A quantitative central limit theorem for the simple symmetric exclusion process

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Abstract A quantitative central limit theorem for the simple symmetric exclusion process (SSEP) on a d-dimensional discrete torus is proven. The argument is based on a comparison of the generators of the density fluctuation field of the SSEP and the generalized Ornstein-Uhlenbeck process, as well as on an infinite-dimensional Berry-Essen bound for the initial particle fluctuations. The obtained rate of convergence is optimal.

Contents

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1 Introduction

We consider the simple symmetric exclusion process (SSEP) on the d-dimensional discrete torus $\mathbb{T}_n^d := \left\{ \frac{2\pi}{2n+1}k : \; k \in \{-n, \ldots, n\}^d \right\}$. This is a continuous time Markov process that describes the evolution of particles located at points of $\mathbb{T}_n^d$, where each side can contain at most one particle. A particle at site $x \in \mathbb{T}_n^d$ attempts to jump to one of the nearest neighboring sides after an exponential waiting time. If the target side is occupied, then the jump does not take place.

As usual, the state space for SSEP is $\{0,1\}^{\mathbb{T}_n^d}$, where $\eta(x) = 1$ provided the side x is occupied by a particle, and $\eta(x) = 0$ otherwise. The generator of the SSEP is defined by

$$\mathcal{G}_n^{EP} F(\eta) := \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n} \left[F(\eta^{x \leftrightarrow x+e_j}) - F(\eta) \right], \quad \eta \in \{0,1\}^{\mathbb{T}_n^d}, \tag{1.1}$$

for each function $F : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$, where

$$\eta^{x \leftrightarrow y}(z) = \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases} \qquad z \in \mathbb{T}_n^d,$$

and $e_j = e_j^n$ denote the canonical vectors of $\mathbb{T}_n^d$. For a function $\rho : \mathbb{T}_n^d \to [0,1]$, we let ν_ρ^n be the product measure on $\{0,1\}^{\mathbb{T}_n^d}$ with marginals given by $\nu_\rho^n \{\eta(x) = 1\} = \rho(x)$, $x \in \mathbb{T}_n^d$. Let $\eta^n = (\eta_t^n)_{t \geq 0}$ be the SSEP with the initial distribution $\nu_{\rho_0^n}^n$, where $\rho_0^n : \mathbb{T}_n^d \to [0,1]$ and the sequence ρ_0^n, $n \geq 1$, converges to a profile $\rho_0 : \mathbb{T} \to [0,1]$ as $n \to \infty$. It is well-known [46, Theorem 2.1] that the hydrodynamic limit of η^n, $n \geq 1$, is given by the solution to the heat equation

$$d\rho_t^\infty = 2\pi^2 \Delta \rho_t^\infty dt \tag{1.2}$$

on $\mathbb{T}^d$ starting from ρ_0.

By [31, 32, 56], also a central limit theorem (CLT) is known. Precisely, it is known that the density fluctuation field

$$\zeta_t^n(x) := (2n+1)^{d/2} \left(\eta_t^n(x) - \rho_t^n(x) \right), \quad x \in \mathbb{T}_n^d,$$

with $\rho_t^n(x) := \mathbb{E}\eta_t^n(x)$ converges to the solution of the linear SPDE

$$d\zeta_t^\infty = 2\pi^2 \Delta \zeta_t^\infty dt + 2\pi \nabla \cdot \left(\sqrt{\rho_t^\infty (1 - \rho_t^\infty)} dW_t \right) \tag{1.3}$$

in the Sobolev space H_{-I} for $I > \frac{d}{2} + 1$ started from ζ_0, where $(dW_t)_{t \geq 0}$ is a d-dimensional space-time white noise, and ζ_0 is a centered Gaussian distribution in H_{-I} with variance $\mathbb{E}\left[\langle \zeta_0, \varphi \rangle^2 \right] = \langle \rho_0(1 - \rho_0)\varphi, \varphi \rangle$ for smooth functions φ on $\mathbb{T}^d$. By [43, Theorem A.1], the discretization error of the heat equation $\|\mathbb{E}\eta^n - \rho^\infty\|_\infty$ behaves like $(2n+1)^{-2}$. Therefore, informally, the CLT corresponds to the expansion

$$\eta_t^n(x) = \rho_t^\infty(x) + (2n+1)^{-d/2} \zeta_t^\infty(x) + (2n+1)^{-(d/2 \wedge 1)} o(1). \tag{1.4}$$

Since the proof given in [56] proceeds via a compactness argument, the martingale central limit theorem, and the Holley and Stroock theory [39, 40], it does not allow the derivation of a quantitative convergence estimate in the central limit theorem, nor in (1.4). This open problem is solved in the present work, with an optimal rate of convergence. It appears that this is the first result proving a quantitative central limit theorem in the context of a non-equilibrium particle system[3].

The proof developed in this work is instead based on the formula

$$\mathbb{E}F(\hat{\rho}_t^n, \hat{\zeta}_t^n) - \mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n}) = \int_0^t \mathbb{E}\left[\left(\mathcal{G}^{FF} - \mathcal{G}^{OU}\right) P_{t-s}^{OU} F(\hat{\rho}_s^n, \hat{\zeta}_s^n)\right] ds, \tag{1.5}$$

see e.g. [24, Lemma 1.2.5], which allows to deduce estimates on the difference of the semigroups $(P_t^{FF})_{t\geq 0}$ and $(P_t^{OU})_{t\geq 0}$ associated with the Markov processes (ρ^n, ζ^n) and $(\rho^\infty, \zeta^\infty)$, from the difference of their generators $\mathcal{G}^{FF}$ and $\mathcal{G}^{OU}$. Here, $\hat{f} = \mathrm{ex}_n f$ denotes smooth interpolation and $(\rho^{\infty,n}, \zeta^{\infty,n})$ is a solution to (1.2) and (1.3) started from the initial particle configuration $(\hat{\rho}_0^n, \hat{\zeta}_0^n)$.

The estimation of the right hand side of (1.5), however, leads to several challenges: Firstly, the difference between generators can be estimated only on sufficiently regular functions $U = P_{t-s}^{OU} F$. Moreover, the obtained errors depend on higher-order derivatives of U, the norms of $\hat{\rho}_s^n$, $\hat{\zeta}_s^n$ in corresponding Sobolev spaces and the expression $B(\zeta_s^n) := [\mathrm{ex}_n(\zeta_s^n \tau \zeta_s^n)]^2$ for the shift operator τ on $\mathbb{T}_n^d$. The differentiability of $P_t^{OU} F$ is a non-trivial problem because the diffusion coefficient $f(\rho^\infty) = \sqrt{\rho^\infty(1-\rho^\infty)}$ in (1.3) is not differentiable. Therefore, the standard approach to the preservation of regularity of infinite-dimensional Kolmogorov equations, by proving the regularity on the level of the corresponding SPDE, cannot be applied. This issue is resolved in this work by a more careful infinite-dimensional analysis based on the fact that the process ζ^∞ is Gaussian. A second important ingredient to this part of the proof is a careful choice of the extension operator ex_n in order to guarantee the differentiability of U at points $(\mathrm{ex}_n \rho, \mathrm{ex}_n \zeta)$ appearing in (1.5), and in order to quantitatively control discretization errors (lattice effects) and interpolation errors, see e.g. Proposition 2.3.

Secondly, the control of the expectation of error terms requires additional path properties of the SSEP compared to the proof of the non-quantified CLT in [56]. For instance, the bound of $\mathbb{E}[B(\zeta_s^n)]$ fundamentally relies on the estimation of the four-point correlation function $\mathbb{E}[\prod_{i=1}^4 (\eta_s^n(x_i) - \rho_s^n(x_i))]$, while only the two point correlation function is used in [56].

Thirdly, quantitative, optimal estimates for the initial fluctuations

$$P_t^{OU} F(\hat{\rho}_0^n, \hat{\zeta}_0^n) - P_t^{OU} F(\rho_0, \zeta_0) = \mathbb{E}F(\rho_t^{\infty,n}, \zeta_t^{\infty,n}) - \mathbb{E}F(\rho_t^\infty, \zeta_t^\infty)$$

are required. Compared to Stein's method in the finite-dimensional context, see e.g., [50, 57], the present situation is more challenging, since the dimension of $\zeta_0^{\infty,n}$ diverges with $n \to \infty$, for observables F that are not assumed to be of the specific form of partial sums. This difficulty is resolved in the present work by carefully controling the constants appearing in the application of Stein's method, thereby proving their independence of the dimension.

We refer the reader to Section A in the appendix for the basic notation. As above, let $(\eta_t^n)_{t\geq 0}$ be the SSEP with the initial distribution $\nu_{\rho_0^n}^n$, $(\rho_t^n)_{t\geq 0}$ its expectation field and $(\zeta_t^n)_{t\geq 0}$ its density fluctuation field for each $n \geq 1$. Let also $(\rho_t^\infty)_{t\geq 0}$ be a solution to the heat equation (1.2) started from ρ_0, and $(\zeta_t^\infty)_{t\geq 0}$ a solution to (1.3) with the initial condition ζ_0. The following theorem is the main result of the paper.

[3] In contrast, in the setting of weakly interacting particle systems, related high order expansions have been obtained in [11].

Theorem 1.1. *Let $J > \frac{d}{2} \vee 2$, $I > d+3$, $\tilde{J} > J+d+5$ and $F \in \mathrm{C}^{1,3}_{l,HS}(H_J, H_{-I})$. Furthermore, assume that $\rho_0 \in H_{\tilde{J}}$ takes values in $[0,1]$ and ρ_0^n is the restriction of ρ_0 to $\mathbb{T}^d_n$ for each $n \geq 1$. Then, for each $T > 0$ there exists a constant C independent of F and n such that*

$$\sup_{t\in[0,T]} \left| \mathbb{E}F(\hat{\rho}^n_t, \hat{\zeta}^n_t) - \mathbb{E}F(\rho^\infty_t, \zeta^\infty_t) \right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|F\|_{\mathrm{C}^{1,3}_{l,HS}}.$$

Remark 1.2. The rate $\frac{1}{n^{\frac{d}{2}\wedge 1}}$ cannot be improved in the statement of Theorem 1.1, since it also includes the discretization error that equals $\frac{1}{n}$.

The following corollary directly follows from Theorem 1.1.

Corollary 1.3. *Under the assumptions of Theorem 1.1, for each $T > 0$ and $m \geq 1$ there exists a constant C such that*

$$\sup_{t\in[0,T]} |\mathbb{E}f(\langle\vec{\varphi}, \zeta^\infty_t\rangle) - \mathbb{E}f(\langle\vec{\varphi}, \zeta^n_t\rangle_n)| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \|f\|_{\mathrm{C}^3_l} \|\vec{\varphi}\|_{\mathrm{C}^{\lceil I\rceil}}$$

for all $n \geq 1$, $f \in \mathrm{C}^3(\mathbb{R}^m)$ and $\vec{\varphi} \in \left(\mathrm{C}^{\lceil I\rceil}(\mathbb{T}^d)\right)^m$.

In [18], the SPDEs

$$d\eta^{n,\delta}_t = \partial_{xx}\eta^{n,\delta}_t dt + \frac{1}{\sqrt{n}}\partial_x \left(\sqrt{\eta^{n,\delta}_t(1-\eta^{n,\delta}_t)} dW^\delta_t \right)$$

have been analyzed as effective models for the one-dimensional SSEP, where $(dW^\delta_t)_{t\geq 0}$ is a mollified 1-dimensional space-time white noise. In appropriate scaling regimes, it was concluded that

$$\mathbb{E}F(\hat{\eta}^n) - \mathbb{E}F(\eta^{n,\delta}) = o\left(n^{-\frac{1}{2}}\right) \tag{1.6}$$

which improves over the deterministic error

$$\mathbb{E}F(\hat{\eta}^n) - \mathbb{E}F(\rho^\infty) = O\left(n^{-\frac{1}{2}}\right).$$

While one would expect (1.6) to be of order $O(n^{-1})$, this was left open in [18] since a quantified CLT for a SSEP was missing, thus giving further motivation for the questions addressed in the present work.

The work is organized as follows. The basic notation and some facts are collected and postponed to the appendix in Section A. Section 2 is devoted to an expansion of generators associated with the particle system and an investigation of some path properties of the system. In particular, the expansion of generators of the SSEP and its density fluctuation field is obtained in Sections 2.1 and 2.2, respectively. Estimates of the expectation of Sobolev norms of $\hat{\rho}^n_s$, $\hat{\zeta}^n_s$ and the control of $\mathbb{E}[B(\zeta^n_s)]$ are obtained in Section 2.3. The aim of Section 3 is to show the regularity of the semigroup associated with the Ornstein-Uhlenbeck process in both variables ρ_0 and ζ_0. The differentiability of U in ζ_0 straightforward follows from the linearity of the SPDE (1.3). Therefore, the main focus of this section is concentrated on the regularity of U with respect to ρ_0. The differentiability of the covariance operator of ζ^∞_t in ρ_0 is obtained in Section 3.2. Then, using a kind of the integration-by-parts formula for Gaussian distributions, we get the differentiability of U. The Berry-Essen bound on the rate of convergence of particle fluctuations $\hat{\zeta}^n_0$ to the Gaussian random distribution ζ^∞_0 in a corresponding Sobolev space is obtained in Section 4. For this, we adapt the finite-dimensional approach, e.g, from [50, 57], to Sobolev spaces. The rest of the appendix is devoted to some properties of pr_n

and ex_n operators, multilinear operators on Sobolev spaces and Frechet differentiable functions defined on Sobolev spaces (see Sections B.1, B.2 and B.3, respectively).

Comments on the literature. For a comprehensive treatment of equilibrium fluctuations, we refer to the monographs [46, 47] and the detailed review of the literature contained therein.

In the case of gradient models and their perturbations, out-of-equilibrium fluctuation results have been established in [10, 15, 45, 55], including the central limit theorem for the weakly asymmetric simple exclusion process in [16, 19], and for the one-dimensional symmetric zero-range process with constant jump rate in [30]. The central limit theorem for the symmetric simple exclusion process was first established in [32, 56]. Several of these works build upon extensions of the equilibrium theories developed by Holley and Stroock [39, 40], as well as the Boltzmann–Gibbs principle [8]. A quantitative form of the Boltzmann–Gibbs principle for independent random walkers, and particle systems with duality has been obtained in [1]. Additionally, non-equilibrium fluctuations for the boundary-driven symmetric SSEP are discussed in [48], the SSEP with a slow bond in [22], and for a tagged particle in SSEP in [43]. In the recent contribution [23] the joint fluctuations of current and occupation time of the one-dimensional non-equilibrium simple symmetric exclusion process have been found. We are not aware of any previous results providing quantitative central limit theorems for out-of-equilibrium fluctuations in these contexts.

For recent advances in the analysis of quantitative fluctuations for non-gradient systems in equilibrium, see [37]. This work also reviews a series of studies establishing the non-quantitative equilibrium central limit theorem for several non-gradient systems.

Recent developments in the quantification of convergence in the law of large numbers for both gradient and non-gradient systems are documented in [36, 51] and the references cited therein. Quantitative estimates of propagation of chaos for mean field systems with singular kernels are provided in [7, 42]. The study of fluctuations in this context has a longstanding history, including works such as [28, 41, 63], with recent contributions in the setting of singular kernels found in [65]. A deep analysis of central limit fluctuations around the Boltzmann equation can be found in [4, 61].

Furthermore, fluctuation corrections of PDEs, leading to stochastic PDEs, and their connection to higher-order fluctuation expansions of particle systems and large deviations, have attracted significant attention in recent years [12, 13, 18, 20, 25–27, 34, 35].

Since its development in [62], Stein's method for the derivation of quantitative estimates on the distance to Gaussians has been an active and fruitful field, an overview of which would go far beyond the scope of this article. We restrict to mentioning a few points of references, where further references to the theory may be found. The main concepts of Stein's method is discussed in the survey article [59]. Careful estimates for multivariate normal approximation with Stein's method are obtained in [50, 57]. An early contribution extending Stein's method to the context of approximations of processes, that is, to infinite dimension is [2]. See also [14] and the references therein for subsequent generalizations. For applications of Stein's method in the context of statistical mechanics, we refer to [17, 21] and the references therein, where Berry-Esseen bounds for Curie-Weiss and mean-field Ising models have been derived. Stein's method in infinite dimension has been developed, for example, in [6, 60] deriving Berry–Esséen type estimate for abstract Wiener measures and in [58] for high-dimensional settings. A significant extension of Stein's method has been achieved by combination with Malliavin calculus in a line of developments [52, 53] and the monograph [54], which, in particular, allows application going beyond observables taking the specific form of partial sums. An extension of admissible functionals has been discussed in [3].

2 Particle system

The goal of this section is to study some properties of the SSEP needed for the proof of the main result. In particular, we expand the generator of the density fluctuation field and show that the leading terms in this expansion coincide with the generator of an Ornstein-Uhlenbeck process.

2.1 Expansion of the generator of the SSEP

We start from the expansion of the generator of the SSEP. Let $\eta^n_t = \left(\eta^n_t(x),\ x \in \mathbb{T}^d_n\right)$, $t \geq 0$, be the SSEP defined on the configuration space $\{0,1\}^{\mathbb{T}^d_n} \subset L_2(\mathbb{T}^d_n)$. Recall that it is a time continuous Markov Process whose generator $\mathcal{G}^{EP}_n$ is defined by (1.1). We extend $(\eta^n_t)_{t\geq 0}$ to a $\mathrm{C}^\infty(\mathbb{T}^d)$-valued process by considering

$$\hat{\eta}^n_t = \mathrm{ex}_n \eta^n_t, \quad t \geq 0.$$

According to (A.8), the restriction of $\hat{\eta}^n_t$ to the set $\mathbb{T}^d_n \subset \mathbb{T}^d$ coincides with η^n_t for each $t \geq 0$ and $n \in \mathbb{N}$. Next note that for each $J \in \mathbb{R}$ and $F \in \mathrm{C}(H_J)$, the process

$$M^F_t := F(\hat{\eta}^n_t) - F(\hat{\eta}^n_0) - \int_0^t \mathcal{G}^{EP}_n \left(F \circ \mathrm{ex}_n\right)(\eta^n_s) ds, \quad t \geq 0,$$

is a right-continuous martingale with respect to the filtration $(\mathcal{F}^{\eta^n}_t)_{t\geq 0}$ generated by η^n. In the next statement, we derive an expansion of the generator $\mathcal{G}^{EP}_n$ that will be used for an expansion of the generator of the density fluctuation field later.

Lemma 2.1. *Let $I > \frac{d}{2} + 1$. Then for each $F \in \mathrm{C}^3_l(H_{-I})$ and $n \geq 1$*

$$\mathcal{G}^{EP}_n \left(F \circ \mathrm{ex}_n\right)(\eta) = 2\pi^2 \langle \Delta_n \mathrm{pr}_n DF(\hat{\eta}), \eta \rangle_n + \frac{4\pi^4}{(2n+1)^{d+2}} \sum_{j=1}^d \left\langle \mathrm{Tr}\left(\partial^{\otimes 2}_{n,j} \mathrm{pr}^{\otimes 2}_n D^2 F(\hat{\eta})\right), [\partial_{n,j}\eta]^2 \right\rangle_n + R^{EP}_n(\eta),$$

for all $\eta \in \{0,1\}^{\mathbb{T}^d_n}$, where

$$\left|R^{EP}_n(\eta)\right| \leq \frac{C_I}{(2n+1)^{2d+1}} \|D^3 F\|_{\mathrm{C}}, \quad \eta \in \{0,1\}^{\mathbb{T}^d_n}.$$

Proof. To prove the lemma, we use the Taylor formula (A.4). For $\eta \in \{0,1\}^{\mathbb{T}^d_n}$ we get

$$\begin{aligned}
\mathcal{G}^{EP}_n \left(F \circ \mathrm{ex}_n\right)(\eta) &= \frac{(2n+1)^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_n} \left(F(\hat{\eta}^{x \leftrightarrow x+e_j}) - F(\hat{\eta})\right) \\
&= \frac{(2n+1)^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_n} DF(\hat{\eta}) \left[\hat{\eta}^{x \leftrightarrow x+e_j} - \hat{\eta}\right] \\
&+ \frac{(2n+1)^2}{4} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_n} D^2 F(\hat{\eta}) \left[\left(\hat{\eta}^{x \leftrightarrow x+e_j} - \hat{\eta}\right)^{\times 2}\right] \\
&+ \frac{(2n+1)^2}{2} \sum_{j=1}^d \sum_{x \in \mathbb{T}^d_n} R_j(x, \hat{\eta}) =: I_1 + I_2 + R^{EP},
\end{aligned}$$

where $\hat{\eta}^{x\leftrightarrow x+e_j} = \mathrm{ex}_n \eta^{x\leftrightarrow x+e_j}$ and

$$|R_j(x,\hat{\eta})| \le \frac{\|D^3 F\|_{\mathrm{C}}}{3!} \|\hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}\|_{H_{-I}}^3.$$

Using the equality (A.10), we first estimate the expression

$$\begin{aligned}\|\hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}\|_{H_{-I}}^2 &= \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^I} \left|\langle \hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}, \varsigma_k\rangle\right|^2 \\ &= \sum_{k\in\mathbb{Z}_n^d} \frac{1}{(1+|k|^2)^I} \left|\langle \eta^{x\leftrightarrow x+e_j} - \eta, \varsigma_k\rangle_n\right|^2\end{aligned}$$

for $x \in \mathbb{T}_n$ and $j \in [d]$. Note that for each $\varphi : \mathbb{T}_n^d \to \mathbb{C}$

$$\begin{aligned}\langle \varphi, \eta^{x\leftrightarrow x+e_j}\rangle_n - \langle \varphi, \eta\rangle_n &= \frac{1}{(2n+1)^d} \sum_{z\in\mathbb{T}_n^d} \varphi(z)\eta^{x\leftrightarrow x+e_j}(z) \\ &- \frac{1}{(2n+1)^d} \sum_{z\in\mathbb{T}_n^d} \varphi(z)\eta(z) \\ &= \frac{1}{(2n+1)^d} \left[\eta(x+e_j) - \eta(x)\right] \varphi(x) \\ &+ \frac{1}{(2n+1)^d} \left[\eta(x) - \eta(x+e_j)\right] \varphi(x+e_j) \\ &= \frac{1}{(2n+1)^d} \left[\eta(x) - \eta(x+e_j)\right] \left[\varphi(x+e_j) - \varphi(x)\right] \\ &= -\frac{4\pi^2}{(2n+1)^{d+2}} \partial_{n,j}\eta(x)\partial_{n,j}\varphi(x).\end{aligned} \tag{2.1}$$

Thus, using (A.7) and Lemma B.1, we estimate

$$\begin{aligned}\left|\langle \eta^{x\leftrightarrow x+e_j} - \eta, \varsigma_k\rangle_n\right| &= \frac{4\pi^2}{(2n+1)^{d+2}} \left|\partial_{n,j}\eta(x)\right| \left|\partial_{n,j}\varsigma_k(x)\right| \\ &= \frac{4\pi^2}{(2n+1)^{d+2}} \left|\partial_{n,j}\eta(x)\right| \left|\mu_{k,j}^n\right| \le \frac{8\pi^2|k_j|}{(2n+1)^{d+1}}\end{aligned}$$

and, consequently,

$$\|\hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}\|_{H_{-I}}^2 \le \frac{64\pi^4}{(2n+1)^{2d+2}} \sum_{k\in\mathbb{Z}_n^d} \frac{|k_j|^2}{(1+|k|^2)^I} \le \frac{64\pi^4 C_I}{(2n+1)^{2d+2}}$$

due to $I > \frac{d}{2} + 1$. This implies the inequality

$$\left|R^{EP}(\eta)\right| \le \frac{(2n+1)^2}{2d} \|D^3 F\|_{\mathrm{C}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_n} \frac{C_I}{(2n+1)^{3d+3}} \le \frac{C_I}{(2n+1)^{2d+1}} \|D^3 F\|_{\mathrm{C}}.$$

In order to rewrite I_1 we use the fact that the derivative $DF(\hat{\eta})$ belongs to the dual space of H_{-I}. Hence, $DF(\hat{\eta}) \in H_I$ and

$$\begin{aligned}DF(\hat{\eta})\left[\hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}\right] &= \langle DF(\hat{\eta}), \hat{\eta}^{x\leftrightarrow x+e_j} - \hat{\eta}\rangle = \left\langle \mathrm{pr}_n DF(\hat{\eta}), \eta^{x\leftrightarrow x+e_j} - \eta\right\rangle_n \\ &= -\frac{4\pi^2}{(2n+1)^{d+2}} \partial_{n,j}\mathrm{pr}_n DF(\hat{\eta})(x)\partial_{n,j}\eta(x),\end{aligned}$$

according to (A.9) and (2.1). This implies

$$I_1 = -\frac{(2n+1)^2}{2} \frac{4\pi^2}{(2n+1)^{d+2}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n^d} \partial_{n,j} \mathrm{pr}_n DF(\hat{\eta})(x) \partial_{n,j} \eta(x)$$
$$= -2\pi^2 \sum_{j=1}^{d} \langle \partial_{n,j} \mathrm{pr}_n DF(\hat{\eta}), \partial_{n,j} \eta \rangle_n = 2\pi^2 \langle \Delta_n \mathrm{pr}_n DF(\hat{\eta}), \eta \rangle_n,$$

where we used the discrete integration by parts formula (A.6).
Since $D^2F(\hat{\eta}) \in \mathcal{L}_2(H_{-I})$, the equality (A.13) yields

$$D^2F(\hat{\eta}) \left[\left(\hat{\eta}^{x \leftrightarrow x+e_j} - \hat{\eta} \right)^{\times 2} \right] = \left\langle \mathrm{pr}_n^{\otimes 2} D^2F(\hat{\eta}), \left(\eta^{x \leftrightarrow x+e_j} - \eta \right)^{\otimes 2} \right\rangle_n.$$

