omega\backslash\{0\}$ with polar coordinates $(r,\theta)$, $r \in (0,1)$, $\theta \in (-\pi,\pi]$, denote by $p_x$ the point of coordinates $(1,\delta\theta)$ on $\Sigma$. Obviously, $\int_\Sigma f(y)\lambda_\Sigma(dy) = \int_\Omega f(p_x)\lambda_\Omega(dx)$ and by Jensen's inequality

$$I := \left(\int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma\right)^2 \leq \int_\Omega (f(x) - f(p_x))^2\,\lambda_\Omega(dx).$$

Define $g(r,\theta) := f(r\cos(\theta), r\sin(\theta))$. Then

$$I \leq \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}(g(r,\theta) - g(1,\delta\theta))^2 r dr d\theta \leq (\sqrt{J_1} + \sqrt{J_2})^2, \tag{3.15}$$

where $J_1 = \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}(g(r,\theta) - g(r,\delta\theta))^2 r dr d\theta$ and $J_2 = \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}(g(r,\delta\theta) - g(1,\delta\theta))^2 r dr d\theta$. On the one hand

$$\begin{aligned} J_1 &= \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}\left(\int_{\delta\theta}^{\theta}\frac{\partial g}{\partial\theta}(r,u)du\right)^2 r dr d\theta \leq \frac{1-\delta}{\pi}\int_0^1\int_{-\pi}^{\pi}|\theta|\int_{-\pi}^{\pi}\left(\frac{\partial g}{\partial\theta}\right)^2(r,u)du\, r dr d\theta \\ &\leq (1-\delta)\pi^2\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}\left(\frac{1}{r}\frac{\partial g}{\partial\theta}\right)^2(r,u)du\, r dr \leq (1-\delta)\pi^2\int_\Omega \|\nabla f\|^2 d\lambda_\Omega. \end{aligned} \tag{3.16}$$

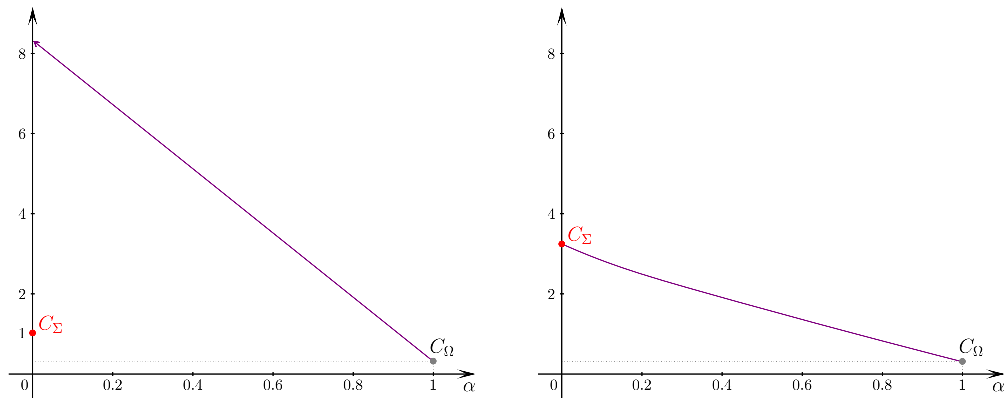
On the other hand

$$J_2 \leq \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}(1-r)\int_r^1\left(\frac{\partial g}{\partial r}\right)^2(s,\delta\theta)ds\, r dr d\theta \leq \frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}\left(\frac{\partial g}{\partial r}\right)^2(s,\delta\theta)\int_0^s(1-r) r dr ds d\theta.$$

For every $s \in [0,1]$, $\int_0^s (1-r) r dr = \frac{s^2}{2} - \frac{s^3}{3} \leq \frac{3s}{16}$, thus

$$J_2 \leq \frac{3}{16\delta\pi}\int_0^1\int_{-\delta\pi}^{\delta\pi}\left(\frac{\partial g}{\partial r}\right)^2(s,u) s ds du \leq \frac{3}{16\delta}\int_\Omega \|\nabla f\|^2 d\lambda_\Omega. \tag{3.17}$$

The proof of the lemma is completed by putting together (3.15), (3.16) and (3.17). □




(a) $\delta = 0.5$

(b) $\delta = 0.9$

Figure 2: The above two figures show the upper estimate given by the r.h.s of (3.12). In the case $\delta = 0.9$ (Figure 2b), the curve interpolates between the extremal constants $C_\Sigma$ and $C_\Omega$, as opposed to the half-sphere case (Figure 2a).

*Proof of Proposition 3.8.* We apply Proposition 2.1 with $C_\Omega = \frac{1}{\sigma_\Omega}$, $C_\Sigma = 4\delta^2$, $K_{\Sigma,\Omega} = \frac{1}{2\delta}$, $K_1(\delta) = \left(\sqrt{1-\delta}\pi + \frac{1}{4}\sqrt{\frac{3}{\delta}}\right)^2$ and $K_2 = 0$. □

For $\delta$ sufficiently large, the map $\alpha \mapsto C_\alpha$ is continuous at $\alpha = 0$. Indeed, by Proposition 2.2, a sufficient condition is $C_\Sigma(\delta) > C_\Omega + K_1(\delta)$, that is

$$4\delta^2 > \frac{1}{\sigma_\Omega} + \left(\sqrt{1-\delta}\pi + \frac{1}{4}\sqrt{\frac{3}{\delta}}\right)^2,$$

which is satisfied for any $\delta \geq 0.862$.

### 3.4 Ball with a needle

Our final example is the unit ball $\Omega = B_1$ of $\mathbb{R}^2$ with a needle $\mathcal{L}$ of length $L$ attached to one point of the boundary, i.e. $\mathcal{L} := \{(x, 0) : 1 \leq x \leq L + 1\}$, see Figure 3. The attachment point and the endpoint of the needle are denoted by $x_0 := (1, 0)$ and $x_L = (L + 1, 0)$, respectively.

In that setting, we define $\overline{\Omega} = \overline{B_1} \cup \mathcal{L}$, $\Sigma = \partial B_1 \cup \mathcal{L}$ and

$$\lambda_\alpha = \alpha\lambda_\Omega + (1-\alpha)\lambda_\Sigma,$$

where $\lambda_\Omega$ is as previously the normalized Lebesgue measure on $\Omega$ and $\lambda_\Sigma = \frac{2\pi}{2\pi+L}\lambda_\partial + \frac{L}{2\pi+L}\lambda_\mathcal{L}$, with $\lambda_\partial$ and $\lambda_\mathcal{L}$ being the normalized Hausdorff measures on $\partial\Omega$ and $\mathcal{L}$, respectively. We choose

$$\mathcal{D}_0 = \left\{ f \in C_0(\overline{\Omega})) \cap C^1(\overline{\Omega} \setminus \{x_0\}) \mid \frac{\partial f}{\partial e_1} + \frac{\partial f}{\partial e_2} + \frac{\partial f}{\partial e_3} = 0 \text{ at } x_0 \right\},$$

where $e_1 = (0, 1)$, $e_2 = (0, -1)$ and $e_3 = (1, 0)$ are the three "tangent" vectors to $\Sigma$ at point $x_0$, and $D := \nabla$, $D^\tau := \sqrt{\beta}\nabla^\tau$, which is well defined in $\Sigma \setminus \{x_0\}$. With this choice, for $\alpha \in [0, 1]$ $(\mathcal{E}_\alpha, \mathcal{D}_0)$ is