Similarly to the computation in (2.1), we get that the right hand side in the expression above equals

$$\frac{16\pi^4}{(2n+1)^{2d+4}} \partial_{n,j}^{\otimes 2} \mathrm{pr}_n^{\otimes 2} D^2F(\hat{\eta})(x,x) \left(\partial_{n,j} \eta(x) \right)^2.$$

Hence,

$$I_2 = \frac{16\pi^4 (2n+1)^2}{4(2n+1)^{2d+4}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_n^d} \partial_{n,j}^{\otimes 2} \mathrm{pr}_n^{\otimes 2} D^2F(\hat{\eta})(x,x) \left(\partial_{n,j} \eta(x) \right)^2$$
$$= \frac{4\pi^4}{(2n+1)^{d+2}} \sum_{j=1}^{d} \left\langle \mathrm{Tr} \left(\partial_{n,j}^{\otimes 2} \mathrm{pr}_n^{\otimes 2} D^2F(\hat{\eta}) \right), (\partial_{n,j}\eta)^2 \right\rangle_n.$$

This completes the proof of the lemma. □

2.2 Density fluctuation field for the SSEP and its generator

The aim of this section is to consider the density fluctuation field

$$\zeta_t^n(x) = (2n+1)^{d/2} (\eta_t^n(x) - \rho_t^n(x)), \ x \in \mathbb{T}_n^d, \quad t \geq 0,$$

for the SSEP and obtain an expansion of its generator. It is easy to see that the process

$$\rho_t^n(x) = \mathbb{E}\eta_t^n(x), \quad x \in \mathbb{T}_n^d, \quad t \geq 0,$$

is a unique solution to the discrete heat equation

$$\rho_t^n(x) = \rho_0^n(x) + 2\pi^2 \int_0^t \Delta_n \rho_s^n(x) ds, \quad x \in \mathbb{T}_n^d, \quad t \geq 0, \tag{2.2}$$

with $\rho_0^n(x) = \mathbb{E}\eta_0^n(x) \in [0,1]$, $x \in \mathbb{T}_n^d$. Moreover, $\rho_t^n \in [0,1]^{\mathbb{T}_n^d} \subset L_2(\mathbb{T}_n^d)$ for all $t \geq 0$. Using the chain rule (see, e.g. Theorem [9, Theorem 2.2.1]) and the discrete integration-by-parts formula, we get for each $F \in \mathrm{C}^1(L_2(\mathbb{T}_n^d))$

$$F(\rho_t^n) = F(\rho_0^n) + 2\pi^2 \int_0^t \langle DF(\rho_s^n), \Delta_n \rho_s^n \rangle_n \, ds$$
$$= F(\rho_0^n) + 2\pi^2 \int_0^t \langle \Delta_n DF(\rho_s^n), \rho_s^n \rangle_n \, ds, \quad t \geq 0.$$

In particular, this implies that (ρ^n_t, η^n_t), $t \geq 0$, is a Markov process with generator

$$2\pi^2 \left\langle \Delta_n D_1 F(\cdot, \eta)(\rho), \rho \right\rangle_n + \left(\mathcal{G}^{EP} F(\rho, \cdot)\right)(\eta).$$

Thus, the process (ρ^n_t, ζ^n_t), $t \geq 0$, is also a Markov process and for each $F \in \mathrm{C}^1(L_2(\mathbb{T}^d_n)^2)$

$$F(\rho^n_t, \zeta^n_t) - F(\rho^n_0, \zeta^n_0) - \int_0^t \mathcal{G}^{FF} F(\rho^n_s, \zeta^n_s) ds, \quad t \geq 0,$$

is a martingale with respect to the filtration $(\mathcal{F}^{\zeta^n}_t)_{t \geq 0}$ generated by the process ζ^n that coincides with $(\mathcal{F}^{\eta^n}_t)_{t \geq 0}$. Here

$$\mathcal{G}^{FF} F(\rho, \zeta) = 2\pi^2 \left\langle \Delta_n D_1 G(\rho, \eta), \rho \right\rangle_n + \mathcal{G}^{EP} G(\rho, \cdot)(\eta),$$

where $G(\rho, \eta) := F(\rho, \zeta)$ and $\eta = \rho + (2n+1)^{-d/2} \zeta$.

Similarly to the previous section, we extend ρ^n_t and ζ^n_t to the domain $\mathbb{T}^d$ by setting

$$\hat{\rho}^n_t := \mathrm{ex}_n \rho^n_t \quad \text{and} \quad \hat{\zeta}^n_t := \mathrm{ex}_n \zeta^n_t \tag{2.3}$$

for all $t \geq 0$ and consider $(\hat{\rho}^n_t, \hat{\zeta}^n_t)$, $t \geq 0$, as a process with values in $H_J \times H_{-I}$ for each $I, J \in \mathbb{R}$. Since $(\hat{\rho}^n, \hat{\zeta}^n)$ is obtained from the Markov process (ρ^n, ζ^n) using injective mappings, it is a Markov process too. Furthermore, for each $F \in \mathrm{C}^1(H_J \times H_{-I})$

$$F(\hat{\rho}^n_t, \hat{\zeta}^n_t) - F(\hat{\rho}^n_0, \hat{\zeta}^n_0) - \int_0^t \mathcal{G}^{FF} \hat{F}(\rho^n_s, \zeta^n_s) ds, \quad t \geq 0, \tag{2.4}$$

is a martingale with respect to $(\mathcal{F}^{\zeta^n}_t)_{t \geq 0}$, where $\hat{F}(\rho, \zeta) = F(\mathrm{ex}_n \rho, \mathrm{ex}_n \zeta)$, $\rho, \zeta \in L_2(\mathbb{T}^d_n)$, and $\hat{F} \in \mathrm{C}^1(L_2(\mathbb{T}^d_n)^2)$, according to Lemma B.22. Note that $\mathcal{G}^{FF} \hat{F}(\rho^n_s, \zeta^n_s)$ can be rewritten as $\mathcal{G}^{FF} \hat{F}(\mathrm{pr}_n \hat{\rho}^n_s, \mathrm{pr}_n \hat{\zeta}^n_s)$ due to Lemma B.3. Consequently, setting

$$\hat{\mathcal{G}}^{FF} F(\rho, \zeta) := \mathcal{G}^{FF} \hat{F}(\mathrm{pr}_n \rho, \mathrm{pr}_n \zeta) = \mathcal{G}^{FF} \left(F \circ \mathrm{ex}^{\times 2}_n\right)(\mathrm{pr}_n \rho, \mathrm{pr}_n \zeta), \quad \rho \in H_J, \quad \zeta \in H_{-I},$$

where $\mathrm{ex}^{\times 2}_n(f, g) = (\mathrm{ex}_n f, \mathrm{ex}_n g)$, $f, g \in L_2(\mathbb{T}^d_n)$, we conclude that for each $F \in \mathrm{C}^1(H_J \times H_I)$

$$F(\hat{\rho}^n_t, \hat{\zeta}^n_t) - F(\hat{\rho}^n_0, \hat{\zeta}^n_0) - \int_0^t \hat{\mathcal{G}}^{FF} F(\hat{\rho}^n_s, \hat{\zeta}^n_s) ds, \quad t \geq 0, \tag{2.5}$$

is a martingale with respect to $(\mathcal{F}^{\zeta^n}_t)_{t \geq 0}$. In particular, the expectation of the martingale in (2.5) equals zero.

We will need the following property for the case of time dependent functions F.

Lemma 2.2. *Let $J, I \in \mathbb{R}$ and $F \in \mathrm{C}^{1,1,1}([0, \infty), H_J, H_{-I})$. Then, for all $t \geq 0$ and $n \in \mathbb{N}$,*

$$\mathbb{E} F_t(\hat{\rho}^n_t, \hat{\zeta}^n_t) = \mathbb{E} F_0(\hat{\rho}^n_0, \hat{\zeta}^n_0) + \int_0^t \mathbb{E}\left[\partial F_s(\hat{\rho}^n_s, \hat{\zeta}^n_s) + \hat{\mathcal{G}}^{FF} F_s(\hat{\rho}^n_s, \hat{\zeta}^n_s)\right] ds, \quad t \geq 0.$$

Proof. Considering a partition $0 = t_0 < t_1 < \ldots < t_m = t$, we get

$$\begin{aligned} \mathbb{E} F_t(\hat{\rho}^n_t, \hat{\zeta}^n_t) - \mathbb{E} F_0(\hat{\rho}^n_0, \hat{\zeta}^n_0) &= \sum_{k=1}^m \mathbb{E}\left[F_{t_i}(\hat{\rho}^n_{t_i}, \hat{\zeta}^n_{t_i}) - F_{t_{i-1}}(\hat{\rho}^n_{t_{i-1}}, \hat{\zeta}^n_{t_{i-1}})\right] \\ &= \sum_{k=1}^m \mathbb{E}\left[F_{t_{i-1}}(\hat{\rho}^n_{t_i}, \hat{\zeta}^n_{t_i}) - F_{t_{i-1}}(\hat{\rho}^n_{t_{i-1}}, \hat{\zeta}^n_{t_{i-1}})\right] \end{aligned}$$

$$+\sum_{k=1}^{m}\mathbb{E}\left[F_{t_i}(\hat\rho^n_{t_i},\hat\zeta^n_{t_i})-F_{t_{i-1}}(\hat\rho^n_{t_i},\hat\zeta^n_{t_i})\right] \tag{2.6}$$
$$=\sum_{k=1}^{m}\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\hat{\mathcal{G}}^{FF}F_{t_{i-1}}(\hat\rho^n_s,\hat\zeta^n_s)ds\right]$$
$$+\sum_{k=1}^{m}\mathbb{E}\left[\int_{t_{i-1}}^{t_i}\partial_s F_s(\hat\rho^n_{t_i},\hat\zeta^n_{t_i})ds\right],$$

where we used (2.5). Trivially, $E_n := \mathrm{ex}_n([0,1]^{\mathbb{T}^d_n})\times\mathrm{ex}_n(n^{d/2}[0,1]^{\mathbb{T}^d_n})$ is a compact subset of $H_J\times H_{-I}$ because it is closed, bounded and finite-dimensional. Moreover, $(\hat\rho^n_t,\hat\zeta^n_t)$, $t\geq 0$, takes values in E_n. We also note that the functions $(s,\rho,\zeta)\mapsto\hat{\mathcal{G}}^{FF}F_s(\rho,\zeta)$ and $(s,\rho,\zeta)\mapsto\partial F_s(\rho,\zeta)$ are continuous and, thus, bounded on $[0,t]\times E_n$. Using the right-continuity of $(\hat\rho^n_t,\hat\zeta^n_t)$, $t\geq 0$, and the dominated convergence theorem, we conclude that the right hand side of (2.6) converges to

$$\int_0^t \mathbb{E}\left[\hat{\mathcal{G}}^{FF}F_s(\hat\rho^n_s,\hat\zeta^n_s)+\partial_s F(\hat\rho^n_s,\hat\zeta^n_s)\right]ds,$$

as the mesh of the partition goes to zero. This completes the proof of the lemma. □

In the next statement, we get an expansion of $\hat{\mathcal{G}}^{FF}_n F(\hat\rho,\hat\zeta)$ needed for its comparison with the generator of an Ornstein-Uhlenbeck process. Let τ^n_j denote the shift operator on $\mathbb{T}^d_n$ defined by $\tau^n_j f(x)=f(x+e^n_j)$.

Proposition 2.3. *Let $J>2$, $I>d+2$, $\tilde I\geq 0$, $\lceil\tilde I\rceil+1+\frac{d}{2}<I$ and $F\in\mathrm{C}^{1,3}_{l,HS}(H_J,H_{-I})$. Then for each $n\geq 1$ there exists a function $R_n:[0,1]^{\mathbb{T}^d_n}\times\mathbb{R}^{\mathbb{T}^d_n}\to\mathbb{R}$ such that*

$$\begin{aligned}\hat{\mathcal{G}}^{FF}_n F(\hat\rho,\hat\zeta) &= 2\pi^2\left\langle\Delta D_1F(\hat\rho,\hat\zeta),\hat\rho\right\rangle+2\pi^2\left\langle\Delta D_2F(\hat\rho,\hat\zeta),\hat\zeta\right\rangle\\ &+4\pi^2\sum_{j=1}^{d}\left\langle\mathrm{Tr}\left(\partial_j^{\otimes 2}D_2^2F(\hat\rho,\hat\zeta)\right),\hat\rho(1-\hat\rho)\right\rangle\\ &+\frac{2\pi^2}{(2n+1)^d}\sum_{j=1}^{d}\left\langle\mathrm{Tr}\left(\partial_j^{\otimes 2}D_2^2F(\hat\rho,\hat\zeta)\right),\mathrm{ex}_n\left[\zeta\tau^n_j\zeta\right]\right\rangle\\ &+R^{FF}_n(\rho,\zeta),\end{aligned}$$

and

$$\left|R^{FF}_n(\rho,\zeta)\right|\leq\frac{C_{J,I,\tilde I}}{n^{\frac{d}{2}\wedge 1}}\|F\|_{C^{1,3}_{l,HS}}\left(1+\|\hat\rho\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}+\|\hat\rho\|_{\mathrm{C}^{\lceil\tilde I\rceil}}\right)\left(1+\|\hat\zeta\|_{H_{-I+2}}+\|\hat\zeta\|_{H_{-\tilde I}}\right)$$

for all $\rho\in[0,1]^{\mathbb{T}^d_n}$ and $\zeta=(2n+1)^{d/2}(\eta-\rho)$, $\eta\in\{0,1\}^{\mathbb{T}^d_n}$.

Proof. Take $\rho\in[0,1]^{\mathbb{T}^d_n}$, $\eta\in\{0,1\}^{\mathbb{T}^d_n}$ and $\zeta:=(2n+1)^{d/2}(\eta-\rho)$. For $\hat G(\rho,\eta):=\hat F(\rho,(2n+1)^{d/2}(\eta-\rho))$, where $\hat F(\rho,\zeta)=F(\hat\rho,\hat\zeta)$, we first rewrite

$$\begin{aligned}\left\langle\Delta_n D_1\hat G(\rho,\eta),\rho\right\rangle_n &= \left\langle\Delta_n D_1\hat F(\rho,(2n+1)^{d/2}(\eta-\rho)),\rho\right\rangle_n\\ &-(2n+1)^{d/2}\left\langle\Delta_n D_2\hat F(\rho,(2n+1)^{d/2}(\eta-\rho)),\rho\right\rangle_n\\ &=\left\langle\Delta_n\mathrm{pr}_nD_1F(\hat\rho,\hat\zeta),\rho\right\rangle_n-(2n+1)^{d/2}\left\langle\Delta_n\mathrm{pr}_nD_2F(\hat\rho,\hat\zeta),\rho\right\rangle_n\end{aligned}$$

due to Lemma B.22. Thus, using Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{G}_n^{FF}\hat{F}(\rho,\zeta) &= 2\pi^2\left\langle \Delta_n \mathrm{pr}_n D_1 F(\hat{\rho},\hat{\zeta}),\rho\right\rangle_n - 2\pi^2(2n+1)^{d/2}\left\langle \Delta_n \mathrm{pr}_n D_2 F(\hat{\rho},\hat{\zeta}),\rho\right\rangle_n \\ &+ 2\pi^2(2n+1)^{d/2}\langle \Delta_n \mathrm{pr}_n D_2 F(\hat{\rho},\hat{\zeta}),\eta\rangle_n \\ &+ \frac{4\pi^4}{(2n+1)^2}\sum_{j=1}^d \left\langle \mathrm{Tr}\left(\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2} D_2^2 F(\hat{\rho},\hat{\zeta})\right),(\partial_{n,j}\eta)^2\right\rangle_n \\ &+ R_n^{EP}(\rho,\eta),\end{aligned} \tag{2.7}$$

where the error term R_n^{EP} satisfies

$$\left|R_n^{EP}(\rho,\eta)\right| \le \frac{C_I}{(2n+1)^{d/2+1}}\|D_2^3 F\|_{\mathrm{C}}.$$

For the first term of the equality (2.7) we have

$$\begin{aligned}&\left|\left\langle \Delta_n \mathrm{pr}_n D_1 F(\hat{\rho},\hat{\zeta}),\rho\right\rangle_n - \left\langle \Delta D_1 F(\hat{\rho},\hat{\zeta}),\hat{\rho}\right\rangle\right| \\ &\qquad = \left|\left\langle \mathrm{ex}_n \Delta_n \mathrm{pr}_n D_1 F(\hat{\rho},\hat{\zeta}),\hat{\rho}\right\rangle - \left\langle \mathrm{pr}_n \Delta D_1 F(\hat{\rho},\hat{\zeta}),\hat{\rho}\right\rangle\right| \\ &\qquad \le \left\|\mathrm{ex}_n \Delta_n \mathrm{pr}_n D_1 F(\hat{\rho},\hat{\zeta}) - \mathrm{pr}_n \Delta D_1 F(\hat{\rho},\hat{\zeta})\right\|_{H_{J-2}}\|\hat{\rho}\|_{H_{-J+2}} \\ &\qquad \le \frac{C}{n}\|D_1 F(\hat{\rho},\hat{\zeta})\|_{H_J}\|\hat{\rho}\| \le \frac{C}{n}\|D_1 F\|_{\mathrm{C}},\end{aligned}$$

according to (A.12), Lemma B.11 and the fact that $J \ge 2$. Similarly, we get

$$\begin{aligned}\left|\langle \Delta_n \mathrm{pr}_n D_2 F(\hat{\rho},\hat{\zeta}),\zeta\rangle_n - \left\langle \Delta D_2 F(\hat{\rho},\hat{\zeta}),\hat{\zeta}\right\rangle\right| &\le \frac{C}{n}\|D_2 F(\hat{\rho},\hat{\zeta})\|_{H_I}\|\hat{\zeta}\|_{H_{-I+2}} \\ &\le \frac{C}{n}\|D_2 F\|_{\mathrm{C}}\|\hat{\zeta}\|_{H_{-I+2}}.\end{aligned}$$

To rewrite the fourth term in (2.7), which will be denoted by I_4, we first set

$$U_{j,n}(\rho,\eta) := \mathrm{Tr}\left(\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2} D_2^2 F(\hat{\rho},\hat{\zeta})\right)$$

and note that

$$\begin{aligned}(\partial_{n,j}\eta(x))^2 &= \frac{(2n+1)^2}{4\pi^2}\left(\eta(x+e_j^n)-\eta(x)\right)^2 \\ &= \frac{(2n+1)^2}{4\pi^2}\left(\eta(x+e_j^n)+\eta(x)-2\eta(x+e_j^n)\eta(x)\right)\end{aligned}$$

for all $x \in \mathbb{T}_n^d$. In terms of the shift operator τ_j^n, we get

$$\begin{aligned}I_4 &= \pi^2\sum_{j=1}^d \left\langle U_{j,n}(\rho,\eta),\tau_j^n\eta+\eta-2\eta\tau_j^n\eta\right\rangle_n \\ &= 2\pi^2\sum_{j=1}^d \left\langle U_{j,n}(\rho,\eta),\rho(1-\rho)\right\rangle_n + \pi^2\sum_{j=1}^d \left\langle U_{j,n}(\rho,\eta),\tau_j^n\eta-\rho\right\rangle_n\end{aligned}$$

$$+\pi^2\sum_{j=1}^{d}\langle U_{j,n}(\rho,\eta),\eta-\rho\rangle_n-2\pi^2\sum_{j=1}^{d}\left\langle U_{j,n}(\rho,\eta),\eta\tau_j^n\eta-\rho^2\right\rangle_n=:\sum_{i=1}^{4}I_{4,i}.$$

We next estimate each term of the right hand side of the equality above.
$(I_{4,1})$: According to Proposition B.18, there exists a function $R^{4,1}_{j,n}:[0,1]^{\mathbb{T}^d_n}\times\{0,1\}^{\mathbb{T}^d_n}\to L_2(\mathbb{T}^d_n)$ such that

$$U_{j,n}(\rho,\eta)=\mathrm{pr}_n\mathrm{Tr}\left(\partial_j^{\otimes 2}D_2^2F(\hat\rho,\hat\zeta)\right)+R^{4,1}_{j,n}(\rho,\eta) \tag{2.8}$$

due to $I>d+2$, where

$$\max_{x\in\mathbb{T}^d_n}\left|R^{4,1}_{j,n}(\rho,\eta)(x)\right|\le\frac{C_I}{n}\sup_{\rho\in[0,1]^{\mathbb{T}^d_n},\zeta\in\mathbb{R}^{\mathbb{T}^d_n}}\|D_2^2F(\hat\rho,\hat\zeta)\|_{\mathcal{L}_2^{HS}(H_{-I})}\le\frac{C_I}{n}\|F\|_{\mathrm{C}^{1,3}_{l,HS}}. \tag{2.9}$$

We also note that

$$\left\|\mathrm{ex}_n\rho^2-(\mathrm{ex}_n\rho)^2\right\|\le\frac{C}{n}\|\mathrm{ex}_n\rho\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}, \tag{2.10}$$

by Lemma B.7. Thus, setting $H(\rho,\eta):=\mathrm{Tr}\left(\partial_j^{\otimes 2}D_2^2F(\hat\rho,\hat\zeta)\right)$ and using (A.12), we can rewrite

$$\begin{aligned}\langle U_{j,n}(\rho,\eta),\rho(1-\rho)\rangle_n&=\langle\mathrm{pr}_nH(\rho,\eta),\rho\rangle_n-\left\langle\mathrm{pr}_nH(\rho,\eta),\rho^2\right\rangle_n+\left\langle R^{4,1}_{j,n}(\rho,\eta),\rho(1-\rho)\right\rangle_n\\&=\langle H(\rho,\eta),\hat\rho\rangle-\left\langle H(\rho,\eta),\mathrm{ex}_n\rho^2\right\rangle+\left\langle R^{4,1}_{j,n}(\rho,\eta),\rho(1-\rho)\right\rangle_n\\&=\langle H(\rho,\eta),\hat\rho(1-\hat\rho)\rangle+R^4_{j,n}(\rho,\eta),\end{aligned}$$

where $R^4_{j,n}(\rho,\eta):=\left\langle R^{4,1}_{j,n}(\rho,\eta),\rho(1-\rho)\right\rangle_n+\left\langle H(\rho,\eta),(\mathrm{ex}_n\rho)^2-\mathrm{ex}_n\rho^2\right\rangle$. Using (2.9) and (2.10), we get

$$\left|R^4_{j,n}(\rho,\eta)\right|\le\frac{C_I}{n}\|F\|_{\mathrm{C}^{1,3}_{l,HS}}+\frac{C}{n}\|H(\rho,\eta)\|\|\mathrm{ex}_n\rho\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}.$$

Since $I>\frac{d}{2}+1$, we can use Lemmas B.2 (iii), B.15 and B.16 to get

$$\begin{aligned}\|H(\rho,\eta)\|&\le C_I\|\partial_j^{\otimes 2}D_2^2F(\hat\rho,\hat\zeta)\|_{\mathcal{L}_2^{HS}(H_{-I+1})}\\&\le C_I\|D_2^2F(\hat\rho,\hat\zeta)\|_{\mathcal{L}_2^{HS}(H_{-I})}\le C_I\|F\|_{\mathrm{C}^{1,3}_{l,HS}}.\end{aligned}$$

Thus,

$$\left|R^4_{j,n}(\rho,\eta)\right|\le\frac{C_I}{n}\left[\|F\|_{\mathrm{C}^{1,3}_{l,HS}}\left(1+\|\hat\rho\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}\right)\right].$$

$(I_{4,3})$: Using (A.12) and Proposition B.18, we get

$$\begin{aligned}\left|\langle U_{j,n}(\rho,\eta),\eta-\rho\rangle_n\right|&=\frac{1}{(2n+1)^{d/2}}\left|\langle U_{j,n}(\rho,\eta),\zeta\rangle_n\right|\\&=\frac{1}{(2n+1)^{d/2}}\left|\left\langle\mathrm{ex}_nU_{j,n}(\rho,\eta),\hat\zeta\right\rangle\right|\\&\le\frac{1}{n^{d/2}}\|\mathrm{ex}_nU_{j,n}(\rho,\eta)\|_{H_{\tilde I}}\|\hat\zeta\|_{H_{-\tilde I}}\\&\le\frac{C_{I,\tilde I}}{n^{d/2}}\|D_2^2F(\hat\rho,\hat\zeta)\|_{\mathcal{L}_2^{HS}(H_{-I})}\|\hat\zeta\|_{H_{-\tilde I}}\end{aligned}$$

$$\leq \frac{C_{I,\tilde{I}}}{n^{d/2}} \|F\|_{\mathrm{C}^{1,3}_{l,HS}} \|\hat{\zeta}\|_{H_{-\tilde{I}}}$$

for each $\tilde{I} \geq 0$ such that $\tilde{I} + 1 + \frac{d}{2} < I$.