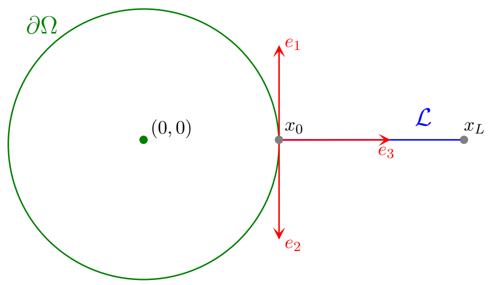



Figure 3: The ball (in green) is denoted by $\Omega$, the boundary of the ball is denoted by $\partial\Omega$ and the needle (in blue) is denoted by $\mathcal{L}$.

a pre-Dirichlet form on $L^2(\overline{\Omega}, \lambda_\alpha)$, whose closure generates Brownian motion on $\Omega$ with sticky boundary diffusion on $\Sigma$, i.e. whose generator is given by

$$A_\alpha(f) = \Delta f \mathbb{I}_\Omega + \beta \Delta_\Sigma f \mathbb{I}_\Sigma - \frac{\alpha}{1-\alpha} \frac{2\pi + L}{\pi} \frac{\partial f}{\partial \nu} \mathbb{I}_{\partial\Omega},$$

with $\Delta_\Sigma$ being the generator of the canonical diffusion on $\Sigma$ with reflecting boundary condition at $x_L$. As before, the optimal Poincaré constant $C_\alpha$ for $A_\alpha$ is given by

$$C_\alpha := \sup_{\substack{f \in \mathcal{D}_0 \\ \mathcal{E}_\alpha(f) > 0}} \frac{\operatorname{Var}_{\lambda_\alpha} f}{\mathcal{E}_\alpha(f)},$$

and let $C_\Omega := C_1$ and $C_\Sigma := C_0$. In this case the following estimate is obtained.

**Proposition 3.11.**

$$C_\alpha \leq \max\left( \frac{1}{\sigma_\Omega} + \frac{3}{8}(1-\alpha), \frac{1}{\beta\gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta(2\pi + L)} \right),$$

*where $\gamma_L > 0$ is the smallest positive solution to*

$$2\cos(\sqrt{\gamma} L)(1 - \cos(\sqrt{\gamma} 2\pi)) + \sin(\sqrt{\gamma} L)\sin(\sqrt{\gamma} 2\pi) = 0. \tag{3.18}$$

*Note that $\gamma_L \leq 1$ for any $L > 0$ and if $L = 2\pi$, $\gamma_{2\pi} = \left(\frac{\arccos(-1/3)}{2\pi}\right)^2 \approx 0.0925$.*

Let us compute the constants needed to apply Proposition 2.1. As we do not expect an inequality of type (2.2) to hold in that case, we set $K_{\Sigma,\Omega} := +\infty$. Moreover, $C_\Sigma$ can be computed exactly as follows.

**Lemma 3.12.** *In this case, $C_\Sigma = \frac{1}{\beta\gamma_L}$.*

*Proof.* The constant $\frac{1}{C_\Sigma}$ is the smallest non-zero eigenvalue $\gamma$ of the following problem:

$$\begin{cases} \beta \Delta^\tau f = -\gamma f & \text{on } \Sigma \backslash \{x_0\}, \\ \frac{\partial f}{\partial \nu} = 0 & \text{at point } x_L, \\ \frac{\partial f}{\partial e_1} + \frac{\partial f}{\partial e_2} + \frac{\partial f}{\partial e_3} = 0 & \text{at point } x_0, \end{cases}$$

where $\Delta^\tau$ is the Laplace-Beltrami operator on $\partial\Omega$ and $\mathcal{L}$. A general solution to that boundary value problem is given by

$$f(x) = \begin{cases} A\cos(\sqrt{\frac{\gamma}{\beta}}y) + B\sin(\sqrt{\frac{\gamma}{\beta}}y) & \text{if } x = (y,0) \in \mathcal{L}, \\ C\cos(\sqrt{\frac{\gamma}{\beta}}\theta) + D\sin(\sqrt{\frac{\gamma}{\beta}}\theta) & \text{if } x = (\cos\theta, \sin\theta) \in \partial\Omega, \end{cases}$$

where $A$, $B$, $C$ and $D$ have to satisfy the continuity assumption of $f$ at point $x_0$ and both boundary conditions, that is:

$$\begin{cases} A = C = C\cos(\sqrt{\dfrac{\gamma}{\beta}}2\pi) + D\sin(\sqrt{\dfrac{\gamma}{\beta}}2\pi), \\ 0 = -A\sin(\sqrt{\dfrac{\gamma}{\beta}}L) + B\cos(\sqrt{\dfrac{\gamma}{\beta}}L), \\ 0 = B + D + C\sin(\sqrt{\dfrac{\gamma}{\beta}}2\pi) - D\cos(\sqrt{\dfrac{\gamma}{\beta}}2\pi). \end{cases}$$

A short computation shows that this system has a non-trivial solution if and only if $\frac{\gamma}{\beta}$ solves (3.18). Therefore, $\frac{1}{C_\Sigma} = \beta\gamma_L$. Obviously, $\gamma = 1$ is a solution to (3.18), thus $\gamma_L \leq 1$. ☐

Next, we look for the constants $K_1$ and $K_2$.

**Lemma 3.13.** *Inequality* (2.3) *holds with* $K_1 = \frac{3}{8}$ *and* $K_2 = \frac{L^2(\pi+L)}{\beta(2\pi+L)}$.

*Proof.* Recall that $\Sigma = \partial\Omega \cup \mathcal{L}$. Let us insert the average of $f$ over $\partial\Omega$ as follows:

$$\begin{aligned} \left(\int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma\right)^2 &\leq 2\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_\partial\right)^2 + 2\left(\int_{\partial\Omega} f d\lambda_\partial - \int_\Sigma f d\lambda_\Sigma\right)^2 \\ &\leq \frac{3}{8}\int_\Omega \|\nabla f\|^2 d\lambda_\Omega + 2\left(\int_{\partial\Omega} f d\lambda_\partial - \int_\Sigma f d\lambda_\Sigma\right)^2, \end{aligned}$$

where the second inequality follows directly from (3.3). Moreover, recalling that $\lambda_\Sigma = \frac{2\pi}{2\pi+L}\lambda_\partial + \frac{L}{2\pi+L}\lambda_\mathcal{L}$

$$\left(\int_{\partial\Omega} f d\lambda_\partial - \int_\Sigma f d\lambda_\Sigma\right)^2 = \frac{L^2}{(2\pi+L)^2}\left(\int_{\partial\Omega} f d\lambda_\partial - \int_\mathcal{L} f d\lambda_\mathcal{L}\right)^2.$$

For every $x = (\cos\theta, \sin\theta) \in \partial\Omega$, with $\theta \in (-\pi, \pi]$, we denote by $p_x$ the point of $\mathcal{L}$ with coordinates $(1 + L - \frac{|\theta|L}{\pi}, 0)$. It follows that

$$\left(\int_{\partial\Omega} f d\lambda_\partial - \int_\mathcal{L} f d\lambda_\mathcal{L}\right)^2 = \left(\int_{\partial\Omega}(f(x) - f(p_x))d\lambda_\partial\right)^2 \leq \int_{\partial\Omega}(f(x) - f(p_x))^2 d\lambda_\partial.$$

Denoting by $\lambda_\partial^+$ and $\lambda_\partial^-$ the normalized Hausdorff measures on $\partial\Omega^+ := \{(x,y) \in \partial\Omega : y > 0\}$ and $\partial\Omega^- := \{(x,y) \in \partial\Omega : y < 0\}$, respectively,

$$\int_{\partial\Omega}(f(x) - f(p_x))^2 d\lambda_\partial = \frac{1}{2}\int_{\partial\Omega^+}(f(x) - f(p_x))^2 d\lambda_\partial^+ + \frac{1}{2}\int_{\partial\Omega^-}(f(x) - f(p_x))^2 d\lambda_\partial^-.$$