$(I_{4,2})$: We first rewrite

$$\begin{aligned}\left\langle U_{j,n}(\rho,\eta), \tau_j^n \eta - \rho \right\rangle_n &= \langle U_{j,n}(\rho,\eta), \eta - \rho \rangle_n - \left\langle U_{j,n}(\rho,\eta), \tau_j^n \eta - \eta \right\rangle_n \\ &= \langle U_{j,n}(\rho,\eta), \eta - \rho \rangle_n - \frac{2\pi}{2n+1} \langle U_{j,n}(\rho,\eta), \partial_{n,j}\eta \rangle_n .\end{aligned}$$

The term $\langle U_{j,n}(\rho,\eta), \eta - \rho\rangle_n$ was estimated above. We now estimate

$$\begin{aligned}\left|\langle U_{j,n}(\rho,\eta), \partial_{n,j}\eta\rangle_n\right| &= \left|\langle \partial_{n,j} U_{j,n}(\rho,\eta), \eta\rangle_n\right| = \left|\langle \mathrm{ex}_n \partial_{n,j} U_{j,n}(\rho,\eta), \hat{\eta}\rangle\right| \\ &\leq \|\mathrm{ex}_n \partial_{n,j} U_{j,n}(\rho,\eta)\| \|\hat{\eta}\|.\end{aligned}$$

Due to the fact that η takes values from $\{0,1\}$ and Corollary B.4, we get $\|\hat{\eta}\| = \|\mathrm{ex}_n \eta\| = \|\eta\|_n \leq 1$. By Lemma B.8 and Proposition B.18, we obtain

$$\|\mathrm{ex}_n \partial_{n,j} U_{j,n}(\rho,\eta)\| \leq \|\mathrm{ex}_n U_{j,n}(\rho,\eta)\|_{H_1} \leq C_I \|F\|_{\mathrm{C}^{1,3}_{l,HS}},$$

where we have used the fact that $I > 2 + \frac{d}{2}$. Thus,

$$\left\langle U_{j,n}(\rho,\eta), \tau_j^n \eta - \rho \right\rangle_n \leq \frac{C_{I,\tilde{I}}}{n^{d/2}} \|F\|_{\mathrm{C}^{1,3}_{l,HS}} \|\hat{\zeta}\|_{H_{-\tilde{I}}} + \frac{C_J}{n} \|F\|_{\mathrm{C}^{1,3}_{l,HS}}.$$

$(I_{4,4})$: Using the equality $\eta = \rho + n^{-d/2}\zeta$, we first rewrite

$$\begin{aligned}\left\langle U_{j,n}(\rho,\eta), \eta\tau_j^n \eta - \rho^2 \right\rangle_n &= \left\langle U_{j,n}(\rho,\eta), \rho\tau_j^n \rho - \rho^2 \right\rangle_n \\ &+ \frac{1}{(2n+1)^{d/2}} \left\langle U_{j,n}(\rho,\eta), \rho\tau_j^n \zeta \right\rangle_n \\ &+ \frac{1}{(2n+1)^{d/2}} \left\langle U_{j,n}(\rho,\eta), \zeta\tau_j^n \rho \right\rangle_n \\ &+ \frac{1}{(2n+1)^{d}} \left\langle U_{j,n}(\rho,\eta), \zeta\tau_j^n \zeta \right\rangle_n .\end{aligned}$$

Let $I_{4,4,i}$, $i \in [4]$, denote the terms in the right hand side of the equality above. We first estimate the term $I_{4,4,1}$ as follows

$$\begin{aligned}|I_{4,4,1}| &= \frac{2\pi}{2n+1} \left|\langle U_{j,n}(\rho,\eta)\rho, \partial_{n,j}\rho\rangle_n\right| \leq \frac{2\pi}{2n+1} \|U_{j,n}(\rho,\eta)\rho\|_n \|\partial_{n,j}\rho\|_n \\ &\leq \frac{2\pi}{2n+1} \|U_{j,n}(\rho,\eta)\|_n \|\partial_{n,j}\rho\|_n = \frac{2\pi}{2n+1} \|\mathrm{ex}_n U_{j,n}(\rho,\eta)\| \|\mathrm{ex}_n \partial_{n,j}\rho\| \\ &\leq \frac{C_I}{n} \|D_2^2 F(\hat{\rho},\hat{\zeta})\|_{\mathcal{L}_2^{HS}(H_{-I})} \|\hat{\rho}\|_{H_1} \\ &\leq \frac{C_I}{n} \|F\|_{\mathrm{C}^{1,3}_{l,HS}} \|\hat{\rho}\|_{H_1},\end{aligned}$$

where we used (A.12) and Proposition B.18. According (A.12) and Lemmas B.9 and B.10, the estimate

$$\begin{aligned}|I_{4,4,2}| &= \frac{1}{(2n+1)^{d/2}} \left|\left\langle U_{j,n}(\rho,\eta)\rho, \tau_j^n \zeta\right\rangle_n\right| = \frac{1}{(2n+1)^{d/2}} \left|\left\langle \mathrm{ex}_n\left(U_{j,n}(\rho,\eta)\rho\right), \mathrm{ex}_n \tau_j^n \zeta\right\rangle\right| \\ &\leq \frac{1}{n^{d/2}} \left\|\mathrm{ex}_n\left(U_{j,n}(\rho,\eta)\rho\right)\right\|_{H_{\tilde{I}}} \|\mathrm{ex}_n \tau_j^n \zeta\|_{H_{-\tilde{I}}}\end{aligned}$$

$$\leq \frac{C_{I,\tilde{J}}}{n^{d/2}} \|\mathrm{ex}_n U_{j,n}(\rho,\eta)\|_{H_{\lceil \tilde{I} \rceil}} \|\hat{\rho}\|_{\mathrm{C}^{\lceil \tilde{I} \rceil}} \|\hat{\zeta}\|_{H_{-\tilde{I}}}$$
$$\leq \frac{C_{I,\tilde{I},\tilde{J}}}{n^{d/2}} \|F\|_{\mathrm{C}^{1,3}_{l,HS}} \|\hat{\rho}\|_{\mathrm{C}^{\lceil \tilde{I} \rceil}} \|\hat{\zeta}\|_{H_{-\tilde{I}}}$$

holds due to $\lceil \tilde{I} \rceil + 1 + \frac{d}{2} < I$. Here we estimated $\|\mathrm{ex}_n U_{j,n}(\rho,\eta)\|_{H_{\tilde{I}}}$ as in $(I_{4,3})$. The term $I_{4,4,3}$, can be estimated similarly to $I_{4,4,2}$ by the same expression. Due to the equality (2.8), we get

$$\begin{aligned} I_{4,4,4} &= \frac{1}{(2n+1)^d} \left\langle U_{j,n}(\rho,\eta), \zeta \tau_j^n \zeta \right\rangle_n \\ &= \frac{1}{(2n+1)^d} \left\langle \mathrm{pr}_n \mathrm{Tr} \left(\partial_j^{\otimes 2} D_2^2 F(\hat{\rho},\hat{\zeta}) \right), \zeta \tau_j^n \zeta \right\rangle_n + \left\langle R_{j,n}^{4,1}(\rho,\eta), (\eta-\rho) \tau_j^n (\eta-\rho) \right\rangle_n . \end{aligned}$$

Now, by (2.9) and the boundedness of η and ρ, we obtain

$$\left| \left\langle R_{j,n}^{4,1}(\rho,\eta), (\eta-\rho) \tau_j^n (\eta-\rho) \right\rangle_n \right| \leq \frac{C_I}{n} \|F\|_{\mathrm{C}^{1,3}_{l,HS}} .$$

This completes the proof of the proposition. □

2.3 Some properties of the density fluctuation field

The goal of this section is to estimate the Sobolev norm of the density fluctuation field and the expectation of the term $\left\langle f, \mathrm{ex}_n \left[\zeta_t^n \tau_j^n \zeta_t^n \right] \right\rangle$ appearing in the expansion of the generator $\mathcal{G}^{FF}$. We first prove an auxiliary statement.

Lemma 2.4. *Let $\rho_0^n \in L_2(\mathbb{T}_n^d)$ take values in $[0,1]$, $\varphi \in \mathrm{C}(\mathbb{T}^d)$ and $(\eta_t^n)_{t \geq 0}$ be the SSEP started from $\eta_0^n = (\eta_0^n(x))_{x \in \mathbb{T}_n^d}$ for each $n \geq 1$, where $\eta_0^n(x)$, $x \in \mathbb{T}_n^d$, are independent random variables with Bernoulli distribution with parameters $\rho_0^n(x)$, $x \in \mathbb{T}_n^d$, respectively. Let also $\rho_t^n(x) = \mathbb{E}\eta_t^n(x)$, $x \in \mathbb{T}_n^d$, $t \geq 0$, and $\zeta_t^n = (2n+1)^{d/2} (\eta_t^n - \rho_t^n)$, $t \geq 0$. Then, for every $t \geq 0$,*

$$\mathbb{E}\left[\langle \mathrm{ex}_n \zeta_t^n, \varphi \rangle^2 \right] \leq \left(1 + 2\pi^2 t \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2 \right) \|\mathrm{pr}_n \varphi\|_{n,\mathrm{C}}^2 .$$

Proof. We set $\varphi_n := \mathrm{pr}_n \varphi$ and rewrite for $n \geq 1$

$$\begin{aligned} \mathbb{E}\left[\langle \mathrm{ex}_n \zeta_t^n, \varphi \rangle^2 \right] = \mathbb{E}\left[\langle \zeta_t^n, \mathrm{pr}_n \varphi \rangle_n^2 \right] &= \frac{1}{(2n+1)^{2d}} \left[\sum_{x \in \mathbb{T}_n^d} \zeta_t^n(x) \varphi_n(x) \right]^2 \\ &= \frac{1}{(2n+1)^{2d}} \sum_{x,y \in \mathbb{T}_n^d} \mathbb{E}\left[\zeta_t^n(x) \zeta_t^n(y) \right] \varphi_n(x) \varphi_n(y) \\ &= \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \mathbb{E}\left[(\eta_t^n(x) - \rho_t^n(x))^2 \right] \varphi_n^2(x) \\ &+ \frac{1}{(2n+1)^d} \sum_{x \neq y \in \mathbb{T}_n^d} \mathbb{E}\left[(\eta_t^n(x) - \rho_t^n(x))(\eta_t^n(y) - \rho_t^n(y)) \right] \varphi_n(x) \varphi_n(y). \end{aligned}$$

The first term of the right hand side of the equality above can be estimated by

$$\frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} \varphi_n^2(x) = \|\mathrm{pr}_n \varphi\|_n^2,$$

due to the fact that $\eta_t^n(x) - \rho_t^n(x) \in [0,1]$ for all $x \in \mathbb{T}_n^d$ and $t \geq 0$. The second term can be estimated by

$$2\pi^2 \sup_{s\in[0,t]} \max_{u\in\mathbb{T}_n^d} |\nabla_n \rho_s^n(u)|^2 \, \|\mathrm{pr}_n \varphi\|_{n,\mathrm{C}}^2 t$$

similarly to the proof of the main theorem in [56, p. 32] (see also Section C for the detailed estimate). Combining both estimates together, we get

$$\begin{aligned}
\mathbb{E}\left[\langle \mathrm{ex}_n \zeta_t^n, \varphi\rangle^2\right] &\leq \|\varphi_n\|_n^2 + 2\pi^2 \sup_{s\in[0,t]} \max_{u\in\mathbb{T}_n^d} |\nabla_n \rho_s^n(u)|^2 \, \|\varphi_n\|_{n,\mathrm{C}}^2 t \\
&\leq \left(1 + 2\pi^2 t \sup_{s\in[0,t]} \|\nabla_n \rho_s^n\|_{n,\mathrm{C}}^2\right) \|\varphi_n\|_{n,\mathrm{C}}^2 \\
&\leq \left(1 + 2\pi^2 t \, \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2\right) \|\varphi_n\|_{n,\mathrm{C}}^2,
\end{aligned}$$

according to the fact that ρ_t^n, $t \geq 0$, is a solution to (2.2) and the maximum principle. This completes the proof of the lemma. □

Lemma 2.5. *Let $I > \frac{d}{2}$. Under the assumptions of Lemma 2.4, for every $t \geq 0$ one has*

$$\mathbb{E}\left[\|\mathrm{ex}_n \zeta_t^n\|_{H_{-I}}^2\right] \leq C_I \left(1 + 2\pi^2 t \, \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2\right).$$

Proof. By the definition of $\|\cdot\|_{H_{-I}}$ and Lemma 2.4, we get

$$\begin{aligned}
\mathbb{E}\left[\|\mathrm{ex}_n \zeta_t^n\|_{H_{-I}}^2\right] &= \sum_{k\in\mathbb{Z}^d} \left(1+|k|^2\right)^{-I} \mathbb{E}\, |\langle \mathrm{ex}_n \zeta_t^n, \tilde{\varsigma}_k\rangle|^2 \\
&\leq \sum_{k\in\mathbb{Z}^d} \left(1+|k|^2\right)^{-I} \left(1 + 2\pi^2 t \, \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2\right) \|\mathrm{pr}_n \tilde{\varsigma}_k\|_{n,\mathrm{C}}^2 \\
&= \sum_{k\in\mathbb{Z}_n^d} \left(1+|k|^2\right)^{-I} \left(1 + 2\pi^2 t \, \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2\right) \|\tilde{\varsigma}_k\|_{n,\mathrm{C}}^2 \\
&\leq C_I \left(1 + 2\pi^2 t \, \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}^2\right),
\end{aligned}$$

where we also used the boundedness of $\tilde{\varsigma}_k$ for the estimate of $\|\tilde{\varsigma}_k\|_{n,\mathrm{C}}$. The proof of the lemma is complete. □

We recall that τ_j^n denotes the shift operator on $\mathbb{T}_n^d$ defined by $\tau_j^n f(x) = \tau_j^n(x + e_j^n)$.

Lemma 2.6. *Let $J > \frac{d}{2}$. Under the assumptions of Lemma 2.4, for every $T > 0$ there exists a constant C depending on J, T and $\sup_{n\geq 1} \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}$ such that for every random variable f in H_J with a finite second moment and defined on the same probability space as ζ^n we have*

$$\left|\frac{1}{(2n+1)^d} \mathbb{E}\left\langle f, \mathrm{ex}_n\left[\zeta_t^n \tau_j^n \zeta_t^n\right]\right\rangle\right| \leq \frac{C}{n^{\frac{d}{2}\wedge 1}} \mathbb{E}\left[\|f\|_{H_J}^2\right]^{\frac{1}{2}}$$

for each $n \geq 1$, $j \in [d]$ and $t \in [0,T]$.

Proof. Using Parseval's identity, (A.9), the Cauchy-Schwarz inequality and (B.2) (i), we get

$$\left|\frac{1}{(2n+1)^d} \mathbb{E}\left\langle f, \mathrm{ex}_n\left[\zeta_t^n \tau_j^n \zeta_t^n\right]\right\rangle\right|^2 = \left|\frac{1}{(2n+1)^d} \mathbb{E}\langle \mathrm{pr}_n f, \zeta_t^n \tau_j^n \zeta_t^n\rangle_n\right|^2$$

$$
\begin{aligned}
&= \frac{1}{(2n+1)^{2d}} \left| \sum_{k\in\mathbb{Z}_n^d} \mathbb{E}\left[\langle \mathrm{pr}_n f, \varsigma_k\rangle_n \langle \varsigma_k, \zeta_t^n \tau_j^n \zeta_t^n\rangle_n\right] \right|^2 \\
&\le \frac{1}{(2n+1)^{2d}} \sum_{k\in\mathbb{Z}_n^d} (1+|k|^2)^J \mathbb{E}\left[|\langle \mathrm{pr}_n f, \varsigma_k\rangle_n|^2 \right] \\
&\qquad \cdot \sum_{k\in\mathbb{Z}_n^d} \frac{1}{(1+|k|^2)^J} \mathbb{E}\left[|\langle \varsigma_k, \zeta_t^n \tau_j^n \zeta_t^n\rangle|^2 \right] \\
&\le \frac{C_J}{(2n+1)^{2d}} \mathbb{E}\left[\|f\|_{H_J}^2\right] \max_{k\in\mathbb{Z}_n^d} \mathbb{E}\left[|\langle \varsigma_k, \zeta_t^n \tau_j^n \zeta_t^n\rangle|^2 \right]
\end{aligned}
$$

since $J > \frac{d}{2}$. We next estimate for each $k \in \mathbb{Z}_n^d$

$$
\begin{aligned}
&\frac{1}{(2n+1)^{2d}} \mathbb{E}\left[|\langle \varsigma_k, \zeta_t^n \tau_j^n \zeta_t^n\rangle|^2 \right] = \frac{1}{(2n+1)^{2d}} \mathbb{E}\left[\langle \varsigma_k, \zeta_t^n \tau_j^n \zeta_t^n\rangle \langle \zeta_t^n \tau_j^n \zeta_t^n, \varsigma_k\rangle \right] \\
&\quad = \frac{1}{(2n+1)^{2d}} \sum_{x,y\in\mathbb{T}_n^d} \varsigma_k(x)\varsigma_{-k}(y) \mathbb{E}\Big[(\eta_t^n(x) - \rho_t^n(x))(\eta_t^n(x+e_j^n) - \rho_t^n(x+e_j^n)) \\
&\qquad\qquad \cdot (\eta_t^n(y) - \rho_t^n(y))(\eta_t^n(y+e_j^n) - \rho_t^n(y+e_j^n)) \Big].
\end{aligned}
$$

Following the observation in [44, Theorem 6.1], that in our setting will follow from similar computations [29], we can bound the expectation above by $\frac{C}{n^2}$ for distinct $x, x+e_j^n, y, y+e_j^n$, where the constant C depends on T and $\sup_{n\ge 1} \|\nabla_n \rho_0^n\|_{n,\mathrm{C}}$. The cardinality of the set

$$
\left\{ (x,y) \in \left(\mathbb{T}_n^d\right)^2 : \ x, x+e_j^n, y, y+e_j^n \text{ are not distinct} \right\}
$$

is bounded by $3(2n+1)^d$. Thus, we can continue the estimate by

$$
\frac{1}{(2n+1)^{2d}} \left[\frac{(2n+1)^{2d} C}{n^2} + 3(2n+1)^d \right] = \frac{C}{n^2} + \frac{3}{(2n+1)^d}.
$$

Consequently, there exists a constant $C > 0$ such that

$$
\left| \frac{1}{(2n+1)^d} \mathbb{E}\left\langle f, \mathrm{ex}_n\left[\zeta_t^n \tau_j^n \zeta_t^n\right]\right\rangle \right|^2 \le \frac{C}{n^{2\wedge d}} \mathbb{E}\left[\|f\|_{H_J}^2\right].
$$

This completes the proof of the statement. □

3 Generalized Ornstein-Uhlenbeck process

The main result of this section is the regularity of the solution U_t, $t \ge 0$, to the infinite-dimensional Kolmogorov backward equation corresponding to the system of SPDEs (1.2), (1.3), which is defined by $U_t := P_t^{OU} F$.

The proof of this regularity faces several challenges due to the form of the diffusion terms in (1.3). Firstly, $\sqrt{\rho(1-\rho)}$ is not differentiable, which prevents from following the usual approach to deduce the regularity of U_t from the regularity of solutions to (1.3) with respect to their initial conditions. Secondly, the variance term $\rho(1-\rho)$ is non-negative only for $\rho \in [0,1]$, and, as a result, the function U_t is well-defined only on a subset of $H_J \times H_{-I}$. This is particularly problematic since the discrete semigroup $(\hat\rho_s^n, \hat\zeta_s^n)$ does not necessarily take values in this domain, since $\hat\rho_s^n$ is not a $[0,1]$-valued function in general. However, this property is crucial for our main approach based on (1.5).

To overcome the latter problem and also to avoid the discussion of the differentiability of U_t at boundary points of its domain, in this section we first approximate $\rho(1-\rho)$ in the SPDE (1.3) by a smooth mollification Φ^ε of the non-negative function $\rho(1-\rho)\vee 0$, such that $\sup_{\varepsilon\in(0,1]}\|\Phi^\varepsilon\|_{C^1}<\infty$. This allows to approximate the function U_t by solutions U_t^ε to Kolmogorov equations that now are well-defined on the complete space $H_J\times H_{-I}$. Then, in Section 5, we compare the corresponding generators on the functions U_t^ε and show that the additional mollification error can be well-controled.

The remaining difficulty of the non-differentiability of the diffusion coefficient $\sqrt{\rho(1-\rho)}$ is addressed in Section 3.2 below.

3.1 Covariance and Itô's formula

In this section, we fix a continuous bounded function $\Phi:\mathbb{R}\to[0,\infty)$ and build a Gaussian process in H_{-I} for some I that will be used for the description of fluctuations of the SSEP. We first consider the heat equation

$$d\rho_t^\infty = 2\pi^2\Delta\rho_t^\infty dt \tag{3.1}$$

in H_J, for some $J\geq 0$, with initial condition $\rho_0\in H_J$. It is well-known that there exists a (continuous) H_J-valued weak solution $(\rho_t^\infty)_{t\geq 0}$ to the heat equation (3.1). The semigroup associated with the PDE (3.1) will be denoted by P_t, $t\geq 0$. In particular,

$$\rho_t^\infty = P_t\rho_0,\quad t\geq 0.$$

We next define the generalized Ornstein-Uhlenbeck process $(\zeta_t^\infty)_{t\geq 0}$ as the variational[4] solution to the SPDE

$$d\zeta_t^\infty = 2\pi^2\Delta\zeta_t^\infty dt + 2\pi\nabla\cdot\left(\sqrt{\Phi(\rho_t^\infty)}dW_t\right), \tag{3.2}$$

where dW is a d-dimensional white noise. The differentiability of the associated semigroup will follow from the differentiability of the variance operator for the Ornstein-Uhlenbeck process whose precise form is described in the next proposition.

Proposition 3.1. *Let Φ be a bounded non-negative continuous function. For each $I>\frac{d}{2}+1$, $\rho_0\in L_2(\mathbb{T}^d)$ and $\zeta_0\in H_{-I}$ there exists a unique continuous H_{-I}-valued variational solution $(\zeta_t^\infty)_{t\geq 0}$ to the SPDE (3.2) started from ζ_0 and*

$$\mathbb{E}\sup_{t\in[0,T]}\|\zeta_t^\infty\|_{H_{-I}}^2<\infty$$

for each $T>0$, where $(\rho_t^\infty)_{t\geq 0}$ solves the heat equation (3.2) with initial condition ρ_0. Moreover, ζ^∞ is a Gaussian process in H_{-I} with expectation

$$m_t(\zeta_0)[\varphi] := \mathbb{E}\langle\varphi,\zeta_t^\infty\rangle = \langle P_t\varphi,\zeta_0\rangle,\quad \varphi\in C^\infty(\mathbb{T}^d), \tag{3.3}$$

and covariance operator

$$\begin{aligned} V_t(\rho)[\varphi,\psi] &:= \operatorname{Cov}(\langle\zeta_t^\infty,\varphi\rangle,\langle\psi,\zeta_t^\infty\rangle) \\ &= 2\pi^2\int_0^t\langle\nabla P_{t-s}\varphi\cdot\nabla P_{t-s}\psi,\Phi(P_s\rho)\rangle\,ds,\quad \varphi,\psi\in C^\infty(\mathbb{T}^d), \end{aligned} \tag{3.4}$$

for each $t>0$.

[4] See [49, Definition 4.2.1]

Proof. The existence and uniqueness of the variational solution to the SPDE (3.2) follows from [49, Theorem 4.2.4] and the fact that $B(t) : L_2(\mathbb{T}^d; \mathbb{R}^d) \to H_{-I}$ defined by

$$B(t)h := 2\pi\nabla \cdot \left(\sqrt{\Phi(\rho_t)}h\right) \tag{3.5}$$

is a Hilbert-Schmidt operator with Hilbert-Schmidt norm

$$\begin{aligned}\|B(t)\|_{HS}^2 &:= \sum_{l=1}^{\infty} \|B(t)h_l\|_{H_{-I}}^2 = 4\pi^2 \sum_{l=1}^{\infty} \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-I} \left|\langle \sqrt{\Phi(\rho_t)}h_l, \nabla\varsigma_k\rangle\right|^2 \\ &= 4\pi^2 \sum_{l=1}^{\infty} \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-I} \left|\langle h_l, \sqrt{\Phi(\rho_t)}k\varsigma_k\rangle\right|^2 \\ &= 4\pi^2 \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-I+1} \|\sqrt{\Phi(\rho_t)}\varsigma_k\|^2 \le 4\pi^2\|\Phi\|_{\mathrm{C}} \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-I+1} < \infty,\end{aligned}$$

where $\{h_l,\ l \ge 1\}$ is an orthonormal basis of $(L_2(\mathbb{T}^d))^d$. Note that the construction of the variational solution is obtained by Galerkin approximation leading to linear SDEs [49, (4.48)]. This implies that the process ζ^∞ is Gaussian in H_{-I} as a limit of Gaussian processes.

Let $T > 0$ and $\varphi \in \mathrm{C}^\infty(\mathbb{T}^d)$ be fixed. Consider $\psi_t := P_{T-t}\varphi \in H_{I+2}$ for all $t \in [0,T]$ and use the martingale problem for ζ^∞ and Itô's formula to get

$$\begin{aligned}\langle \psi_t, \zeta_t^\infty\rangle &= \langle \psi_0, \zeta_0\rangle + \int_0^t \langle \partial_s\psi_s, \zeta_s^\infty\rangle\, ds + 2\pi^2 \int_0^t \langle \Delta\psi_s, \zeta_s^\infty\rangle ds + \text{mart.} \\ &= \langle P_T\varphi, \zeta_0\rangle + \text{mart.}\end{aligned}$$

for all $t \in [0,T]$ a.s. Thus, taking the expectation and setting $t = T$, we get

$$m_T(\zeta_0)[\varphi] = \mathbb{E}\,\langle \varphi, \zeta_T^\infty\rangle = \langle P_T\varphi, \zeta_0\rangle\,. \tag{3.6}$$

Similarly, we compute

$$\begin{aligned}\langle \psi_t, \zeta_t^\infty\rangle^2 &= \langle \psi_0, \zeta_0\rangle^2 + 2\int_0^t \langle \psi_s, \zeta_s^\infty\rangle\,\langle \partial_s\psi_s, \zeta_s^\infty\rangle\, ds + 4\pi^2 \int_0^t \langle \psi_s, \zeta_s^\infty\rangle\langle \Delta\psi_s, \zeta_s^\infty\rangle ds \\ &\quad + 2\pi^2 \int_0^t \left\langle |\nabla\psi_s|^2, \Phi(\rho_s^\infty)\right\rangle ds + \text{mart.} \\ &= \langle P_T\varphi, \zeta_0\rangle^2 + 2\pi^2 \int_0^t \left\langle |\nabla\psi_s|^2, \Phi(\rho_s^\infty)\right\rangle ds + \text{mart.}\end{aligned}$$

Therefore, using (3.6), we obtain for $t = T$

$$\begin{aligned}\mathrm{Var}\,\langle \varphi, \zeta_T^\infty\rangle &= \mathrm{Var}\,\langle \psi_T, \zeta_T^\infty\rangle = \mathbb{E}\left[\langle \psi_T, \zeta_T^\infty\rangle^2\right] - \left[\mathbb{E}\langle \psi_T, \zeta_T^\infty\rangle\right]^2 \\ &= \langle P_T\varphi, \zeta_0\rangle^2 + 2\pi^2 \int_0^T \left\langle |\nabla\psi_s|^2, \Phi(\rho_s^\infty)\right\rangle ds - \langle P_T\varphi, \zeta_0\rangle^2 \\ &= 2\pi^2 \int_0^T \left\langle |\nabla P_{T-s}\varphi|^2, \Phi(\rho_s^\infty)\right\rangle ds.\end{aligned}$$

The expression for the covariance operator $V_t(\rho)$ follows from the polarization equality. This completes the proof of the proposition. □

Remark 3.2. The statement of the theorem remains valid if $\Phi(\rho_t)$ is replaced by Φ_t for each measurable locally bounded function $\Phi : [0,\infty) \to L_2(\mathbb{T}^d)$ with $\Phi_t \ge 0$ for all $t \ge 0$.

Lemma 3.3. *Let $\Phi^n : [0,\infty) \to L_2(\mathbb{T}^d)$, $n \in \mathbb{N}_0$, be locally bounded functions such that $\Phi_t^n \geq 0$ for all $t \geq 0$, $n \in \mathbb{N}_0$ and*

$$\sup_{t\in[0,T]} \|\Phi_t^n - \Phi_t^0\| \to 0, \quad n \to \infty,$$

for each $T > 0$. Additionally assume that $\zeta^n \to \zeta^0$ in H_{-I} for some $I > \frac{d}{2} + 1$ and $t_n \to t_0$ in $[0,\infty)$ as $n \to \infty$. Let also $(\zeta_t^{\infty,n})_{t\geq 0}$ be a (variational) solution to

$$d\zeta_t^{\infty,n} = 2\pi^2 \Delta \zeta_t^{\infty,n} dt + 2\pi \nabla \cdot \left(\sqrt{\Phi_t^n} dW_t \right)$$

started from ζ^n for every $n \in \mathbb{N}_0$. Then $\mathrm{Law}\zeta_{t_n}^{\infty,n} \to \mathrm{Law}\zeta_t^{\infty,0}$ in the 2-Wasserstein topology on the space of probability measures on H_{-I} with a finite second moment as $n \to \infty$. In particular, for each $F \in \mathrm{C}_l^1(H_{-I})$

$$\mathbb{E}F(\zeta_{t_n}^{\infty,n}) \to \mathbb{E}F(\zeta_t^{\infty,0})$$

as $n \to \infty$.

Proof. We will first show that $\zeta_{t_n}^{\infty,n} \to \zeta_{t_0}^{\infty,0}$ in distribution as $n \to \infty$, using [5, Example 3.8.15]. For this we will show that the means $\mathbb{E}\zeta_{t_n}^{\infty,n}$ converge to $\mathbb{E}\zeta_{t_0}^{\infty,0}$ in H_{-I}, the covariance operators $V_{t_n}^n$ of $\zeta_{t_n}^{\infty,n}$ converge to the covariance operator $V_{t_0}^0$ of $\zeta_{t_0}^{\infty,0}$ in $\mathcal{L}_2(H_I)$ and $\mathbb{E}[\|\zeta_{t_n}^{\infty,n}\|_{H_{-I}}^2] \to \mathbb{E}[\|\zeta_{t_0}^{\infty,0}\|_{H_{-I}}^2]$.
By Proposition 3.1 and Remark 3.2, we get

$$\begin{aligned}\left\|\mathbb{E}\zeta_{t_n}^{\infty,n} - \mathbb{E}\zeta_{t_0}^{\infty,0}\right\|_{H_{-I}} &= \left\|P_{t_n}\zeta^n - P_{t_0}\zeta^0\right\|_{H_{-I}} \\ &\leq \left\|P_{t_n}\left(\zeta^n - \zeta^0\right)\right\|_{H_{-I}} + \left\|P_{t_n}\zeta^0 - P_{t_0}\zeta^0\right\|_{H_{-I}} \\ &\leq \|\zeta^n - \zeta^0\|_{H_{-I}} + \left\|P_{t_n}\zeta^0 - P_{t_0}\zeta^0\right\|_{H_{-I}} \to 0\end{aligned}$$

as $n \to \infty$. We similarly estimate

$$\left\|V_{t_n}^n - V_{t_0}^0\right\|_{\mathcal{L}_2} \leq \|V_{t_n}^n - V_{t_n}^0\|_{\mathcal{L}_2} + \|V_{t_n}^0 - V_{t_0}^0\|_{\mathcal{L}_2}.$$

The fact that $\|V_{t_n}^0 - V_{t_0}^0\|_{\mathcal{L}_2} \to 0$ follows from the continuity of $(\zeta_t^{\infty,0})_{t\geq 0}$ in H_{-I} and [5, Example 3.8.15]. Next, using Proposition 3.1 and Remark 3.2 again, we estimate

$$\begin{aligned}\left\|V_{t_n}^n - V_{t_n}^0\right\|_{\mathcal{L}_2}^2 &\leq \left\|V_{t_n}^n - V_{t_n}^0\right\|_{\mathcal{L}_2^{HS}}^2 \\ &= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{-I}(1+|l|^2)^{-I} \left|V_{t_n}^n(\tilde{\varsigma}_k,\tilde{\varsigma}_l) - V_{t_n}^0(\tilde{\varsigma}_k,\tilde{\varsigma}_l)\right|^2 \\ &\leq 4\pi^4 t_n \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{-I}(1+|l|^2)^{-I} \\ &\qquad \cdot \int_0^{t_n} \left|\left\langle \nabla P_{t_n-s}\tilde{\varsigma}_k \cdot \nabla P_{t_n-s}\tilde{\varsigma}_l, \Phi_s^n - \Phi_s^0 \right\rangle\right|^2 ds.\end{aligned}$$

According to the fact that

$$P_t\tilde{\varsigma}_k = e^{-2\pi^2|k|^2 t}\tilde{\varsigma}_k, \quad k \in \mathbb{Z}^d,$$

we get

$$\nabla P_{t_n-s}\tilde{\varsigma}_k \cdot \nabla P_{t_n-s}\tilde{\varsigma}_l = -e^{-2\pi^2(|k|^2+|l|^2)(t_n-s)} k \cdot l \tilde{\varsigma}_k\tilde{\varsigma}_l.$$

We now separately estimate for $k, l \in \mathbb{Z}^d$ and a bounded measurable function $f : [0,t] \to L_2(\mathbb{T}^d)$

$$\begin{aligned}\int_0^{t_n} \langle \nabla P_{t_n-s}\tilde{\varsigma}_k \cdot \nabla P_{t_n-s}\tilde{\varsigma}_l, f_s\rangle^2\, ds &\le |k|^2|l|^2 \int_0^{t_n} e^{-4\pi^2(|k|^2+|l|^2)(t_n-s)}\langle \tilde{\varsigma}_k\tilde{\varsigma}_l, f_s\rangle^2 ds \\ &\le |k|^2|l|^2 \sup_{s\in[0,t_n]} \|f_s\|^2 \int_0^{t_n} e^{-4\pi^2(|k|^2+|l|^2)(t_n-s)}ds \\ &\le \frac{\sup_{s\in[0,t_n]}\|f_s\|^2}{4\pi^2}\frac{|k|^2|l|^2}{|k|^2+|l|^2}.\end{aligned} \tag{3.7}$$

Hence, due to the fact that $I > \frac{d}{2}+1$, we conclude that

$$\left\|V^n_{t_n} - V^0_{t_n}\right\|^2_{\mathcal{L}_2^{HS}} \le C_I t_n \sup_{s\in[0,t_n]} \|\Phi^n_s - \Phi^0_s\| \to 0$$

as $n \to \infty$. The convergence of the second moments $\mathbb{E}[\|\zeta^{\infty,n}_{t_n}\|^2_{H_{-I}}]$ to the second moment $\mathbb{E}[\|\zeta^{\infty,0}_t\|^2_{H_{-I}}]$ can be proved similarly. Hence, by [5, Example 3.8.15], $\zeta^{\infty,n}_{t_n} \to \zeta^{\infty,0}_{t_0}$ in H_{-I} in distribution as $n \to \infty$. Now, using the fact that $\mathbb{E}[\|\zeta^{\infty,n}_{t_n}\|^2_{H_{-I}}] \to \mathbb{E}[\|\zeta^{\infty,0}_t\|^2_{H_{-I}}]$ as $n \to \infty$, we can conclude that $\mathrm{Law}\zeta^{\infty,n}_{t_n} \to \mathrm{Law}\zeta^{\infty,0}_{t_0}$ in the 2-Wasserstein topology on the space of probability measures on H_{-I}, by [64, Theorem I.6.9]. This easily implies the second part of the lemma. □

Remark 3.4. According to the definition of variational solutions, we have

$$\rho^\infty_t = \rho_0 + 2\pi^2 \int_0^t \Delta\rho^\infty_s ds, \quad t \ge 0,$$

in H_{J-2} and

$$\zeta^\infty_t = \zeta_0 + 2\pi^2 \int_0^t \Delta\zeta^\infty_s ds + 2\pi \int_0^t B(s)dW_s, \quad t \ge 0,$$

in H_{-I-2}.

We will provide here the Itô formula for the process $(\rho^\infty, \zeta^\infty)$. Note that while Itô's formula for Hilbert space valued processes is available in the literature, we need to obtain the resulting Itô -correction term in a particular form. We therefore include the result.

Lemma 3.5. *Let $I > \frac{d}{2}+1$, $J \ge 0$, $F \in \mathrm{C}^{1,1,2}([0,\infty), H_{J-2}, H_{-I-2})$, $D_2^2 F$ take values in $\mathcal{L}_2^{HS}(H_{-I-2})$ and $(\rho^\infty_t, \zeta^\infty_t)$, $t \ge 0$, be a solution in $H_J \times H_{-I}$ to (3.1), (3.2) started from $(\rho_0, \zeta_0) \in H_J \times H_{-I}$. Then*

$$\begin{aligned}F_t(\rho^\infty_t, \zeta^\infty_t) = F_0(\rho_0,\zeta_0) &+ 2\pi \int_0^t \langle D_2F_s(\rho^\infty_s,\zeta^\infty_s), B(s)dW_s\rangle \\ &+ \int_0^t \partial F_s(\rho^\infty_s,\zeta^\infty_s)ds + 2\pi^2 \int_0^t \langle \Delta D_1F_s(\rho^\infty_s,\zeta^\infty_s), \rho^\infty_s\rangle\, ds \\ &+ 2\pi^2 \int_0^t \langle \Delta D_2F_s(\rho^\infty_s,\zeta^\infty_s), \zeta^\infty_s\rangle\, ds \\ &+ 2\pi^2 \int_0^t \sum_{j=1}^d \left\langle \mathrm{Tr}\left(\partial_j^{\otimes 2} D_2^2F_s(\rho^\infty_s,\zeta^\infty_s)\right), \Phi(\rho^\infty_s)\right\rangle ds\end{aligned}$$

for all $t \ge 0$, where $(B(t))_{t\ge0}$ is defined by (3.5).

Proof. We first note that according to the assumptions on J and I, the process $(\rho_t^\infty, \zeta_t^\infty)$, $t \geq 0$, has a continuous version in $H_J \times H_{-I}$ and thus the identities of Remark 3.4 hold in the spaces H_{J-2} and H_{-I-2}, respectively. Using then the infinite-dimensional Itô formula[5] in the Hilbert space $H_{J-2} \times H_{-I-2}$, we get

$$\begin{aligned} F_t(\rho_t^\infty, \zeta_t^\infty) &= F_0(\rho_0, \zeta_0) + 2\pi \int_0^t \langle D_2 F_s(\rho_s^\infty, \zeta_s^\infty), B(s) dW_s \rangle \\ &+ \int_0^t \partial F_s(\rho_s^\infty, \zeta_s^\infty) ds + 2\pi^2 \int_0^t \langle D_1 F_s(\rho_s^\infty, \zeta_s^\infty), \Delta \rho_s^\infty \rangle \, ds \\ &+ 2\pi^2 \int_0^t \langle D_2 F_s(\rho_s^\infty, \zeta_s^\infty), \Delta \zeta_s^\infty \rangle \, ds \\ &+ \frac{1}{2} \int_0^t \operatorname{trace} \left[D_2^2 F_s(\rho_s^\infty, \zeta_s^\infty) B(s) B^*(s) \right] ds, \end{aligned} \tag{3.8}$$

where $B^*(s) : H_{-I-2} \to (L_2(\mathbb{T}^d))^d$ is the adjoint operator to $B(s)$ and $U(s) := D_3^2 F_s(\rho_s, \zeta_s) B(s) B^*(s)$ is interpreted as a bounded linear operator on H_{-I-2} defined by

$$\langle U(s) \varsigma_k, \varsigma_l \rangle_{H_{-I-2}} = D_2^2 F_s(\rho_s, \zeta_s) \left[B(s) B^*(s) \varsigma_k, \varsigma_{-l} \right], \quad k, l \in \mathbb{Z}^d.$$

We next rewrite the last term in the right hand side of (3.8). For this, we take the orthonormal basis $\left\{ \left(1 + |k|^2\right)^{(I+2)/2} \varsigma_k, \ k \in \mathbb{Z}^d \right\}$ on H_{-I-2} and compute

$$\operatorname{trace}\left[U(s)\right] = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{I+2} D_2^2 F_s(\rho_s, \zeta_s) \left[B(s) B^*(s) \varsigma_k, \varsigma_{-k} \right].$$

Taking also an orthonormal basis $\{h_l = (h_l^j)_{j \in [d]}, \ l \in \mathbb{N}\}$ on $(L_2(\mathbb{T}^d))^d$, we can expand $B^*(s)\varsigma_k$ in the Fourier series

$$\begin{aligned} B^*(s)\varsigma_k &= \sum_{l=1}^\infty \langle B^*(s) \varsigma_k, h_l \rangle h_l = \sum_{l=1}^\infty \langle \varsigma_k, B(s) h_l \rangle_{H_{-I}} h_l \\ &= (1 + |k|^2)^{-I-2} \sum_{l=1}^\infty \langle \varsigma_k, B(s) h_l \rangle h_l \\ &= 2\pi (1 + |k|^2)^{-I-2} \sum_{j=1}^d \sum_{l=1}^\infty \left\langle \varsigma_k, \partial_j \left(\sqrt{\Phi(\rho_s)} h_l^j \right) \right\rangle h_l \\ &= -2\pi (1 + |k|^2)^{-I-2} \sum_{j=1}^d \sum_{l=1}^\infty \mathrm{i} k_j \left\langle \varsigma_k \sqrt{\Phi(\rho_s)}, h_l^j \right\rangle h_l \\ &= \left(-2\pi \mathrm{i} (1 + |k|^2)^{-I-2} k_j \varsigma_k \sqrt{\Phi(\rho_s)} \right)_{j \in [d]}. \end{aligned}$$

Thus,

$$B(s) B^*(s) \varsigma_k = -4\pi^2 (1 + |k|^2)^{-I-2} \mathrm{i} \sum_{j=1}^d k_j \partial_j \left(\varsigma_k \Phi(\rho_s) \right)$$

and, consequently,

$$\operatorname{trace}\left[U(s)\right] = -4\pi^2 \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \mathrm{i} k_j D_2^2 F_s(\rho_s, \zeta_s) \left[\partial_j \left(\varsigma_k \Phi(\rho_s) \right), \varsigma_{-k} \right]$$

[5] see e.g. [33, Theorem 2.10]

$$= 4\pi^2 \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}^d} D_2^2 F_s(\rho_s, \zeta_s) \left[\partial_j \left(\varsigma_k \Phi(\rho_s)\right), \partial_j \varsigma_{-k}\right].$$

Using the expansion of $\Phi(\rho_s)$ in the Fourier series

$$\Phi(\rho_s) = \sum_{l\in\mathbb{Z}^d} \langle \Phi(\rho_s), \varsigma_l\rangle \varsigma_l = \sum_{l\in\mathbb{Z}^d} \langle \varsigma_{-l}, \Phi(\rho_s)\rangle \varsigma_l$$
$$= \sum_{l\in\mathbb{Z}^d} \langle \varsigma_l, \Phi(\rho_s)\rangle \varsigma_{-l},$$

we get

$$\operatorname{trace}[U(s)] = 4\pi^2 \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} D_2^2 F_s(\rho_s, \zeta_s) \left[\partial_j \varsigma_{k-l}, \partial_j \varsigma_{-k}\right] \langle \varsigma_l, \Phi(\rho_s)\rangle$$
$$= 4\pi^2 \sum_{j=1}^{d} \left\langle \sum_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^d} D_2^2 F_s(\rho_s, \zeta_s) \left[\partial_j \varsigma_{k-l}, \partial_j \varsigma_{-k}\right] \varsigma_l, \Phi(\rho_s) \right\rangle$$
$$= 4\pi^2 \sum_{j=1}^{d} \left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2} D_2^2 F_s(\rho_s, \zeta_s)\right), \Phi(\rho_s)\right\rangle,$$

according to Lemma B.15. This completes the proof of the lemma. □

3.2 Differentiability of the Ornstein-Uhlenbeck semigroup

Let $(\rho_t^\infty)_{t\geq 0}$ and $(\eta_t^\infty)_{t\geq 0}$ be solutions to (3.1) and (3.2) with a bounded continuous function $\Phi : \mathbb{R} \to [0, \infty)$, respectively. In this section, we consider these processes as functions of their initial conditions $\rho := \rho_0^\infty$ and $\zeta := \zeta_0^\infty$ and study the differentiability of

$$U_t^\Phi(\rho, \zeta) := \mathbb{E} F(\rho_t^\infty, \zeta_t^\infty)$$

with respect to (ρ, ζ) for $F \in \mathrm{C}_{l,HS}^{1,3}(H_J, H_{-I})$.

The fact that U^Φ is three times continuously differentiable with respect to ζ directly follows from the linearity of ζ^∞ in ζ, see the proof of Proposition 3.9 below. Hence, the main challenge is the regularity of U^Φ with respect to ρ. The main difficulty is that the diffusion term $\sqrt{\Phi(\rho)}$ is not differentiable, and, therefore, we cannot follow the usual approach to conclude the differentiability of U^Φ from the differentiability of the solution ζ^∞ to the SPDE (3.2) as function of its initial condition. This is solved in this section by exploiting the Gaussianity of ζ^∞ together with an infinite-dimensional integration-by-parts formula.

We start from the following auxiliary statements.

Lemma 3.6. *Let $I > \frac{d}{2} + 1$, $J > \frac{d}{2}$, $\zeta \in H_{-I}$ be fixed and $\Phi \in \mathrm{C}_b^2(\mathbb{R})$. Let also $(\zeta_t^\infty)_{t\geq 0}$ be a solution to (3.2) started from ζ, where $(\rho_t^\infty)_{t\geq 0}$ is a solution to the heat equation (3.1) with the initial condition $\rho_0^\infty = \rho \in H_J$. Then for each $t > 0$ the covariance $V_t(\rho)$ of ζ_t^∞ can be extended to an element in $\mathcal{L}_2^{HS}(H_I)$ also denoted by $V_t(\rho)$. Moreover, the map V_t belongs to $\mathrm{C}_b^1\left(H_J; \mathcal{L}_2^{HS}(H_I)\right)$ and its derivative at $\rho \in H_J$ in direction $h \in H_J$ is given by*

$$DV_t(\rho)[h][\varphi, \psi] = 2\pi^2 \int_0^t \langle \nabla P_{t-s}\varphi \cdot \nabla P_{t-s}\psi, \Phi'\left(P_s\rho\right) P_s h\rangle\, ds \tag{3.9}$$

for all $\varphi, \psi \in H_J$ and

$$\|DV_t(\rho)[h]\|_{\mathcal{L}_2^{HS}(H_I)} \leq tC_I \|\Phi'\|_{\mathrm{C}} \|h\|_{H_J}. \tag{3.10}$$

Proof. Using Hölder's inequality and Proposition 3.1, we get

$$|V_t(\rho)[\varphi,\psi]| \le \|\varphi\|_{H_I}\|\psi\|_{H_I}\mathbb{E}\left[\|\zeta_t^\infty\|_{H_{-I}}^2\right] \le C_{\rho,I,\zeta}\|\varphi\|_{H_I}\|\psi\|_{H_I}$$

for all $\varphi,\psi\in \mathrm{C}^\infty(\mathbb{T}^d)$. This implies that $V_t(\rho)$ can be extended to a continuous multilinear operator on $(H_I)^2$. Using Proposition 3.1 again, following the proof of Lemma 3.3 and applying the estimate (3.7), we can show the boundedness of the Hilbert-Schmidt norm of $V_t(\rho)$ given by

$$\|V_t(\rho)\|_{\mathcal{L}_2^{HS}}^2 \le C_I t \sup_{s\in[0,t]}\|\Phi(\rho_s^\infty)\|^2 \le C_I t\|\Phi\|_{\mathrm{C}}^2. \tag{3.11}$$

To get the (Lipschitz) continuity of $V_t : H_J \to \mathcal{L}_2^{HS}(H_I)$, we can also follow the proof of Lemma 3.3 and use the estimate (3.7) to get for $\rho,\tilde\rho\in H_J$

$$\begin{aligned}\|V_t(\rho)-V_t(\tilde\rho)\|_{\mathcal{L}_2^{HS}}^2 &\le C_I t\sup_{s\in[0,t]}\|\Phi(P_s\rho)-\Phi(P_s\tilde\rho)\|\\ &\le \|\Phi'\|_{\mathrm{C}}\,\|P_s\rho-P_s\tilde\rho\| \le \|\Phi'\|_{\mathrm{C}}\|\rho-\tilde\rho\|\\ &\le \|\Phi'\|_{\mathrm{C}}\|\rho-\tilde\rho\|_{H_J}.\end{aligned}$$

We next check the differentiability of V_t at $\rho\in H_J$ and show that its derivative is given by

$$DV_t(\rho)[h][\varphi,\psi] = 2\pi^2\int_0^t \langle \nabla P_{t-s}\varphi\cdot\nabla P_{t-s}\psi, \Phi'(P_s\rho)P_sh\rangle\,ds.$$

Similarly as above, we estimate

$$\begin{aligned}&\|V_t(\rho+h)-V_t(\rho)-DV_t(\rho)[h]\|_{\mathcal{L}_2^{HS}}^2\\ &\quad=\sum_{k,l\in\mathbb{Z}^d}(1+|k|^2)^{-I}(1+|l|^2)^{-I}\,|V_t(\rho+h)[\tilde\varsigma_k,\tilde\varsigma_l]-V_t(\rho)[\tilde\varsigma_k,\tilde\varsigma_l]-DV_t(\rho)[h][\tilde\varsigma_k,\tilde\varsigma_l]|^2\\ &\quad\le tC_I\sup_{s\in[0,t]}\|\Phi\left(P_s\rho+P_sh\right)-\Phi\left(P_s\rho\right)-\Phi'(P_s\rho)P_sh\|^2 \le tC_I\|\Phi''\|_{\mathrm{C}}\|(P_sh)^2\|^2\\ &\quad\le tC_{J,I}\|\Phi''\|_{\mathrm{C}}\|h\|_{H_J}^4,\end{aligned}$$

where we used Taylor's expansion for Φ, (3.7) and

$$\left\|(P_sh)^2\right\|^2 \le \left\|(P_sh)^2\right\|_{\mathrm{C}}^2 = \|P_sh\|_{\mathrm{C}}^4 \le C_J\|h\|_{\mathrm{C}}^4 \le C_J\|h\|_{H_J}^4$$

due to $J>\frac{d}{2}$.
The boundedness of DV_t follows from (3.9) and (3.7). Indeed, for each $h\in H_J$, one has

$$\begin{aligned}\|DV_t(\rho)[h]\|_{\mathcal{L}_2^{HS}}^2 &\le 4\pi^4 t\sum_{k,l\in\mathbb{Z}^d}(1+|k|^2)^{-I}(1+|l|^2)^{-I}\\ &\qquad\cdot\int_0^t\left|\langle\nabla P_{t-s}\tilde\varsigma_k\cdot\nabla P_{t-s}\tilde\varsigma_l,\Phi'(P_s\rho)P_sh\rangle\right|^2ds\\ &\le tC_I\sup_{s\in[0,t]}\|\Phi'(P_s\rho)P_sh\|^2 \le tC_I\|\Phi'\|_{\mathrm{C}}\|P_sh\|^2\\ &= tC_I\|\Phi'\|_{\mathrm{C}}\|h\|^2 \le tC_I\|\Phi'\|_{\mathrm{C}}\|h\|_{H_J}^2.\end{aligned}$$

The continuity of DV_t can be checked similarly. This completes the proof of the statement. □

Lemma 3.7. *Under the assumptions of Lemma 3.6, one has*

$$\sup_{\rho\in H_J} \mathbb{E}\left[\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|_{H_{-I}}^2\right] \to 0$$

as $n\to\infty$.