Moreover, for any $\mathcal{C}^1$-function $g : [-\pi, L] \to \mathbb{R}$,

$$\frac{1}{\pi}\int_0^\pi \left|g(-\theta) - g(L - \tfrac{\theta L}{\pi})\right|^2 d\theta \leq \frac{\pi + L}{2}\int_{-\pi}^L |g'(t)|^2 dt,$$

so we deduce, identifying $\partial\Omega^+$ with $[-\pi, 0]$ and $\mathcal{L}$ with $[0, L]$, that

$$\int_{\partial\Omega^+} (f(x) - f(p_x))^2 d\lambda_\partial^+ \leq \frac{\pi + L}{2} \left( \pi \int_{\partial\Omega^+} \|\nabla^\tau f\|^2 d\lambda_\partial^+ + L \int_{\mathcal{L}} \|\nabla^\tau f\|^2 d\lambda_{\mathcal{L}} \right)$$

and using symmetry to deal with $\partial\Omega^-$, we obtain

$$\begin{aligned}\int_{\partial\Omega} (f(x) - f(p_x))^2 d\lambda_\partial &\leq \frac{\pi + L}{4} \left( \pi \int_{\partial\Omega^+} \|\nabla^\tau f\|^2 d\lambda_\partial^+ + \pi \int_{\partial\Omega^-} \|\nabla^\tau f\|^2 d\lambda_\partial^- + 2L \int_{\mathcal{L}} \|\nabla^\tau f\|^2 d\lambda_{\mathcal{L}} \right) \\ &\leq \frac{(\pi + L)(2\pi + L)}{2} \int_\Sigma \|\nabla^\tau f\|^2 d\lambda_\Sigma.\end{aligned}$$

Putting together the above inequalities, we get

$$\left( \int_\Omega f d\lambda_\Omega - \int_\Sigma f d\lambda_\Sigma \right)^2 \leq \frac{3}{8} \int_\Omega \|\nabla f\|^2 d\lambda_\Omega + 2 \frac{L^2}{(2\pi + L)^2} \frac{(\pi + L)(2\pi + L)}{2\beta} \int_\Sigma \beta \|\nabla^\tau f\|^2 d\lambda_\Sigma$$

which leads to inequality (2.3) with $K_1 = \frac{3}{8}$ and $K_2 = \frac{L^2(\pi+L)}{\beta(2\pi+L)}$. ☐

*Proof of Proposition 3.11.* Since $K_{\Sigma,\Omega} = \infty$, we immediately get from Proposition 2.1 that

$$C_\alpha \leq \max\left(C_\Omega + (1-\alpha)K_1, \alpha K_2, C_\Sigma + \alpha K_2\right) = \max\left(C_\Omega + (1-\alpha)K_1, C_\Sigma + \alpha K_2\right).$$

Therefore,

$$C_\alpha \leq \max\left( \frac{1}{\sigma_\Omega} + \frac{3}{8}(1-\alpha), \frac{1}{\beta\gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta(2\pi + L)} \right), \tag{3.19}$$

where $\sigma_\Omega \approx 3.39$. ☐

**Remark 3.14.** If $\beta$ is large enough, that is if the diffusion velocity is larger on $\Sigma$ than on $\Omega$, then the first term in (3.19) dominates. Precisely, if $\beta \geq \sigma_\Omega \left( \frac{1}{\gamma_L} + \frac{L^2(\pi+L)}{2\pi+L} \right)$, then (3.19) rewrites for any $\alpha$

$$C_\alpha \leq \frac{1}{\sigma_\Omega} + \frac{3}{8}(1-\alpha).$$

Conversely, if $\beta \leq \frac{1}{\gamma_L} \left( \frac{1}{\sigma_\Omega} + \frac{3}{8} \right)^{-1}$, then (3.19) rewrites for any $\alpha$

$$C_\alpha \leq \frac{1}{\beta\gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta(2\pi + L)}.$$

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