Proof. We rewrite

$$\begin{aligned}\mathbb{E}\left[\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|_{H_{-I}}^2\right] &= \sum_{k\notin\mathbb{Z}_n^d} (1+|k|^2)^{-I}\mathbb{E}\left[\langle\zeta_t^\infty, \tilde\varsigma_k\rangle^2\right] \\ &= \sum_{k\notin\mathbb{Z}_n^d} (1+|k|^2)^{-I}\left(V_t(\rho)[\tilde\varsigma_k,\tilde\varsigma_k] + \mathbb{E}\left[\langle\tilde\varsigma_k,\zeta_t^\infty\rangle\right]^2\right).\end{aligned}$$

Using Proposition 3.1 and following the proof of Lemma 3.3, in particular (3.7), the expression above can be estimated as follows

$$\begin{aligned}&\sum_{k\notin\mathbb{Z}_n^d} (1+|k|^2)^{-I}\left(V_t(\rho)[\tilde\varsigma_k,\tilde\varsigma_k] + (m_t(\zeta;\tilde\varsigma_k))^2\right) \\ &\quad\le \sum_{k\notin\mathbb{Z}_n^d} (1+|k|^2)^{-I}\left(\pi^2 t\|\Phi\|_{\mathrm{C}}\frac{|k|^4}{2|k|^2} + \langle\zeta, P_t\tilde\varsigma_k\rangle^2\right) \\ &\quad\le \pi^2 t\|\Phi\|_{\mathrm{C}}\sum_{k\notin\mathbb{Z}_n^d}(1+|k|^2)^{-I+1} + \sum_{k\notin\mathbb{Z}_n^d}(1+|k|^2)^{-I}e^{-8\pi^2|k|^2 t}\langle\zeta,\tilde\varsigma_k\rangle^2.\end{aligned}$$

This implies the uniform convergence of $\mathbb{E}\left[\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|_{H_{-I}}^2\right]$ to zero as $n\to\infty$. □

Define for $A\in\mathcal{L}_2^{HS}(H_{-I})$ and $B\in\mathcal{L}_2^{HS}(H_I)$

$$A:B := \sum_{k,l\in\mathbb{Z}^d} A[\tilde\varsigma_k,\tilde\varsigma_l]B[\tilde\varsigma_k,\tilde\varsigma_l]$$

and note that

$$|A:B| \le \|A\|_{\mathcal{L}_2^{HS}(H_{-I})}\|B\|_{\mathcal{L}_2^{HS}(H_I)},$$

according to (B.13).

Proposition 3.8. *Let* $I>\frac{d}{2}+1$, $J>\frac{d}{2}$ *and* $\zeta\in H_{-I}$ *be fixed. Let also* $(\zeta_t^\infty)_{t\ge0}$ *be a solution to* (3.2) *started from* ζ, *where* $(\rho_t^\infty)_{t\ge0}$ *is a solution to the heat equation (3.1) with the initial condition* $\rho_0^\infty=\rho\in H_J$. *Then for each* $F\in\mathrm{C}_l^2(H_{-I})$ *with bounded uniformly continuous second derivative in the space* $\mathcal{L}_2^{HS}(H_{-I})$ *and* $t\ge0$ *the function*

$$U_t(\rho) := \mathbb{E}F(\zeta_t^\infty), \quad \rho\in H_J,$$

belongs to $\mathrm{C}_l^1(H_J)$ *and for each* $\rho\in H_J$ *and* $t>0$

$$DU_t(\rho)[h] = \frac{1}{2}\mathbb{E}\left[D^2F(\zeta_t^\infty) : DV_t(\rho)\,[h]\right], \quad h\in H_J, \tag{3.12}$$

where $V_t(\rho)$ *is the covariance operator of* ζ_t^∞ *defined by (3.4).*

Proof. Let $t>0$ be fixed. We will show the differentiability of U_t on H_J, using the differentiability of the variance V_t that follows from Lemma 3.6. Define the sequence of functions

$$U^n(\rho) := \mathbb{E}F\left(\mathrm{pr}_n\zeta_t^\infty\right), \quad n\ge1,$$

and show that they are continuously differentiable on H_J and their derivatives converge uniformly to a continuous function $\tilde{U}$. By [9, Theorem 3.6.1], we will conclude that $U_t \in \mathrm{C}^1(H_J)$ and $DU = \tilde{U}$.

Setting $\xi^n := (\langle \zeta_t^\infty, \tilde{\varsigma}_k \rangle - m_k)_{k \in \mathbb{Z}_n^d}$ for $m_k = \mathbb{E}\langle \zeta_t^\infty, \tilde{\varsigma}_k \rangle$, we can represent U^n as follows

$$U^n(\rho) = \mathbb{E} f_n\left(m^n + \xi^n\right),$$

where

$$f_n(z) = F\left(\chi_n(z)\right), \quad z \in \mathbb{R}^{\mathbb{Z}_n^d},$$

$\chi_n(z) := \sum_{k \in \mathbb{Z}_n^d} z_k \tilde{\varsigma}_k$ and $m^n := (m_k)_{k \in \mathbb{Z}_n^d}$. Note that $\xi^n := (\langle \zeta_t^\infty, \tilde{\varsigma}_k \rangle - m_k)_{k \in \mathbb{Z}_n^d}$ is a centered Gaussian vector with covariance matrix

$$V^n(\rho) := \left(V_t(\rho)[\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right)_{k,l \in \mathbb{Z}_n^d}$$

that is non-negatively defined and symmetric. By the differentiability of F, the function f_n belongs to $\mathrm{C}_l^2(\mathbb{R}^{\mathbb{Z}_n^d})$ and

$$\frac{\partial f_n}{\partial z_k} = DF(\chi_n)[\tilde{\varsigma}_k], \quad \frac{\partial^2 f_n}{\partial z_k \partial z_l} = D^2 F(\chi_n)[\tilde{\varsigma}_k, \tilde{\varsigma}_l], \quad k, l \in \mathbb{Z}_n^d. \tag{3.13}$$

Using the spectral decomposition theorem, there exists a square-root $\sqrt{V^n(\rho)}$ of $V^n(\rho)$, that is a (unique) non-negatively defined symmetric matrix such that $(\sqrt{V^n(\rho)})^2 = V^n(\rho)$. Thus, we can define $\xi^n = \sqrt{V^n(\rho)}\tilde{\xi}^n$ for a standard Gaussian vector $\tilde{\xi}^n = (\tilde{\xi}_k)_{k \in \mathbb{Z}_n^d}$. Therefore, the differentiability of U^n will follow from the differentiability of $\rho \mapsto \mathbb{E} f_n\big(m^n + \sqrt{V^n(\rho)}\tilde{\xi}^n\big)$.

Let Sym_n denote the Hilbert space of symmetric matrices $(A_{k,l})_{k,l \in \mathbb{Z}_n^d}$ with real entries and be equipped with the inner product

$$A : B := \sum_{k,l \in \mathbb{Z}_n^d} A_{k,l} B_{k,l}.$$

The open subset of positively defined matrices from Sym_n will be denoted by Sym_n^+. Note that the square-root function $\sqrt{\cdot} : \mathrm{Sym}_n^+ \to \mathrm{Sym}_n^+$ is continuously differentiable and its derivative $(D\sqrt{A})[B]$ in a direction $B \in \mathrm{Sym}_n$ satisfies

$$\left(D\sqrt{A}\right)[B]\sqrt{A} + \sqrt{A}\left(D\sqrt{A}\right)[B] = B, \tag{3.14}$$

according to the expression for the derivative of the product $A = \sqrt{A}\sqrt{A}$. We next consider for each $\delta > 0$ a continuously differentiable function $G_\delta(A) := \delta I + A$, $A \in \mathrm{Sym}_n^{-\delta}$, with values in Sym_n^+, where I is the identity matrix and

$$\mathrm{Sym}_n^{-\delta} := \left\{ A \in \mathrm{Sym}_n : \ Ax \cdot x > -\delta |x|^2, \quad x \in \mathbb{R}^{\mathbb{Z}_n^d} \setminus \{0\} \right\}$$

is an open subset of Sym_n. Then $G_\delta \in \mathrm{C}^1(\mathrm{Sym}_n^{-\delta}; \mathrm{Sym}_n^+)$ and, consequently, the function

$$K_{n,\delta}(A) := \mathbb{E} f_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right), \quad A \in \mathrm{Sym}_n^{-\delta},$$

is continuously differentiable with derivative in a direction $B \in \mathrm{Sym}_n$ given by

$$DK_{n,\delta}(A)[B] = \mathbb{E}\left[Df_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right) \cdot \left(\left(D\sqrt{G_\delta}(A)\right)[B]\tilde{\xi}^n\right)\right]$$

due to [9, Theorem 2.2.1] and the dominated convergence theorem. Using the integration-by-parts formula for a Gaussian vector (see Lemma C.1), we get

$$DK_{n,\delta}(A)[B] = \mathbb{E}\left[D^2 f_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right) : \left(\left(D\sqrt{G_\delta}(A)\right)[B]\sqrt{G_\delta(A)}\right)\right].$$

Next, by the equality $A : (BR) = A : (RB)$ for $A, B, R \in \mathrm{Sym}_n$ and (3.14), we have

$$\begin{aligned} DK_{n,\delta}(A)[B] &= \frac{1}{2}\mathbb{E}\left[D^2 f_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right) : \left(\left(D\sqrt{G_\delta}(A)\right)[B]\sqrt{G_\delta(A)}\right)\right] \\ &+ \frac{1}{2}\mathbb{E}\left[D^2 f_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right) : \left(\sqrt{G_\delta(A)}\left(D\sqrt{G_\delta}(A)\right)[B]\right)\right] \\ &= \frac{1}{2}\mathbb{E}\left[D^2 f_n\left(m^n + \sqrt{G_\delta(A)}\tilde{\xi}^n\right) : B\right] \end{aligned} \tag{3.15}$$

for all $A \in \mathrm{Sym}_n^{-\delta}$ and $B \in \mathrm{Sym}_n$. By the differentiability of the composition and the expression (3.15), we conclude that the function $\mathbb{E}f_n\left(m^n + \sqrt{\delta I + V^n}\tilde{\xi}^n\right)$ is continuously differentiable and

$$\begin{aligned} DU^{n,\delta}(\rho) &:= D\mathbb{E}f_n\left(m^n + \sqrt{\delta I + V^n(\rho)}\tilde{\xi}^n\right) \\ &= \frac{1}{2}\mathbb{E}\left[D^2 f_n\left(m^n + \sqrt{\delta I + V^n(\rho)}\tilde{\xi}^n\right) : DV^n(\rho)\right], \quad \rho \in H_J, \end{aligned}$$

for all $\delta > 0$. Now, taking $\delta \to 0+$, and using [9, Theorem 3.6.1] and the dominated convergence theorem, we get that $U^n \in \mathrm{C}^1(H_J)$ and

$$\begin{aligned} DU^n(\rho)[h] &= D\mathbb{E}f_n\left(m^n + \sqrt{V^n(\rho)}\tilde{\xi}^n\right)[h] \\ &= \frac{1}{2}\mathbb{E}\left[D^2 f_n(m^n + \xi^n) : DV^n(\rho)[h]\right], \quad \rho, h \in H_J. \end{aligned}$$

Note that the assumptions of [9, Theorem 3.6.1] require the uniform convergence of the sequence $DU^{n,\delta}$ to DU^n as $\delta \to 0$. We will show this property for a more complicated sequence of derivatives at the end of this proof. The uniform convergence in the present case can be obtained similarly.

In order to show the differentiability of U_t, we will use [9, Theorem 3.6.1] again. We first note that

$$U^n(\rho) \to U_t(\rho)$$

as $n \to \infty$ for each $\rho \in H_J$, by the dominated convergence theorem and the fact that $\mathrm{pr}_n\zeta_t^\infty \to \zeta_t^\infty$ a.s. in H_{-I} as $n \to \infty$. We will next rewrite the derivative DU_n via the derivatives D^2F and DV_t in the corresponding spaces. Using (3.13) and

$$DV^n_{k,l}(\rho)[h] = DV_t(\rho)[h][\tilde{\varsigma}_k, \tilde{\varsigma}_l],$$

we obtain

$$\begin{aligned} DU^n(\rho)[h] &= \frac{1}{2}\sum_{k,l\in\mathbb{Z}^d_n}\mathbb{E}\left[D^2F(\mathrm{pr}_n\zeta_t^\infty)[\tilde{\varsigma}_k, \tilde{\varsigma}_l]DV_t(\rho)[h][\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right] \\ &= \frac{1}{2}\mathbb{E}\left[D^2F(\mathrm{pr}_n\zeta_t^\infty) : \mathrm{pr}_n^{\otimes 2}DV_t(\rho)[h]\right] \end{aligned}$$

for all $\rho, h \in H_J$. We next note that $D^2F(\zeta) \in \mathcal{L}_2^{HS}(H_{-I})$ and $DV_t(\rho)[h] \in \mathcal{L}_2^{HS}(H_I)$, by Lemma 3.6. Hence, $D^2F(\zeta) : DV_t(\rho)[h]$ is well defined for each $\zeta \in H_{-I}$, $\rho \in H_J$ and $h \in H_J$. We will show that $DU^n \to DU$ uniformly. Using (B.13), we get

$$|DU^n(\rho)[h] - DU(\rho)[h]|^2 \leq \frac{1}{2}\mathbb{E}\left[\left|D^2F(\mathrm{pr}_n\zeta_t^\infty) : \left(\mathrm{pr}_n^{\otimes 2}DV_t(\rho)[h] - DV_t(\rho)[h]\right)\right|^2\right]$$

$$+\frac{1}{2}\mathbb{E}\left[\left|\left(D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right) : V_t(\rho)[h]\right|^2\right]$$
$$\leq \frac{1}{2}\left\|\mathrm{pr}_n^{\otimes 2} DV_t(\rho)[h] - DV_t(\rho)[h]\right\|^2_{\mathcal{L}_2^{HS}(H_I)} \mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\right]$$
$$+\frac{1}{2}\|DV_t(\rho)[h]\|^2_{\mathcal{L}_2^{HS}(H_I)} \mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\right]$$

for all $h, \rho \in H_J$. By (3.10),

$$\|DV(\rho)[h]\|^2_{\mathcal{L}_2^{HS}(H_I)} \leq tC_I\|\Phi'\|_{\mathrm{C}}\|h\|^2_{H_J}, \quad \rho, h \in H_J.$$

Moreover, similarly to the proof of (3.10), we conclude

$$\begin{aligned}\left\|\tilde{V}^n(\rho)[h] - V(\rho)[h]\right\|^2_{\mathcal{L}_2^{HS}(H_I)} &= \sum_{k,l\notin\mathbb{Z}_n^d} (1+|k|^2)^{-I}(1+|l|^2)^{-I}\left|DV(\rho)[h][\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right|^2 \\ &\leq \sum_{k,l\notin\mathbb{Z}_n^d} (1+|k|^2)^{I+1}(1+|l|^2)^{-I+1}t\|\Phi'\|_{\mathrm{C}}\|h\|^2_{H_J} \\ &= \varepsilon_n t\|\Phi'\|_{\mathrm{C}}\|h\|^2_{H_J}, \quad \rho, h \in H_J,\end{aligned}$$

where $\varepsilon_n := \sum_{k,l\notin\mathbb{Z}_n^d}(1+|k|^2)^{-I+1}(1+|l|^2)^{-I+1} \to 0$ as $n\to\infty$, due to $I > \frac{d}{2}+1$. We next fix arbitrary $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\|D^2F(\zeta) - D^2F(\zeta')\|_{\mathcal{L}_2^{HS}(H_{-I})} < \varepsilon$$

for all $\zeta, \zeta' \in H_{-I}$ satisfying $\|\zeta - \zeta'\|_{H_{-I}} < \delta$, according to the uniform continuity of D^2F. We also take $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sup_{\rho\in H_J}\mathbb{E}\left[\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|^2_{H_{-I}}\right] < \varepsilon,$$

by Lemma 3.7. Then, using Chebyshev's inequality, we get

$$\begin{aligned}&\mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\right] \\ &\quad= \mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\mathbb{I}_{\left\{\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|_{H_{-I}}\geq\delta\right\}}\right] \\ &\quad+ \mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\mathbb{I}_{\left\{\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|_{H_{-I}}\leq\delta\right\}}\right] \\ &\quad\leq \frac{4}{\delta^2}\sup_{\zeta\in H_{-I}}\|D^2F(\zeta)\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\mathbb{E}\left[\|\mathrm{pr}_n\zeta_t^\infty - \zeta_t^\infty\|^2_{H_{-I}}\right] + \varepsilon^2 \\ &\quad\leq \frac{4}{\delta^2}\sup_{\zeta\in H_{-I}}\|D^2F(\zeta)\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\varepsilon + \varepsilon^2.\end{aligned}$$

This shows that

$$\sup_{\rho\in H_J}\mathbb{E}\left[\left\|D^2F(\mathrm{pr}_n\zeta_t^\infty) - D^2F(\zeta_t^\infty)\right\|^2_{\mathcal{L}_2^{HS}(H_{-I})}\right] \to 0$$

as $n\to\infty$. Consequently,

$$\sup_{\rho\in H_J}\|DU^n(\rho) - DU(\rho)\|_{H_{-J}} = \sup_{\rho\in H_J}\sup_{\|h\|_J\leq 1}|DU^n(\rho)[h] - DU(\rho)[h]| \to 0.$$

The boundedness of DU follows from the expression (3.12), (B.13) and Lemma 3.6. This completes the proof of the proposition. □

We next define for a function $F \in \mathrm{C}^{1,2}_{l,HS}(H_J, H_{-I})$ the differential operator

$$\mathcal{G}^{OU,\Phi} F(\rho,\zeta) = 2\pi^2 \left\langle \Delta D_1 F(\rho,\zeta), \rho \right\rangle + 2\pi^2 \left\langle \Delta D_2 F(\rho,\zeta), \zeta \right\rangle \\ + 2\pi^2 \sum_{j=1}^{d} \left\langle \mathrm{Tr}\left(\partial_j^{\otimes 2} D_2^2 F(\rho,\zeta)\right), \Phi(\rho) \right\rangle, \quad \rho \in H_{J+2}, \ \ \zeta \in H_{-I+2}.$$

Proposition 3.9. *Let $I > \frac{d}{2} + 1$, $J > \frac{d}{2}$, $\Phi \in \mathrm{C}^2_b(\mathbb{R})$, $F \in \mathrm{C}^{2,4}_l(H_J, H_{-I})$ and $D_2^2 F$ be bounded and uniformly continuous in $\mathcal{L}_2^{HS}(H_{-I})$. Let also*

$$U_t(\rho,\zeta) := \mathbb{E}F(\rho_t^\infty, \zeta_t^\infty), \quad t \geq 0,$$

for $\rho \in H_J$, $\zeta \in H_{-I}$, where $(\rho_t^\infty, \zeta_t^\infty)$, $t \geq 0$, is a solution in $H_J \times H_{-I}$ to (3.1), (3.2) started from (ρ,ζ). Then the function U belongs to $\mathrm{C}^{0,1,3}([0,\infty), H_J, H_{-I})$ and for each $T > 0$

$$\sup_{t\in[0,T]} \|U_t\|_{\mathrm{C}^{1,3}_{l,HS}} \leq \mathrm{C}_{I,T} \left(\|\Phi'\|_{\mathrm{C}} + 1\right) \|F\|_{\mathrm{C}^{1,3}_{l,HS}}. \tag{3.16}$$

Moreover, if $I > \frac{d}{2} + 3$, then for each $\rho \in H_{J+2}$ and $\zeta \in H_{-I+2}$ the map $t \mapsto U_t(\rho,\zeta)$ is continuously differentiable,

$$\partial U_t(\rho,\zeta) = \mathcal{G}^{OU,\Phi} U_t(\rho,\zeta), \quad t > 0, \ \rho \in H_{J+2}, \ \zeta \in H_{-I+2}, \tag{3.17}$$

and $\partial U \in \mathrm{C}\left([0,\infty) \times H_{J+2} \times H_{-I+2}\right)$.

Proof. To prove the proposition, we will split the dependence of ρ_t^∞ and ζ_t^∞ on the initial condition for the heat equation (3.1), extending U_t by

$$\tilde{U}_t(\tilde{\rho}, \rho, \zeta) := \mathbb{E}F(\tilde{\rho}_t^\infty, \zeta_t^\infty),$$

where $(\tilde{\rho}_t^\infty)_{t\geq 0}$ is a solution to (3.1) started from $\tilde{\rho} \in H_J$ and $(\zeta_t^\infty)_{t\geq 0}$ is a solution to (3.2) started from $\zeta \in H_{-I}$ with the diffusion coefficient depending on the solution $(\rho_t^\infty)_{t\geq 0}$ to the heat equation (3.1) started from $\rho \in H_J$. Then

$$U_t(\rho,\zeta) = \tilde{U}_t(\rho,\rho,\zeta)$$

for all $t \geq 0$, $\rho \in H_J$ and $\zeta \in H_{-I}$.

The continuity of $\tilde{U}$ directly follows from the mean-value theorem (see [9, Theorem 3.3.2]), the continuity $(t,\tilde{\rho}) \mapsto \tilde{\rho}_t^\infty$ as a map from $[0,\infty) \times H_J$ to H_J and the continuity of $(t,\rho,\zeta) \mapsto \mathrm{Law}\zeta_t^\infty$ in the 2-Wasserstein topology as a map from $[0,\infty) \times H_J \times H_{-I}$ to the space of probability distributions on H_{-I} with a finite second moment, by Lemma 3.3.

We next show that $\tilde{U}_t$ is differentiable on H_{-I} with respect to the third variable and its derivative in a direction $h \in H_{-I}$ equals

$$D_3\tilde{U}_t(\tilde{\rho},\rho,\zeta)[h] = \mathbb{E}\left[D_2 F(\tilde{\rho}_t^\infty, \zeta_t^\infty)[P_t h]\right] \tag{3.18}$$

for all $\tilde{\rho}, \rho \in H_J$, $\zeta \in H_{-I}$ and $t \geq 0$. Using the differentiability of F, we get

$$\left|\tilde{U}_t(\tilde{\rho},\rho,\zeta+h) - \tilde{U}_t(\tilde{\rho},\rho,\zeta) - D_3\tilde{U}_t(\tilde{\rho},\rho,\zeta)[h]\right| \\ = \left|\mathbb{E}\left[F(\tilde{\rho}_t^\infty, \zeta_t^{\infty,h}) - F(\tilde{\rho}_t^\infty, \zeta_t^\infty) - D_2F(\tilde{\rho}_t^\infty, \zeta_t^\infty)[P_t h]\right]\right| \\ = \left|\mathbb{E}\left[F(\tilde{\rho}_t^\infty, \zeta_t^\infty + P_t h) - F(\tilde{\rho}_t^\infty, \zeta_t^\infty) - D_2F(\tilde{\rho}_t^\infty, \zeta_t^\infty)[P_t h]\right]\right|,$$

where $(\zeta_t^{\infty,h})_{t\geq 0}$ is a solution to (3.2) started from $\zeta + h$ and, by the linearity of (3.2), $\zeta_t^{\infty,h} = \zeta_t^\infty + P_t h$. Consequently, we can estimate the right hand side of the equality above by $\frac{1}{2}\|D_2^2 F\|_{\mathrm{C}}\|P_t h\|_{H_{-I}}^2$, according to [9, Theorem 5.6.1]. The continuity $D_3\tilde{U}$ can be proved similarly to the continuity of $\tilde{U}$. Similarly, we can also prove that $\tilde{U}$ is continuously differentiable with respect to ζ to the third order and continuously differentiable with respect to $\tilde{\rho}$. Moreover, the derivatives have a similar structure as in (3.18). Hence they are uniformly bounded.

The continuous differentiability of $\tilde{U}$ with respect to ρ and the boundedness of its derivative follows from Proposition 3.8. Thus, $U \in \mathrm{C}_l^{0,1,3}([0,\infty), H_J, H_{-I})$, by [9, Proposition 2.6.2]. The fact that $D_2^2 U_t(\rho,\zeta) \in \mathcal{L}_2^{HS}(H_{-I})$ follows from the estimate

$$\begin{aligned}
\|D_2^2 U_t(\rho,\zeta)\|_{\mathcal{L}_2^{HS}(H_{-I})}^2 &= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^I (1+|l|^2)^I \left|\mathbb{E}\left[D_2^2 F(\rho_t^\infty,\zeta_t^\infty)[P_t\tilde{\varsigma}_k, P_t\tilde{\varsigma}_l]\right]\right|^2 \\
&= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^I (1+|l|^2)^I e^{-4\pi^2 t\left(|k|^2+|l|^2\right)} \left|\mathbb{E}\left[D_2^2 F(\rho_t^\infty,\zeta_t^\infty)[\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right]\right|^2 \\
&\leq \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^I (1+|l|^2)^I \mathbb{E}\left[\left|D_2^2 F(\rho_t^\infty,\zeta_t^\infty)[\tilde{\varsigma}_k,\tilde{\varsigma}_l]\right|^2\right] \\
&= \mathbb{E}\left[\left\|D_2^2 F(\rho_t^\infty,\zeta_t^\infty)\right\|_{\mathcal{L}_2^{HS}(H_{-I})}^2\right] \leq \sup_{\rho\in H_J,\zeta\in H_{-I}} \left\|D_2^2 F(\rho,\zeta)\right\|_{\mathcal{L}_2^{HS}(H_{-I})}^2.
\end{aligned}$$

The bound (3.16) follows from the latter inequality, direct estimates of the derivatives $D_1\tilde{U}$, $D_3^m\tilde{U}$, $m\in[3]$, that satisfy expressions similar to (3.18), Lemma 3.6 and Proposition 3.8.

Let $\rho \in H_{J+2}$, $\zeta \in H_{-I+2}$ and $(\rho_t^\infty,\zeta_t^\infty)$, $t\geq 0$, be a solution to (3.1), (3.2) started from (ρ,ζ). By Proposition 3.1, the process $(\rho^\infty,\zeta^\infty)$ takes values in $H_{J+2}\times H_{-I+2}$. Using the Markov property of $(\rho^\infty,\zeta^\infty)$ and Lemma 3.5, we get for each $t\geq 0$ and $\varepsilon>0$

$$\begin{aligned}
U_{t+\varepsilon}(\rho,\zeta) - U_t(\rho,\zeta) &= \mathbb{E}\left[U_t(\rho_\varepsilon^\infty,\zeta_\varepsilon^\infty)\right] - U_t(\rho,\zeta) \\
&= 2\pi^2\int_0^\varepsilon \mathbb{E}\left\langle \Delta D_2 U_t(\rho_s^\infty,\zeta_s^\infty), \zeta_s^\infty\right\rangle ds \\
&+ 2\pi^2\int_0^t \mathbb{E}\left\langle \Delta D_1 U_t(\rho_s^\infty,\zeta_s^\infty), \rho_s^\infty\right\rangle ds \\
&+ 2\pi^2\int_0^t \sum_{j=1}^d \mathbb{E}\left\langle \mathrm{Tr}\left(\partial_j^{\otimes 2} D_2^2 U_t(\rho_s^\infty,\zeta_s^\infty)\right), \Phi(\rho_s^\infty)\right\rangle ds.
\end{aligned}$$

By the continuity of $(\rho^\infty,\zeta^\infty)$ in $H_{J+2}\times H_{-I+2}$, the fact that $U\in \mathrm{C}_l^{0,1,2}([0,\infty),H_J,H_{-I})$, the estimate (3.16) and Lemmas B.15, B.16 with the observation that $\Phi(\rho_t^\infty)$, $t\geq 0$, is continuous in $L_2(\mathbb{T}^d)$, we get

$$\lim_{\varepsilon\to 0+}\frac{U_{t+\varepsilon}(\rho,\zeta)-U_t(\rho,\zeta)}{\varepsilon} = \mathcal{G}^{OU,\Phi}U_t(\rho,\zeta).$$

Taking into account that the right derivative of $(U_t(\rho,\zeta))_{t\geq 0}$ with respect to t is continuous, we conclude that $(U_t(\rho,\zeta))_{t\geq 0}$ is continuously differentiable (in t) and the equality (3.17) holds. The continuity of ∂U follows from (3.17). □

4 Berry-Esseen bound for the initial fluctuations

The main result of this section is a quantified CLT for the fluctuations of the random initialization of the SSEP $(\eta_t^n)_{t\geq 0}$. Recall that η_0^n has the distribution $\nu_{\rho_0^n}^n$ that is the product measure on $\{0,1\}^{\mathbb{T}_n^d}$ with marginals given by $\nu_{\rho_0^n}^n\{\eta(x)=1\} = \rho_0^n(x)$, $x \in \mathbb{T}_n^d$, for a function $\rho_0^n : \mathbb{T}_n^d \to [0,1]$. We define the multilinear operator

$$A_\rho[\varphi, \psi] = \langle \rho(1-\rho)\varphi, \psi\rangle, \quad \varphi, \psi \in H_I, \tag{4.1}$$

for $\rho \in L_2(\mathbb{T}^d)$ taking values in $[0,1]$ and $I > \frac{d}{2}$, and note that it is a trace class operator since

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^I} A[\tilde{\varsigma}_k, \tilde{\varsigma}_k] &= \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^I} \langle \rho(1-\rho)\tilde{\varsigma}_k, \tilde{\varsigma}_k\rangle \\ &\le \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^I} \left(\|\rho\| + \|\rho\|^2\right) < \infty.\end{aligned}$$

Thus, by [38, Proposition 3.15], there exists a centered Gaussian random variable ζ in H_{-I} with covariance A_ρ, that is,

$$\mathbb{E}\left[\langle \zeta, \varphi\rangle\langle \psi, \zeta\rangle\right] = A_\rho[\varphi, \psi], \quad \varphi, \psi \in H_I.$$

In the next statement we obtain a rate of convergence for the fluctuation density field η_0^n of the SSEP, started with distribution $\nu_{\rho_0^n}^n$, to a Gaussian random variable with covariance operator A_{ρ_0}. Since in this section, we do not work with processes but only with their initial conditions, we will drop the time-dependence in the notation throughout this section.

Proposition 4.1. *Let $I > \frac{d}{2}+1$ and $\rho \in \mathrm{C}^1(\mathbb{T}^d)$. Assume that ζ is a centered Gaussian random variable in H_{-I} with covariance operator A_ρ. Let also $\rho_n \in L_2(\mathbb{T}_n^d)$, η_n have distribution $\nu_{\rho_n}^n$ and $\zeta_n = (2n+1)^{d/2}(\eta_n - \rho_n)$ for each $n \ge 1$. Then for each $F \in \mathrm{C}^3_{l,HS}(H_{-I})$ and $n \ge 1$*

$$|\mathbb{E}F(\mathrm{ex}_n\zeta_n) - \mathbb{E}F(\zeta)| \le C_I \left(\frac{1}{n^{1\wedge\frac{d}{2}}}(1 + \|\nabla\rho\|_{\mathrm{C}}) + \|\rho_n - \rho\|_n\right)\|F\|_{\mathrm{C}^3_{l,HS}}.$$

Proof. Using the triangle inequality, it is enough to estimate $\mathbb{E}F(\mathrm{ex}_n\zeta_n) - \mathbb{E}F(\mathrm{pr}_n\zeta)$ and $\mathbb{E}F(\mathrm{pr}_n\zeta) - \mathbb{E}F(\zeta)$. By the mean value theorem (see [9, Theorem 3.3.2]), we obtain

$$|\mathbb{E}F(\mathrm{pr}_n\zeta) - \mathbb{E}F(\zeta)| \le \|DF\|_{\mathrm{C}}\mathbb{E}\|\mathrm{pr}_n\zeta - \zeta\|_{H_{-I}}.$$

Note that ζ has a version that belongs to H_{-I+1} due to the fact that $I - 1 > \frac{d}{2}$ and [38, Proposition 3.15]. Thus, $\mathbb{E}[\|\zeta\|_{-I+1}] < \infty$. Then using Lemma B.2, we get

$$\mathbb{E}\|\mathrm{pr}_n\zeta - \zeta\|_{H_{-I}} \le \frac{C_I}{n}\mathbb{E}\|\mathrm{pr}_n\zeta - \zeta\|_{H_{-I+1}} \le \frac{C_I}{n}\mathbb{E}\|\zeta\|_{H_{-I+1}} \le \frac{C_I}{n}.$$

We next estimate $R_I^n := |\mathbb{E}F(\mathrm{pr}_n\zeta) - \mathbb{E}F(\mathrm{ex}_n\zeta_n)|$ by adopting Stein's method, see e.g., [50, 57] and the survey paper [59]. While in these contributions, Stein's method is developed for finite-dimensional random variables, the dimension of ζ_n diverges to infinity for $n \to \infty$. Therefore, we need to carefully control the dependency of the occurring constants, and to control them uniformly with respect to the dimension.

Let $n \ge 1$ be fixed. An important step in the estimation of R_I^n is the identification of the (finite-dimensional) random variable $\mathrm{ex}_n\zeta_n$ taking values in the Sobolev space H_{-I} with a random variable X taking values in a Euclidean space, and to then build an exchangeable pair (X, X'). This will allow to apply the general finite-dimensional result from [50, Theorem 3]. We will identify $\mathrm{ex}_n\zeta_n$ with its coordinates with respect to the

basis $\tilde{\varsigma}''_k := \frac{1}{(1+|k|^2)^{I/2}}\tilde{\varsigma}_k$, $k \in \mathbb{Z}^d$, of H_{-I} by defining $X_k := \langle \zeta_n, \tilde{\varsigma}''_k \rangle_n$ for $k \in \mathbb{Z}^d_n$. Then $X = (X_k)_{k\in\mathbb{Z}^d_n}$ is a random variable in $\mathbb{R}^{\mathbb{Z}^d_n}$. In particular, $|X|^2 = \|\mathrm{ex}_n\zeta_n\|^2_{H_{-I}}$.
The standard approach for the construction of an exchangeable pair for a random vector with independent coordinates is to replace a randomly chosen coordinate by an independent one with the same distribution. Note that the coordinates of X are not independent. However, we have the independence of the fluctuations $\zeta_n(x)$, $x \in \mathbb{T}^d_n$. Therefore, we will replace $\zeta_n(x)$, $x \in \mathbb{T}^d_n$, by an independent copy for a randomly chosen x. Let $\tilde{\zeta}_n$ be an independent copy of ζ_n and γ be a uniformly distributed random variable on $\mathbb{T}^d_n$ that is independent of ζ_n and $\tilde{\zeta}_n$. Define

$$\zeta'_n(x) := \zeta_n(x)\mathbb{I}_{\{\gamma\neq x\}} + \tilde{\zeta}_n(x)\mathbb{I}_{\{\gamma=x\}}, \quad x \in \mathbb{T}^d_n,$$

and

$$\zeta'_n(x) := \zeta_n(x)\mathbb{I}_{\{\gamma\neq x\}} + \tilde{\zeta}_n(x)\mathbb{I}_{\{\gamma=x\}}, \quad x \in \mathbb{T}^d_n,$$

and

$$X'_k := \langle \zeta'_n, \tilde{\varsigma}''_k \rangle_n = X_k + \frac{1}{(2n+1)^d}\left(\tilde{\zeta}_n(\gamma) - \zeta_n(\gamma)\right)\tilde{\varsigma}''_k(\gamma)$$

for each $k \in \mathbb{Z}^d_n$. Trivially, (X, X') is an exchangeable pair, that is, (X, X') and (X', X) have the same distribution.
We also need to replace the function $F : H_{-I} \to \mathbb{R}$ by a function $f_n : \mathbb{R}^{\mathbb{T}^d_n} \to \mathbb{R}$ such that $F(\mathrm{ex}_n\zeta_n) = f_n(X)$. Trivially, we have to take $f_n := F \circ \kappa_n$, where $\kappa_n(z) := \sum_{k\in\mathbb{Z}^d} z_k\tilde{\varsigma}'_k$ for $z = (z_k)_{k\in\mathbb{Z}^d_n}$ and $\tilde{\varsigma}'_k := \left(1+|k|^2\right)^{I/2}\tilde{\varsigma}_k$, $k \in \mathbb{Z}^d_n$. In particular, $f \in \mathrm{C}^3(\mathbb{R}^{\mathbb{T}^d_n})$ and

$$\frac{\partial f_n}{\partial z_k} = DF\left(\kappa_n\right)[\tilde{\varsigma}'_k] \quad \text{and} \quad \frac{\partial^2 f_n}{\partial z_k \partial z_l} = D^2F\left(\kappa_n\right)[\tilde{\varsigma}'_k, \tilde{\varsigma}'_l]$$

for all $k, l \in \mathbb{Z}^d_n$.
Next, for every $k \in \mathbb{Z}^d_n$ we compute

$$\begin{aligned}\mathbb{E}\left[X'_k - X_k|X\right] &= \frac{1}{(2n+1)^d}\mathbb{E}\left[\left(\tilde{\zeta}_n(\gamma) - \zeta_n(\gamma)\right)\tilde{\varsigma}''_k(\gamma)|\zeta\right] \\ &= \frac{1}{(2n+1)^{2d}}\sum_{x\in\mathbb{Z}^d_n}\left(-\zeta_n(x)\right)\tilde{\varsigma}''_k(x) = -\frac{1}{(2n+1)^d}X.\end{aligned}$$

Moreover, for each $k, l \in \mathbb{Z}^d_n$

$$\begin{aligned}&\mathbb{E}\left[\left(X'_k - X_k\right)\left(X'_l - X_l\right)|X\right] \\ &\quad = \frac{1}{(2n+1)^{2d}}\sum_{x,y\in\mathbb{Z}^d_n}\mathbb{E}\left[(\zeta'_n(x) - \zeta_n(x))(\zeta'_n(y) - \zeta_n(y))|\zeta\right]\tilde{\varsigma}''_k(x)\tilde{\varsigma}''_l(y) \\ &\quad = \frac{1}{(2n+1)^{2d}}\sum_{x,y\in\mathbb{Z}^d_n}\mathbb{E}\left[(\tilde{\zeta}_n(x) - \zeta_n(x))(\tilde{\zeta}_n(y) - \zeta_n(y))\mathbb{I}_{\{\gamma=x,\gamma=y\}}|\zeta\right]\tilde{\varsigma}''_k(x)\tilde{\varsigma}''_l(y). \\ &\quad = \frac{1}{(2n+1)^{3d}}\sum_{x\in\mathbb{Z}^d_n}\mathbb{E}\left[(\tilde{\zeta}_n(x) - \zeta_n(x))^2|\zeta\right]\tilde{\varsigma}''_k(x)\tilde{\varsigma}''_l(x).\end{aligned}$$

Due to the equality

$$\mathbb{E}\left[(\tilde{\zeta}_n(x) - \zeta_n(x))^2|\zeta\right] = \mathbb{E}\left[(\zeta_n(x))^2\right] + \zeta^2_n(x) = (2n+1)^d\rho_n(x)(1-\rho_n(x)) + \zeta^2_n(x),$$

we get

$$\begin{aligned}\mathbb{E}\left[(X_k'-X_k)\left(X_l'-X_l\right)|X\right] &= \frac{1}{(2n+1)^{2d}} \sum_{x\in\mathbb{Z}_n^d} \rho_n(x)(1-\rho_n(x))\tilde{\varsigma}_k''(x)\tilde{\varsigma}_l''(x) \\ &\quad + \frac{1}{(2n+1)^{3d}} \sum_{x\in\mathbb{Z}_n^d} \zeta_n^2(x)\tilde{\varsigma}_k''(x)\tilde{\varsigma}_l''(x) \\ &= \frac{1}{(2n+1)^d}\left\langle \rho_n(1-\rho_n)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\right\rangle_n + \frac{1}{(2n+1)^{2d}}\left\langle \zeta_n^2\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\right\rangle_n .\end{aligned}$$

Note that the entries of the covariance matrix $\Sigma = (\Sigma_{k,l})_{k,l\in\mathbb{Z}_n^d}$ of the random vector

$$Z = (\langle \mathrm{pr}_n\zeta, \tilde{\varsigma}_k''\rangle)_{k\in\mathbb{Z}_n^d} = (\langle \zeta, \tilde{\varsigma}_k''\rangle)_{k\in\mathbb{Z}_n^d}$$

are given by

$$\Sigma_{k,l} = \langle \rho(1-\rho)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle.$$

We thus rewrite

$$\begin{aligned}\mathbb{E}\left[(X_k'-X_k)\left(X_l'-X_l\right)|X\right] &= \frac{2}{(2n+1)^d}\langle \rho(1-\rho)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle \\ &\quad + \frac{1}{(2n+1)^d}\left[\left\langle \left((\eta_n-\rho_n)^2-\rho_n(1-\rho_n)\right)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\right\rangle_n\right] \\ &\quad + \frac{2}{(2n+1)^d}\left[\langle \rho_n(1-\rho_n)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle_n - \langle \rho(1-\rho)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle\right] \\ &= \frac{2}{(2n+1)^d}\Sigma_{k,l} + \frac{1}{(2n+1)^d}E_{k,l}^*,\end{aligned}$$

where

$$\begin{aligned}E_{k,l}^* &= \left\langle \left((\eta_n-\rho_n)^2-\rho_n(1-\rho_n)\right)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\right\rangle_n \\ &\quad + 2\left[\langle \rho_n(1-\rho_n)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle_n - \langle \rho(1-\rho)\tilde{\varsigma}_k'', \tilde{\varsigma}_l''\rangle\right] \\ &=: E_{k,l}^{1*} + 2E_{k,l}^{2*}.\end{aligned}$$

Using [50, Theorem 3], we get

$$\begin{aligned}|\mathbb{E}F(\mathrm{ex}_n\zeta_n) - \mathbb{E}F(\mathrm{pr}_n\zeta)| &= |\mathbb{E}f_n(X) - \mathbb{E}f_n(Z)| \\ &\le \frac{(2n+1)^d}{4(2n+1)^d}\|D^2f_n\|_{\mathrm{C}(\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d}))}\mathbb{E}\|E^*\|_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})} \\ &\quad + \frac{(2n+1)^d}{9}\|D^3f_n\|_{\mathrm{C}(\mathcal{L}_3(\mathbb{R}^{\mathbb{T}_n^d}))}\mathbb{E}\left|X'-X\right|^3,\end{aligned}$$

We next estimate each term in the right hand side of the inequality above. We start from

$$\begin{aligned}\|D^2f_n\|^2_{\mathrm{C}(\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d}))} &= \sup_{z\in\mathbb{R}^{\mathbb{Z}_n^d}} \sum_{k,l\in\mathbb{Z}_n^d}\left(\frac{\partial^2 f_n}{\partial z_k\partial z_l}(z)\right)^2 \\ &= \sup_{z\in\mathbb{R}^{\mathbb{Z}_n^d}} \sum_{k,l\in\mathbb{Z}_n^d}\left(D^2F\left(\kappa_n(z)\right)[\tilde{\varsigma}_k', \tilde{\varsigma}_l']\right)^2 \\ &= \sup_{z\in\mathbb{R}^{\mathbb{Z}_n^d}} \sum_{k,l\in\mathbb{Z}_n^d}(1+|k|^2)^I(1+|l|^2)^I\left(D^2F\left(\kappa_n(z)\right)[\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right)\end{aligned}$$

$$\leq \sup_{g\in H_{-I}} \|D^2F(g)\|^2_{\mathcal{L}_2^{HS}(H_{-I})} = \|D^2F\|^2_{C(\mathcal{L}_2^{HS}(H_{-I}))}.$$

Using Hölder's inequality and then Jensen's inequality, we get

$$\mathbb{E}\left[\|E^*\|_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right]^2 \leq 2\mathbb{E}\left[\|E^{1*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right] + 8\mathbb{E}\left[\|E^{2*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right].$$

Rewriting

$$\mathbb{E}\left[\|E^{1*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right] = \sum_{k,l\in\mathbb{Z}_n^d} \mathbb{E}\left[\left\langle\left((\eta_n-\rho_n)^2-\rho_n(1-\rho_n)\right)\tilde{\varsigma}_k'',\tilde{\varsigma}_l''\right\rangle_n^2\right]$$

$$= \frac{1}{(2n+1)^{2d}}\sum_{k,l\in\mathbb{Z}_n^d}\mathbb{E}\left[\sum_{x\in\mathbb{T}_n^d}\left((\eta_n(x)-\rho_n(x))^2-\rho_n(x)(1-\rho_n(x))\right)\tilde{\varsigma}_k''(x)\tilde{\varsigma}_l''(x)\right]^2$$

and using the independence of $\eta_n(x)$, $x\in\mathbb{T}_n^d$, and the equality $\mathbb{E}\left[(\eta_n(x)-\rho_n(x))^2\right] = \rho_n(x)(1-\rho_n(x))$, we get

$$\mathbb{E}\left[\|E^{1*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right] = \frac{1}{(2n+1)^{2d}}\sum_{k,l\in\mathbb{Z}_n^d}\frac{1}{(1+|k|^2)^I(1+|l|^2)^I}$$
$$\cdot\sum_{x\in\mathbb{T}_n^d}\mathbb{E}\left[\left((\eta_n(x)-\rho_n(x))^2-\rho_n(x)(1-\rho_n(x))\right)^2\right]\tilde{\varsigma}_k^2(x)\tilde{\varsigma}_l^2(x)$$
$$\leq \frac{16}{(2n+1)^{2d}}\sum_{k,l\in\mathbb{Z}_n^d}\frac{1}{(1+|k|^2)^I(1+|l|^2)^I}$$
$$\cdot\sum_{x\in\mathbb{T}_n^d}\mathbb{E}\left[\left((\eta_n(x)-\rho_n(x))^2-\rho_n(x)(1-\rho_n(x))\right)^2\right]$$
$$\leq \frac{C_I}{(2n+1)^d}$$

due to the boundedness of η_n, ρ_n and the fact that $I > \frac{d}{2}$. We now consider

$$\mathbb{E}\left[\|E^{2*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right] = \sum_{k,l\in\mathbb{Z}_n^d}\left(\langle\rho_n(1-\rho_n)\tilde{\varsigma}_k'',\tilde{\varsigma}_l''\rangle_n - \langle\rho(1-\rho)\tilde{\varsigma}_k'',\tilde{\varsigma}_l''\rangle\right)^2$$
$$= \sum_{k,l\in\mathbb{Z}_n^d}\frac{1}{(1+|k|^2)^I(1+|l|^2)^I}\left(\langle\rho_n(1-\rho_n)\tilde{\varsigma}_k,\tilde{\varsigma}_l\rangle_n - \langle\rho(1-\rho)\tilde{\varsigma}_k,\tilde{\varsigma}_l\rangle\right)^2.$$

To estimate the sum in the right hand side, we rewrite for $\varphi\in \mathrm{C}(\mathbb{T}^d)$

$$|\langle\rho_n(1-\rho_n),\varphi\rangle_n - \langle\rho(1-\rho),\varphi\rangle|$$
$$= \left|\frac{1}{(2n+1)^d}\sum_{x\in\mathbb{T}_n^d}\rho_n(x)(1-\rho_n(x))\varphi(x) - \int_{\mathbb{T}^d}\rho(y)(1-\rho(y))\varphi(y)dy\right| \tag{4.2}$$
$$= \left|\int_{\mathbb{T}^d}\bar{\rho}_n(y)(1-\bar{\rho}_n(y))\bar{\varphi}_n(y)dy - \int_{\mathbb{T}^d}\rho(y)(1-\rho(y))\varphi(y)dy\right|,$$

where

$$\bar{\rho}_n = \sum_{x\in\mathbb{T}_n^d}\rho_n(x)\mathbb{I}_{\pi_x^n} \quad\text{and}\quad \bar{\varphi}_n = \sum_{x\in\mathbb{T}_n^d}\varphi(x)\mathbb{I}_{\pi_x^n}$$

for $\pi_x^n = \prod_{j=1}^d \left[x_j, x_j + \frac{2\pi}{2n+1}\right)$. Using the triangle inequality, we can bound the right hand side of (4.2) by

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\bar\rho_n(y) - \rho(y)| \, (1 - \bar\rho_n(y)) \, |\bar\varphi_n(y)| \, dy \\
&\quad + \int_{\mathbb{T}^d} \rho(y) \, |\bar\rho_n(y) - \rho(y)| \, |\bar\varphi_n(y)| \, dy \\
&\quad + \int_{\mathbb{T}^d} \rho(y)(1 - \rho(y)) \, |\varphi(y) - \bar\varphi_n(y)| \, dy \\
&\quad \leq 2\|\varphi\|_{\mathrm{C}} \int_{\mathbb{T}^d} |\bar\rho_n(y) - \rho(y)| \, dy + \frac{C}{n} \|\nabla\varphi\|_{\mathrm{C}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}\left[\|E^{2*}\|^2_{\mathcal{L}_2^{HS}(\mathbb{R}^{\mathbb{T}_n^d})}\right] &\leq \sum_{k,l \in \mathbb{Z}_n^d} \frac{1}{(1+|k|^2)^I (1+|l|^2)^I} \\
&\qquad \cdot \left(\|\tilde\varsigma_k \tilde\varsigma_l\|_{\mathrm{C}} \int_{\mathbb{T}^d} |\bar\rho_n(y) - \rho(y)| \, dy + \frac{C}{n} \|\nabla\left(\tilde\varsigma_k \tilde\varsigma_l\right)\|_{\mathrm{C}} \right)^2 \\
&\leq C_I \left(\frac{1}{n} + \int_{\mathbb{T}^d} |\bar\rho_n(y) - \rho(y)| \, dy \right)^2 \\
&\leq C_I \left(\frac{1}{n}(1 + \|\nabla\rho\|_{\mathrm{C}}) + \|\rho_n - \rho\|_n \right)^2.
\end{aligned}$$

We now estimate

$$\begin{aligned}
\|D^3 f_n\|_{\mathrm{C}(\mathcal{L}_3(\mathbb{R}^{\mathbb{T}_n^d})} &= \sup_{z \in \mathbb{R}^{\mathbb{Z}_n^d}} \sup_{|a^i| \leq 1} \left| D^3 f_n(z)(a_1, a_2, a_3) \right| \\
&= \sup_{z \in \mathbb{R}^{\mathbb{Z}_n^d}} \sup_{|a^i| \leq 1} \left| \sum_{k,l,i} \frac{\partial^3 f_n}{\partial z_k \partial z_l \partial z_i}(z) a_k^1 a_l^2 a_i^3 \right| \\
&= \sup_{z \in \mathbb{R}^{\mathbb{Z}_n^d}} \sup_{|a^i| \leq 1} \left| \sum_{k,l,i} D^3 F(\kappa_n(z)) \, [\tilde\varsigma_k', \tilde\varsigma_l', \tilde\varsigma_i'] \, a_k^1 a_l^2 a_i^3 \right| \\
&= \sup_{z \in \mathbb{R}^{\mathbb{Z}_n^d}} \sup_{|a^i| \leq 1} \left| D^3 F(\kappa_n(z)) \left[\iota_n(a^1), \iota_n(a^2), \iota_n(a^3)\right] \right|,
\end{aligned}$$

where

$$\iota_n(a) = \sum_{k \in \mathbb{Z}_n^d} a_k \tilde\varsigma_k' \in L_2(\mathbb{T}^d)$$

for $a \in \mathbb{R}^{\mathbb{Z}_n^d}$. Due to the identity

$$\|\iota(a)\|^2_{H_{-I}} = \sum_{k \in \mathbb{Z}_n^d} \frac{1}{(1+|k|^2)^I} \left(1+|k|^2\right)^I a_k^2 = \sum_{k \in \mathbb{Z}_n^d} a_k^2 = |a|^2,$$

we get

$$\|D^3 f_n\|_{\mathrm{C}} \leq \sup_{z \in \mathbb{R}^{\mathbb{Z}_n^d}} \sup_{\|g_i\|_{H_{-I}} \leq 1} \left| D^3 F(\chi_n(z)) \, [g_1, g_2, g_3] \right| \leq \|D^3 F\|_{\mathrm{C}(\mathcal{L}_3(H_{-I}))}.$$

It only remains to estimate

$$\begin{aligned}\mathbb{E}\left[|X'-X|^3\right] &= \frac{1}{(2n+1)^{3d}}\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}_n^d}\left|\tilde{\zeta}_n(\gamma)-\zeta_n(\gamma)\right|^2|\varsigma_k''(\gamma)|^2\right)^{\frac{3}{2}}\right]\\ &= \frac{1}{(2n+1)^{3d}}\mathbb{E}\left[\left|\tilde{\zeta}_n(\gamma)-\zeta_n(\gamma)\right|^3\left(\sum_{k\in\mathbb{Z}_n^d}\frac{1}{(1+|k|^2)^I}\right)^{\frac{3}{2}}\right]\\ &= \frac{C_I}{(2n+1)^{3d}}\mathbb{E}\left[\left|\tilde{\zeta}_n(\gamma)-\zeta_n(\gamma)\right|^3\right]\\ &\le \frac{C_I}{(2n+1)^{3d/2}}\end{aligned}$$

due to the bound $\left|\zeta_n(\gamma)-\tilde{\zeta}_n(\gamma)\right| \le 2(2n+1)^{d/2}$ for all $x\in\mathbb{T}_n^d$. Combining all estimates together, we get the statement of the proposition. □

5 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. We will do so under more general assumptions on ρ_0^n than in the statement of the result. Namely, we assume that the initial conditions ρ_0^n are arbitrary functions from $L_2(\mathbb{T}_n^d)$ taking values in $[0,1]$ such that $\sup_{n\ge1}\|\mathrm{ex}_n\rho_0^n\|_{H_{\tilde{J}}}<\infty$. Additionally, let $J>\frac{d}{2}\vee 2$, $\tilde{I}>\frac{d}{2}+1$, $I>\tilde{I}+\frac{d}{2}+2$ and $\tilde{J}>(\tilde{I}\vee(\frac{d}{2}+4))+\frac{d}{2}+1$. We will show that for each $T>0$ there exists a constant C independent of F and n such that

$$\sup_{t\in[0,T]}\left|\mathbb{E}F(\hat{\rho}_t^n,\hat{\zeta}_t^n)-\mathbb{E}F(\rho_t^\infty,\eta_t^\infty)\right|\le C\|F\|_{\mathrm{C}^{1,3}_{l,HS}}\left(\frac{1}{n^{\frac{d}{2}\wedge 1}}+\|\hat{\rho}_0^n-\rho_0\|_{H_J}\right).$$

Using the inequality above and Lemma B.6, this immediately yields Theorem 1.1.
We first assume that $F\in\mathrm{C}^{2,4}_{l,HS}(H_J,H_{-I})$ and D_2^2F is uniformly continuous in $\mathcal{L}_2^{HS}(H_{-I})$.
Let $t\in(0,T]$ be fixed. To compare the difference $\mathbb{E}\left[F(\rho_t^\infty,\zeta_t^\infty)\right]-\mathbb{E}[F(\hat{\rho}_t^n,\hat{\zeta}_t^n)]$, we will use the expression (1.5). Since $U_{t-s}(\hat{\rho}_s^n,\hat{\zeta}_s^n)$ is not well-defined there if $\hat{\rho}_s^n$ takes values outside $[0,1]$, we will first replace the process ζ^∞ by a solutions to the SPDE (3.2) with Φ being a mollification of $f(x):=x(1-x)\vee 0$, $x\in\mathbb{R}$. More precisely, we take a non-negative function $\phi\in\mathrm{C}^2(\mathbb{R})$ such that $\mathrm{supp}\phi\in[-1,1]$ and $\int_{\mathbb{R}}\phi(x)dx=1$. Then for each $\varepsilon>0$ we define $\phi_\varepsilon:=\frac{1}{\varepsilon}\phi(\varepsilon\cdot)$ and $\Phi_\varepsilon:=\phi_\varepsilon * f$. Let

$$U_t^\varepsilon(\rho,\zeta):=\mathbb{E}F(\rho_t^\infty,\zeta_t^{\infty,\varepsilon}),$$

where $(\rho^\infty,\zeta^{\infty,\varepsilon})$ is a solution to (3.1), (3.2) in $H_J\times H_{-\tilde{I}}$ started from (ρ_0,ζ_0) with Φ replaced by Φ_ε. Since $I>\frac{d}{2}+3$ and $J>\frac{d}{2}+1$, we can use Proposition 3.8 to conclude that $U^\varepsilon,\partial U^\varepsilon,D_1U^\varepsilon\in\mathrm{C}([0,\infty)\times H_{J+2}\times H_{-I+2})$ and $U_t^\varepsilon\in\mathrm{C}^{1,3}_{l,HS}(H_J,H_{-I})$ for each $t>0$. Thus, by Lemma 2.2 and Proposition 3.9, we get

$$\begin{aligned}\mathbb{E}F(\hat{\rho}_t^n,\hat{\zeta}_t^n) &= \mathbb{E}U_{t-t}^\varepsilon(\hat{\rho}_t^n,\hat{\zeta}_t^n)\\ &= \mathbb{E}U_t^\varepsilon(\hat{\rho}_0^n,\hat{\zeta}_0^n)+\int_0^t\mathbb{E}\left[\hat{\mathcal{G}}^{FF}U_{t-s}^\varepsilon(\hat{\rho}_s^n,\hat{\zeta}_s^n)-\partial U_{t-s}^\varepsilon(\hat{\rho}_t^n,\hat{\zeta}_t^n)\right]ds\\ &= \mathbb{E}U_t^\varepsilon(\hat{\rho}_0^n,\hat{\zeta}_0^n)+\int_0^t\mathbb{E}\left[\hat{\mathcal{G}}^{FF}U_{t-s}^\varepsilon(\hat{\rho}_s^n,\hat{\zeta}_s^n)-\mathcal{G}^{OU,\Phi_\varepsilon}U_{t-s}^\varepsilon(\hat{\rho}_t^n,\hat{\zeta}_t^n)\right]ds.\end{aligned}$$

Applying Proposition 2.3, we obtain

$$\begin{aligned}
&\mathbb{E}F(\hat{\rho}^n_t,\hat{\zeta}^n_t)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\\
&=\frac{2\pi^2}{(2n+1)^d}\sum_{j=1}^d\int_0^t\mathbb{E}\left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right),\mathrm{ex}_n\left[\zeta^n_s\tau^n_j\zeta^n_s\right]\right\rangle ds\\
&+4\pi^2\int_0^t\sum_{j=1}^d\mathbb{E}\Bigg[\left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right),\hat{\rho}^n_s(1-\hat{\rho}^n_s)\right\rangle\\
&\qquad-\left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right),\Phi_\varepsilon\left(\hat{\rho}^n_s\right)\right\rangle\Bigg]ds\\
&+\int_0^t\mathbb{E}R^n_{t-s}(\rho^n_s,\zeta^n_s)ds,
\end{aligned}$$

where

$$|R^n_s(\rho,\zeta)|\le\frac{C_{J,I,\tilde{I},T}}{n^{\frac{d}{2}\wedge 1}}\|U^\varepsilon_{t-s}\|_{\mathrm{C}^{1,3}_{l,HS}}\left(1+\|\hat{\rho}\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}+\|\hat{\rho}\|_{C^{\lceil\tilde{I}\rceil}}\right)\left(1+\|\hat{\zeta}\|_{H_{-\tilde{I}}}\right)$$

for all $\rho\in[0,1]^{\mathbb{T}^d_n}$ and $\zeta=(2n+1)^{d/2}(\eta-\rho)$, $\eta\in\{0,1\}^{\mathbb{T}^d_n}$. Consequently,

$$\begin{aligned}
&\left|\mathbb{E}F(\hat{\rho}^n_t,\hat{\zeta}^n_t)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|\\
&\le\frac{2\pi^2}{(2n+1)^d}\sum_{j=1}^d\int_0^t\left|\mathbb{E}\left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right),\mathrm{ex}_n\left[\zeta^n_s\tau^n_j\zeta^n_s\right]\right\rangle\right|ds\\
&+2\pi^2\int_0^t\sum_{j=1}^d\mathbb{E}\left|\left\langle \operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right),\hat{\rho}^n_s(1-\hat{\rho}^n_s)-\Phi_\varepsilon\left(\hat{\rho}^n_s\right)\right\rangle\right|ds \qquad (5.1)\\
&+\frac{C_{J,I,\tilde{I},T}}{n^{\frac{d}{2}\wedge 1}}\int_0^t\|U^\varepsilon_{t-s}\|_{\mathrm{C}^{1,3}_{l,HS}}\left(1+\|\hat{\rho}^n_s\|^2_{\mathrm{C}^{\lceil d/2\rceil+4}}+\|\hat{\rho}^n_s\|_{C^{\lceil\tilde{I}\rceil}}\right)\left(1+\mathbb{E}\|\hat{\zeta}^n_s\|_{H_{-\tilde{I}}}\right)ds.
\end{aligned}$$

We next note that the function $f^{n,j,\varepsilon}_s:=\operatorname{Tr}\left(\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\right)$ belongs to $H_{\tilde{I}}$ due to $\tilde{I}+1+\frac{d}{2}<I$ and

$$\begin{aligned}
\|f^{n,j,\varepsilon}_s\|_{H_{\tilde{I}}}&\le C\|\partial_j^{\otimes 2}D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\|_{\mathcal{L}^{HS}_2(H_{-I+1})} \qquad (5.2)\\
&\le C\|D_2^2U^\varepsilon_{t-s}(\hat{\rho}^n_s,\hat{\zeta}^n_s)\|_{\mathcal{L}^{HS}_2(H_{-I})}\le C_{I,T}(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}},
\end{aligned}$$

according to Proposition 3.9 and Lemmas B.15 and B.16. Thus, by Lemma 2.6 (recall that $\tilde{I}>\frac{d}{2}$), the first term of (5.1) can be estimated by

$$\frac{\tilde{C}}{n^{\frac{d}{2}\wedge 1}}\mathbb{E}\left[\|f^{n,j,\varepsilon}_s\|^2_{H_{\tilde{I}}}\right]^{1/2}\le\frac{\tilde{C}}{n^{\frac{d}{2}\wedge 1}}C_{I,T}(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}},$$

where the constant $\tilde{C}$ depends on $\tilde{J}$, d and $\sup_{n\ge1}\|\nabla\rho^n_0\|_{n,\mathrm{C}}$. Note that the finiteness of $\sup_{n\ge1}\|\nabla\rho^n_0\|_{n,\mathrm{C}}$ follows from

$$\|\nabla_n\rho^n_0\|_{n,\mathrm{C}}\le\sum_{j=1}^d\|\partial_{n,j}\rho^n_0\|_{n,\mathrm{C}}\le\sum_{j=1}^d\|\mathrm{ex}_n\partial_{n,j}\rho^n_0\|_{\mathrm{C}}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{d} \|\mathrm{ex}_n \partial_{n,j} \rho_0^n\|_{H_{\tilde{J}-1}} \leq \sum_{j=1}^{d} \|\mathrm{ex}_n \rho_0^n\|_{H_{\tilde{J}}} \\
&= d \|\hat{\rho}_0^n\|_{H_{\tilde{J}}}
\end{aligned} \tag{5.3}
$$

and the assumption (i) of the theorem, where starting from the second inequality in the estimate above we used the interpolation property (A.8) of ex_n, the Sobolev embedding theorem and then Lemma B.8.

We next estimate the second term of the right hand side of (5.1). We note that $|\Phi_\varepsilon(x) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{R}$. Thus,

$$
\begin{aligned}
&\int_0^t \sum_{j=1}^{d} \left| \left\langle \mathrm{Tr}\left(\partial_j^{\otimes 2} D_2^2 U_{t-s}^{\varepsilon}(\hat{\rho}_s^n, \hat{\zeta}_s^n) \right), \hat{\rho}_s^n(1-\hat{\rho}_s^n) - \Phi_\varepsilon\left(\hat{\rho}_s^n\right) \right\rangle \right| ds \\
&\qquad \leq \int_0^t \sum_{j=1}^{d} \left| \left\langle f_s^{n,j,\varepsilon}, f\left(\hat{\rho}_s^n\right) - \Phi_\varepsilon\left(\hat{\rho}_s^n\right) \right\rangle \right| ds \\
&\qquad + \int_0^t \sum_{j=1}^{d} \left| \left\langle f_s^{n,j,\varepsilon}, \hat{\rho}_s^n(1-\hat{\rho}_s^n) \mathbb{I}_{\{\hat{\rho}_s^n \notin [0,1]\}} \right\rangle \right| ds.
\end{aligned} \tag{5.4}
$$

The first term of the right hand side of (5.4) can be estimated by $\varepsilon C_{I,T}(\|\Phi_\varepsilon'\|_{\mathrm{C}} + 1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}}$, according to the Cauchy-Schwarz inequality and (5.2). Next, for each $s \in [0,t]$ we have

$$
\begin{aligned}
\left| \left\langle f_s^{n,j,\varepsilon}, \hat{\rho}_s^n(1-\hat{\rho}_s^n)\mathbb{I}_{\{\hat{\rho}_s^n \notin [0,1]\}} \right\rangle \right| &\leq \|f_s^{n,j,\varepsilon}\|_{\mathrm{C}} \left\| \hat{\rho}_s^n \mathbb{I}_{\{\hat{\rho}_s^n \notin [0,1]\}} \right\| \left\| (1-\hat{\rho}_s^n)\mathbb{I}_{\{\hat{\rho}_s^n \notin [0,1]\}} \right\| \\
&\leq \|f_s^{n,j,\varepsilon}\|_{\tilde{I}} \left\| \hat{\rho}_s^n \mathbb{I}_{\{\hat{\rho}_s^n < 0\}} \right\| \|1-\hat{\rho}_s^n\|.
\end{aligned}
$$

Consider the convex function $\psi(x) := |x|\mathbb{I}_{\{x<0\}}$, $x \in \mathbb{R}$, and note that it satisfies the triangle inequality $\psi(x+y) \leq \psi(x) + \psi(y)$, $x, y \in \mathbb{R}$. Thus,

$$
\begin{aligned}
\left\| \hat{\rho}_s^n \mathbb{I}_{\{\hat{\rho}_s^n < 0\}} \right\| &= \|\psi\left(\hat{\rho}_s^n\right)\| \leq \|\psi\left(\hat{\rho}_s^n - \rho_s^\infty\right)\| + \|\psi\left(\rho_s^\infty\right)\| \\
&\leq \|\hat{\rho}_s^n - \rho_s^\infty\| + 0,
\end{aligned}
$$

since $\rho_s^\infty \geq 0$. Now, using the triangle inequality, Corollary B.12 and Lemma B.2, we get

$$
\begin{aligned}
\|\hat{\rho}_s^n - \rho_s^\infty\| &\leq \|\hat{\rho}_s^n - \mathrm{pr}_n \rho_s^\infty\| + \|\mathrm{pr}_n \rho_s^\infty - \rho_s^\infty\| \\
&\leq C_T \|\hat{\rho}_0^n - \rho_0\| + \frac{C_T}{n} \|\rho_0\|_{H_2}.
\end{aligned}
$$

Consequently,

$$
\left\| \hat{\rho}_s^n \mathbb{I}_{\{\hat{\rho}_s^n < 0\}} \right\| \leq C_T \|\hat{\rho}_0^n - \rho_0\| + \frac{C_T}{n} \|\rho_0\|_{H_2}
$$

for all $s \in [0,t]$. Note that

$$
\|1 - \hat{\rho}_s^n\| = \|1 - \rho_s^n\|_n \leq 1,
$$

according to the maximum principle. This shows that the second term in the right hand side of (5.4) is estimated by

$$
C_{I,T}(\|\Phi_\varepsilon'\|_{\mathrm{C}} + 1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}} \left(\|\hat{\rho}_0^n - \rho_0\| + \frac{1}{n} \|\rho_0\|_{H_2} \right).
$$

To estimate the third term of the right hand side of (5.1), we use Proposition 3.9 to control $\|U^\varepsilon_{t-s}\|_{\mathrm{C}^{1,3}_{l,HS}}$ by $C_{I,T}(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}}$. Next, recall that the sequence $\|\hat{\rho}^n_0\|_{H_{\tilde{J}}}$, $n\geq 1$, is bounded. Since trivially $\|\hat{\rho}^n_t\|_{H_{\tilde{J}}}\leq\|\hat{\rho}^n_0\|_{H_{\tilde{J}}}$ for all $t\geq 0$, we get that $\|\hat{\rho}^n_s\|_{\mathrm{C}^{\lceil d/2\rceil+4}}$ and $\|\hat{\rho}^n_s\|_{\mathrm{C}^{\lceil \tilde{I}\rceil}}$ are uniformly bounded in $n\geq 1$ and $s\in[0,t]$, due to the Sobolev embedding theorem and the fact that $\tilde{J}>\lceil d/2\rceil+4+\frac{d}{2}$ and $\tilde{J}>\lceil\tilde{I}\rceil+\frac{d}{2}$. Using Lemma 2.5 and (5.3), we get

$$\mathbb{E}\left[\|\mathrm{ex}_n\zeta^n_t\|^2_{H_{-\tilde{I}}}\right]<C_{\tilde{I}}\left(1+2\pi^2dt\|\hat{\rho}^n_0\|_{H_{\tilde{J}}}\right).$$

This completes the proof of the fact that

$$\begin{aligned}&\left|\mathbb{E}F(\hat{\rho}^n_t,\hat{\zeta}^n_t)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|\\&\qquad\leq C(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}}\left(\frac{1}{n^{\frac{d}{2}\wedge 1}}\left(1+\|\rho_0\|_{H_2}\right)+\|\hat{\rho}^n_0-\rho_0\|+\varepsilon\right),\end{aligned}\tag{5.5}$$

where the constant C depends on $J,\tilde{J},I,\tilde{I},T$ and $\sup_{n\geq 1}\|\hat{\rho}^n_0\|_{H_{\tilde{J}}}$.

We next estimate the difference $\mathbb{E}U^\varepsilon_t(\rho_0,\zeta_0)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)$. By the triangle inequality and the mean-value theorem, we get

$$\begin{aligned}\left|\mathbb{E}U^\varepsilon_t(\rho_0,\zeta_0)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|&\leq\left|\mathbb{E}U^\varepsilon_t(\rho_0,\zeta_0)-\mathbb{E}U^\varepsilon_t(\rho_0,\hat{\zeta}^n_0)\right|\\&\quad+\left|\mathbb{E}U^\varepsilon_t(\rho_0,\hat{\zeta}^n_0)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|\\&\leq\left|\mathbb{E}U^\varepsilon_t(\rho_0,\hat{\zeta}^n_0)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|\\&\quad+\|D_1U^\varepsilon_t\|_{\mathrm{C}}\|\rho_0-\hat{\rho}^n_0\|_{H_J}.\end{aligned}\tag{5.6}$$

Recall that

$$\|U^\varepsilon_t\|_{\mathrm{C}^{1,3}_{l,HS}}\leq C_{I,T}(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}},$$

according to Proposition 3.9. Moreover, by Proposition 4.1,

$$\left|\mathbb{E}U^\varepsilon_t(\rho_0,\zeta_0)-\mathbb{E}U^\varepsilon_t(\rho_0,\hat{\zeta}^n_0)\right|\leq C_I\left(\frac{1}{n^{1\wedge\frac{d}{2}}}(1+\|\nabla\rho_0\|_{\mathrm{C}})+\|\rho^n_0-\rho_0\|_n\right)\|U^\varepsilon_t\|_{\mathrm{C}^3_{l,HS}}.$$

We can also estimate $\|\nabla\rho_0\|_{\mathrm{C}}\leq\|\rho_0\|_{H_J}$ and

$$\|\rho^n_0-\rho_0\|_n\leq\|\hat{\rho}^n_0-\rho_0\|_{\mathrm{C}}\leq\|\hat{\rho}^n_0-\rho_0\|_{H_J}.$$

Consequently,

$$\begin{aligned}&\left|\mathbb{E}U^\varepsilon_t(\rho_0,\zeta_0)-\mathbb{E}U^\varepsilon_t(\hat{\rho}^n_0,\hat{\zeta}^n_0)\right|\\&\qquad\leq C_{I,T}(\|\Phi'_\varepsilon\|_{\mathrm{C}}+1)\|F\|_{\mathrm{C}^{1,3}_{l,HS}}\left(\frac{1}{n^{1\wedge\frac{d}{2}}}(1+\|\rho_0\|_{H_J})+\|\hat{\rho}^n_0-\rho_0\|_{H_J}\right).\end{aligned}\tag{5.7}$$

Combining the inequalities (5.5), (5.7) and using the uniform bound of $\|\Phi'_\varepsilon\|_{\mathrm{C}}$ in ε, we get that there exists a constant C independent of n, ε, t and F such that

$$\left|\mathbb{E}F(\hat{\rho}^n_t,\hat{\zeta}^n_t)-\mathbb{E}F(\rho^\infty_t,\zeta^{\infty,\varepsilon}_t)\right|\leq Ct\|F\|_{\mathrm{C}^{1,3}_{l,HS}}\left(\frac{1}{n^{1\wedge\frac{d}{2}}}+\|\hat{\rho}^n_0-\rho_0\|_{H_J}+\varepsilon\right).\tag{5.8}$$

Now, making $\varepsilon\to 0+$ and using Lemma 3.3, we get the required estimate for $F\in\mathrm{C}^{2,4}_{l,HS}(H_J,H_{-I})$ with uniformly continuous second order derivative D_2F in $\mathcal{L}^{HS}_2(H_{-I})$. Since the constant C is independent of F in the inequality (5.8), we can cover the case $F\in\mathrm{C}^{1,3}_{l,HS}(H_J,H_{-I})$ by an pointwise approximation argument. This completes the proof of Theorem 1.1.

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A Notation and basic facts

The goal of this section is to introduce the basic notation that are used throughout this work.

A.1 Continuous spaces

Recall that $\mathbb{T}^d$ denotes the d-dimensional torus $(\mathbb{R}/\{2\pi k - \pi : \ k \in \mathbb{Z}\})^d$. Let $\mathrm{C}(E)$ be the space of continuous functions on E and $\mathrm{C}^m(E)$ be a subspace of $\mathrm{C}(E)$ consisting of all m-times continuously differentiable functions for $m \in \mathbb{N} \cup \{\infty\}$, where $E = \mathbb{T}^d$ or $\mathbb{R}^d$. We equip $\mathrm{C}(\mathbb{T}^d)$ and $\mathrm{C}^m(\mathbb{T}^d)$ with the standard uniform norms denoted by $\|\cdot\|_{\mathrm{C}}$ and $\|\cdot\|_{\mathrm{C}^m}$, respectively. For $f \in \mathrm{C}^m(\mathbb{T}^d)$ we write $\partial_j^m f$ for its partial derivative of m-th order with respect to the j-th coordinate. As usual, we also set

$$\Delta f := \sum_{j=1}^d \partial_j^2 f \quad \text{and} \quad \nabla f := (\partial_j f)_{j \in [d]} .$$

The set of all functions from $\mathrm{C}^m(\mathbb{R}^d)$ that have bounded derivatives to the m-th order is denoted by $\mathrm{C}^m_l(\mathbb{R}^d)$. The subset of $\mathrm{C}^m_l(\mathbb{R}^d)$ consisting of all bounded functions is denoted by $\mathrm{C}^m_b(\mathbb{R}^d)$.

Sobolev spaces. Let $L_2(\mathbb{T}^d)$ denote the Hilbert space of square-integrable real-valued functions on $\mathbb{T}^d$ with respect to the Lebesgue measure. The inner product on $L_2(\mathbb{T}^d)$ associated with the normalized Lebasgue measure is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$. To define a basis on $L_2(\mathbb{T}^d)$ we split $\mathbb{Z}^d \setminus \{0\}$ on two disjoint subsets $\mathbb{Z}^d_1$ and $\mathbb{Z}^d_2$ such that $\mathbb{Z}^d_1 = -\mathbb{Z}^d_2$ and take

$$\tilde{\varsigma}_k = \begin{cases} 2\cos k \cdot x, & k \in \mathbb{Z}^d_1, \\ 2 \sin k \cdot x, & k \in \mathbb{Z}^d_2, \\ 1, & k = 0, \end{cases}$$

for all $k \in \mathbb{Z}^d$. We also consider the complex-valued functions

$$\varsigma_k = \left(e^{\mathrm{i}k \cdot x}, \ x \in \mathbb{T}^d\right), \quad k \in \mathbb{Z}^d,$$

that form an orthonormal basis in the Hilbert space of all square-integrable complex-valued functions on $\mathbb{T}^d$ equipped with the standard inner product, denoted also by $\langle \cdot, \cdot \rangle$. Since for each $f \in L_2(\mathbb{T}^d)$ and $n \in \mathbb{N}$

$$\mathrm{pr}_n f := \sum_{k \in \mathbb{Z}^d_n} \langle f, \varsigma_k \rangle \varsigma_k = \sum_{k \in \mathbb{Z}^d_n} \langle f, \tilde{\varsigma}_k \rangle \tilde{\varsigma}_k \in L_2(\mathbb{T}^d),$$

where $\mathbb{Z}_n^d = \{-n, \dots, n\}^d$, the set of functions $\{\tilde{\varsigma}_k, k \in \mathbb{Z}^d\}$ is an orthonormal basis in $L_2(\mathbb{T}^d)$. To simplify many computations later on, we will prefer to work with $\{\varsigma_k, k \in \mathbb{Z}\}$. For $J \geq 0$ we define the Sobolev space

$$H_J := \left\{ f \in L_2(\mathbb{T}^d) : \ \|f\|_{H_J}^2 := \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^J |\langle f, \varsigma_k \rangle|^2 < \infty \right\}$$

and H_{-J} as the completion of $L_2(\mathbb{T}^d)$ with respect to the norm

$$\|f\|_{H_{-J}}^2 := \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^{-J} |\langle f, \varsigma_k \rangle|^2 .$$

It is well-known that $H_J \subset L_2(\mathbb{T}^d) \subset H_{-J}$ for each $J > 0$ and H_{-J} is the dual space to H_J with respect to the relation $\langle \cdot, \cdot \rangle$. We also note that the operators ∂_j and Δ can be naturally defined on H_J for each $J \in \mathbb{R}$. Moreover, $\partial_j : H_J \to H_{J-1}$ and $\Delta : H_J \to H_{J-2}$ are bounded linear operators and

$$\|f\|_{H_J} = \left\|(1 + \Delta)^J f\right\|$$

for each $J \in \mathbb{R}$.

Multilinear operators. Let $(E_i, \| \cdot \|_{E_i})$, $i \in [2]$, be arbitrary Banach spaces. The set of all continuous symmetric multilinear operators from E_1^m to E_2 is denoted by $\mathcal{L}_m(E_1; E_2)$. We equip $\mathcal{L}_m(E_1; E_2)$ with the norm

$$\|A\|_{\mathcal{L}_m} := \sup_{\|x_j\|_{E_1} \leq 1} \|A[x_1, \dots, x_m]\|_{E_2},$$

which makes it a Banach space (see e.g. [9, Section 1.8] for more details). If $E_2 = \mathbb{R}$, we simply write $\mathcal{L}_m(E_1)$ instead of $\mathcal{L}_m(E_1; \mathbb{R})$. If E_1 is a separable Hilbert space with an orthonormal basis $\{z_l, l \in \mathbb{N}\}$ and $E_2 = \mathbb{R}$, we define the space of Hilbert-Schmidt multilinear operators by

$$\mathcal{L}_m^{HS}(E_1) := \left\{ A \in \mathcal{L}_m(E_1) : \|A\|_{\mathcal{L}_m^{HS}}^2 := \sum_{(l_j) \in \mathbb{N}^m} |A[z_{l_1}, \dots, z_{l_m}]|^2 < \infty \right\}.$$

Note that the space $\mathcal{L}_m^{HS}(E_1)$ can be defined iteratively as the space of all Hilbert-Schmidt operators from E_1 to $\mathcal{L}_{m-1}^{HS}(E_1)$ for $m \geq 2$, where $\mathcal{L}_1^{HS}(E_1)$ is identified with E_1 via the Riesz representation theorem, and then $\| \cdot \|_{\mathcal{L}_m^{HS}}$ coincides with the usual Hilbert-Schmidt norm. In particular, for $E_1 = H_J$ and $m = 2$ one has

$$\begin{aligned} \|A\|_{\mathcal{L}_2^{HS}}^2 &:= \sum_{k,l \in \mathbb{Z}^d} (1 + |k|^2)^{-J} (1 + |l|^2)^{-J} \left|A[\varsigma_k, \varsigma_l]\right|^2 \\ &= \sum_{k,l \in \mathbb{Z}^d} (1 + |k|^2)^{-J} (1 + |l|^2)^{-J} \left|A[\tilde{\varsigma}_k, \tilde{\varsigma}_l]\right|^2 . \end{aligned} \tag{A.1}$$

A simple computation shows that $\mathcal{L}_m(H_J)$ is continuously embedded into $\mathcal{L}_m^{HS}(H_I)$ for $I > J + d/2$, i.e. the restriction of $A \in \mathcal{L}_m(H_J)$ to $(H_I)^m$ belongs to $\mathcal{L}_m^{HS}(H_I)$ and

$$\|A\|_{\mathcal{L}_m^{HS}(H_I)} \leq C_{I,J,m} \|A\|_{\mathcal{L}_m(H_J)}, \tag{A.2}$$

where the constant $C_{I,J,m}$ depends on I, J and m.

For each $J \in \mathbb{R}$ and $A \in \mathcal{L}_2(H_J)$ we define the symmetric multilinear operator

$$\partial_j^{\otimes 2} A[f,g] = A[\partial_j f, \partial_j g], \quad f,g \in H_{J+1},$$

that belongs to $\mathcal{L}_2(H_{J+1})$. Moreover, it is easily seen that

$$\left\|\partial_j^{\otimes 2} A\right\|_{\mathcal{L}_2(H_{J+1})} \leq \|A\|_{\mathcal{L}_2(H_J)}.$$

We will need a bounded linear operator $\mathrm{Tr} : \mathcal{L}_2^{HS}(H_{-J}) \to H_I$ such that $\mathrm{Tr} K_a = a(x,x)$, where K_a denotes the kernel multilinear operator for a kernel $a : \left(\mathbb{T}^d\right)^2 \to \mathbb{R}$. Since the δ_x-function belongs to H_{-J} for $J > \frac{d}{2}$, we define the function $\mathrm{Tr} A : \mathbb{T}^d \to \mathbb{R}$ by

$$\mathrm{Tr} A(x) = A[\delta_x, \delta_x].$$

It is continuous and, by Lemma B.15 below, belongs to H_I for each $I < J - \frac{d}{2}$.

Derivatives on Banach spaces. Let $\mathrm{C}(E_1; E_2)$ be the space of continuous functions from a Banach space E_1 to a Banach space E_2. The subspace of $\mathrm{C}(E_1; E_2)$ of m-times continuously Frechet differentiable functions[6] is denoted by $\mathrm{C}^m(E_1; E_2)$. The subspace of $\mathrm{C}^m(E_1; E_2)$ of all bounded functions together with their derivatives to the m-th order is denoted by $\mathrm{C}_b^m(E_1; E_2)$. We will simply write $\mathrm{C}(E_1)$, $\mathrm{C}^m(E_1)$, $\mathrm{C}_b^m(E_1)$ instead of $\mathrm{C}(E_1; \mathbb{R})$, $\mathrm{C}^m(E_1; \mathbb{R})$, $\mathrm{C}_b^m(E_1; \mathbb{R})$, respectively. Note that for each $k \in [m] := \{1, \ldots, m\}$ the k-th derivative $D^k F(x)$ of $F \in \mathrm{C}^m(E_1; E_2)$ at $x \in E_1$ can be identified with a continuous symmetric multilinear operator from $\mathcal{L}_k(E_1; E_2)$. The set of functions F from $\mathrm{C}^m(E_1)$ whose derivatives $D^k F$ are bounded (in $\|\cdot\|_{\mathcal{L}_k}$-norm) functions for all $k \in [m]$ is denoted by $\mathrm{C}_l^m(E_1)$. Note that functions from $\mathrm{C}_l^m(E_1)$ are not bounded in general but they are of linear growth. The semi-norm on $\mathrm{C}_l^m(E_1)$ is defined by

$$\|F\|_{\mathrm{C}_l^m} := \sum_{k=1}^{m} \sup_{x \in E_1} \|D^k F(x)\|_{\mathcal{L}_k}.$$

If additionally $m \geq 2$ and $D^2 F$ is an $\mathcal{L}_2^{HS}(E_1)$-valued bounded function, we write $D^2 F \in \mathrm{C}_{l,HS}^m(E_1)$ and define

$$\|F\|_{\mathrm{C}_{l,HS}^m} := \|F\|_{\mathrm{C}_l^m} + \sup_{x \in E_1} \|D^2 F(x)\|_{\mathcal{L}_2^{HS}}. \tag{A.3}$$

We often identify $DF(x)$ with an element from H_{-J} for each $F \in \mathrm{C}^1(H_J)$ using the dual relation $\langle \cdot, \cdot \rangle$ between H_J and H_{-J}.

Remark A.1. Note that $F \in \mathrm{C}_l^{m_1,m_2}(H_J, H_{-I'}) \subset \mathrm{C}_{l,HS}^{m_1,m_2}(H_J, H_{-I})$ for $I' > I + \frac{d}{2}$, according to (A.2) and Lemma B.20 below. Thus, the assumption on the boundedness of the Hilbert-Schmidt norm of $D_2 F$ can be replaced with the differentiability of F in a larger Sobolev space.

Set $x^{\times m} := (x, \ldots, x) \in E_1^m$ for $x \in E_1$. A function $F \in \mathrm{C}^{m+1}(E_1; E_2)$ with bounded derivative $D^{m+1} F$ can be expanded into the Taylor series

$$F(x) = \sum_{k=0}^{m} \frac{1}{k!} D^k F(x_0) \left[(x - x_0)^{\times k}\right] + R_m(x, x_0), \quad x \in E_1, \tag{A.4}$$

where

$$\|R_m(x, x_0)\|_{E_2} \leq \frac{1}{(m+1)!} \|D^{m+1} F\|_{\mathcal{L}_{m+1}} \|x - x_0\|_{E_1}^{m+1},$$

[6] See [9, Section 5]

according to [9, Theorem 5.6.2].

The subspace of $\mathrm{C}(E_1 \times \ldots \times E_j)$ of all functions that are m_i-times continuously differentiable with respect to the i-th variable will be denoted by $\mathrm{C}^{m_1,\ldots,m_2}(E_1,\ldots,E_j)$ and D_i^k, $i \in [j]$, will denote the corresponding partial derivatives of the k-th order. We similarly introduce $\mathrm{C}_l^{m_1,\ldots,m_j}(E_1,\ldots,E_j)$, $\mathrm{C}^{1,m_1,\ldots,m_j}([0,\infty),E_1,\ldots,E_j)$ and $\|F\|_{\mathrm{C}_l^{m_1,\ldots,m_j}}$. If $F \in \mathrm{C}([0,\infty) \times E_1 \times \ldots \times E_j)$ and it is differentiable with respect to the first (time) variable, we use a special notation ∂F for its time derivative. In this case, all other derivatives, if they exist, are denoted by $D_1, D_2, ..., D_j$, respectively. Note that $\mathrm{C}^{m,m}(E_1,E_2) = C^m(E_1 \times E_2)$, according to [9, Proposition 2.6.2].

A.2 Discrete spaces

We recall that $\mathbb{T}_n^d := \left\{ \frac{2\pi k}{2n+1} : \ k \in \mathbb{Z}_n^d \right\}$ is the d-dimensional torus[7], and is considered as a subset of $\mathbb{T}^d$. The space of functions from $\mathbb{T}_n^d$ to $\mathbb{R}$ equipped with the inner product

$$\langle f, g \rangle_n = \frac{1}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} f(x) g(x)$$

is denoted by $L_2(\mathbb{T}_n^d)$. The corresponding norm on $L_2(\mathbb{T}_n^d)$ and the maximum norm are denoted by $\|\cdot\|_n$ and $\|\cdot\|_{n,\mathrm{C}}$, respectively.

Following [46, Section 5.6], we can write

$$f(x) = \sum_{k \in \mathbb{Z}_n^d} \langle f, \varsigma_k \rangle_n \varsigma_k(x) = \sum_{k \in \mathbb{Z}_n^d} \langle f, \tilde{\varsigma}_k \rangle_n \tilde{\varsigma}_k, \quad x \in \mathbb{T}_n^d, \tag{A.5}$$

for each function $f \in L_2(\mathbb{T}_n^d)$ due to the equality $\langle \varsigma_k, \varsigma_l \rangle_n = \delta_{k,l}$ for $k, l \in \mathbb{Z}_n^d$, where $\delta_{k,l}$ is the Kronecker-Delta.

The discrete differential operators on $L_2(\mathbb{T}_n^d)$ are defined by

$$\begin{aligned} \partial_{n,j} f(x) &:= \frac{2n+1}{2\pi} \left(f(x + e_j^n) - f(x) \right), \quad x \in \mathbb{T}_n^d, \\ \nabla_n f &:= (\partial_{n,j} f)_{j \in [d]} \end{aligned}$$

and

$$\Delta_n f(x) = \frac{(2n+1)^2}{4\pi^2} \sum_{j=1}^d \left(f(x + e_j) + f(x - e_j) - 2f(x) \right), \quad x \in \mathbb{T}_n^d,$$

where $e_j = e_j^n$ denote the canonical vectors, that is, $e_j^n = \left(\frac{2\pi}{2n+1} \mathbb{I}_{\{i=j\}} \right)_{i \in [d]}$ and we used the normalization constant $\frac{1}{2\pi}$ for the sake of conformity with the continuous derivatives. A simple computation shows that

$$\langle \Delta_n f, g \rangle_n = \langle f, \Delta_n g \rangle_n \quad \text{and} \quad \langle \partial_{n,j} f, g \rangle_n = -\langle f, \partial_{n,j} g \rangle_n \tag{A.6}$$

for each $f, g \in L_2(\mathbb{T}_n^d)$. We also note that for each $k \in \mathbb{Z}_n^d$ the equalities

$$\partial_{n,j} \varsigma_k = \mu_{k,j}^n \varsigma_k \quad \text{and} \quad \Delta_n \varsigma_k = -\lambda_k^n \varsigma_k \tag{A.7}$$

[7] The choice of the scale for the torus is motivated by our argument that relies on the discrete/continuous Fourier expansion. In particular, to simplify the notation, we removed the constant 2π from the exponent in the standard Fourier basis by rescaling the torus. The odd number of points in any direction will allow us easily to jump between complex-valued exponential basis and the real-valued cos-sin basis.

hold with $\mu^n_{k,j} := \frac{(2n+1)}{2\pi}\left(e^{\mathrm{i}\frac{2\pi k_j}{2n+1}} - 1\right)$ and $\lambda^n_k := \frac{(2n+1)^2}{2\pi^2}\sum_{j=1}^d \left[1 - \cos\frac{2\pi k_j}{2n+1}\right]$.

For a function $f : \left(\mathbb{T}^d_n\right)^2 \to \mathbb{R}$, we define

$$\partial^{\otimes 2}_{n,j} f(x_1, x_2) = \left(\partial_{n,j} g(x_1, \cdot)\right)(x_2), \quad (x_1, x_2) \in \left(\mathbb{T}^d_n\right)^2,$$

where $g(x_1, x_2) = \left(\partial_{n,j} g(\cdot, x_2)\right)(x_1)$, and $\mathrm{Tr} f : \mathbb{T}^d_n \to \mathbb{R}$ by

$$\mathrm{Tr} f(x) = f(x, x), \quad x \in \mathbb{T}^d_n.$$

A.2.1 Projection and extension operators

Recall the expansion (A.5) for $f \in L_2(\mathbb{T}^d_n)$. Since the right hand side of this expansion is a well-defined smooth real-valued function on $\mathbb{T}^d$, we will use it for the interpolation of f. More precisely, for $f \in L_2(\mathbb{T}^d_n)$ define

$$\mathrm{ex}_n f(x) = \sum_{k \in \mathbb{Z}^d_n} \langle f, \varsigma_k \rangle_n \varsigma_k(x) = \sum_{k \in \mathbb{Z}^d_n} \langle f, \tilde{\varsigma}_k \rangle_n \tilde{\varsigma}_k(x), \quad x \in \mathbb{T}^d.$$

By (A.5), we have

$$\mathrm{ex}_n f(x) = f(x), \quad x \in \mathbb{T}^d_n. \tag{A.8}$$

Considering a function f defined on $\mathbb{T}^d$, we will write $\mathrm{ex}_n f$ for ex_n applied to the restriction of f to $\mathbb{T}^d_n$.

We will also need a kind of inverse operation to ex_n that will allow to transform elements from H_J to functions on $\mathbb{T}^d_n$ for every $J \in \mathbb{R}$. For the sake of this, we will use the usual projection operator

$$\mathrm{pr}_n g = \sum_{k \in \mathbb{Z}^d_n} \langle g, \varsigma_k \rangle \varsigma_k = \sum_{k \in \mathbb{Z}^d_n} \langle g, \tilde{\varsigma}_k \rangle \tilde{\varsigma}_k.$$

For every $g \in H_J$, the function $\mathrm{pr}_n g$ is well-defined and smooth on $\mathbb{T}^d$. Therefore, its restriction to $\mathbb{T}^d_n$ is well-defined as well, and is also denoted by $\mathrm{pr}_n g$.

The equality

$$\langle \mathrm{ex}_n f, g \rangle = \langle f, \mathrm{pr}_n g \rangle_n \tag{A.9}$$

easily follows from the definitions of ex_n and pr_n for every $f \in L_2(\mathbb{T}^d_n)$ and $g \in H_J$. Thus, it will be often used to replace the discrete inner product by the continuous one and vice versa. In particular, the equality (A.9) implies that

$$\langle \mathrm{ex}_n f, \varsigma_k \rangle = \begin{cases} \langle f, \varsigma_k \rangle_n, & \text{if } k \in \mathbb{Z}^d_n, \\ 0, & \text{otherwise,} \end{cases} \tag{A.10}$$

for each $k \in \mathbb{Z}^d$. One can also easily see that

$$\mathrm{pr}_n \mathrm{ex}_n f = f \quad \text{and} \quad \mathrm{ex}_n \mathrm{pr}_n g = \mathrm{pr}_n g \tag{A.11}$$

for all $f \in L_2(\mathbb{T}^d_n)$ and $g \in H_J$. Thus, combining (A.9) and (A.11), we obtain

$$\langle \mathrm{ex}_n f_1, \mathrm{ex}_n f_2 \rangle = \langle f_1, f_2 \rangle_n \quad \text{and} \quad \langle \mathrm{pr}_n g_1, \mathrm{pr}_n g_2 \rangle = \langle \mathrm{pr}_n g_1, \mathrm{pr}_n g_2 \rangle_n \tag{A.12}$$

for all $f_1, f_2 \in L_2(\mathbb{T}^d_n)$ and $g_1, g_2 \in H_J$. With some abuse of notation, we set $\hat{f} := \mathrm{ex}_n f$ and $\check{g} := \mathrm{pr}_n g$.

For $A \in \mathcal{L}_m(H_J)$ we similarly define $\mathrm{pr}_n^{\otimes m} A : \left(\mathbb{T}_n^d\right)^m \to \mathbb{R}$ by

$$\mathrm{pr}_n^{\otimes m} A = \sum_{k \in (\mathbb{Z}_n^d)^m} A[\tilde{\varsigma}_{\times k}] \tilde{\varsigma}_k,$$

where

$$\tilde{\varsigma}_k = \bigotimes_{j=1}^m \tilde{\varsigma}_{k_j} \quad \text{and} \quad \tilde{\varsigma}_{\times k} = (\tilde{\varsigma}_{k_j})_{j \in [m]}$$

for $k = (k_j)_{j \in [m]}$. Similarly to $\mathrm{pr}_n f$, we will also consider $\mathrm{pr}_n^{\otimes} A$ as a smooth function on $\mathbb{T}^d$, that is defined by the same expression. For $f = (f_j)_{j \in [m]} \in (L_2(\mathbb{T}_n^d))^m$, let also

$$\mathrm{ex}_n^{\times m} f = (\mathrm{ex}_n f_j)_{j \in [m]} .$$

A simple computation yields the equality

$$A \left[\mathrm{ex}_n^{\times m} f\right] = \left\langle \mathrm{pr}_n^{\otimes m} A, f^{\otimes m} \right\rangle_n , \tag{A.13}$$

where $f^{\otimes m}(x) := \prod_{j=1}^m f_j(x_j)$, $x = (x_j)_{j \in [m]} \in \left(\mathbb{T}_n^d\right)^m$ and $\langle \cdot, \cdot \rangle_n$ is the discrete inner product on $L_2(\mathbb{T}_n^{md})$. We will also identify $\mathrm{pr}_n^{\otimes m} A$ with the symmetric multilinear operator

$$\mathrm{pr}_n^{\otimes m} A[f] = \left\langle \mathrm{pr}_n^{\otimes m} A, f^{\otimes m} \right\rangle, \quad f = (f_j)_{j \in [m]} \in H_J^m.$$

A.3 Further notation and comments

The natural filtration generated by a càdlàg process X_t, $t \geq 0$, is denoted by $(\mathcal{F}_t^X)_{t \geq 0}$. The distribution of a random variable ξ in a Banach space is denoted by $\mathrm{Law}\xi$.

A constant C in estimates below will be changed from line to line. Parameters on which C depends will be listed as its subscripts, e.g. $C_{J,I}$ will mean that the constant depends on parameters J, I. Since the dimension d is fixed, we will not further point out the dependence on d in constants.

B Some operators on Sobolev spaces

In this section, we will prove some basic properties of pr_n, ex_n and multilinear operators on Sobolev spaces.

B.1 pr_n and ex_n operators

Recall that $\varsigma_k(x) = e^{\mathrm{i} k \cdot x}$, $x \in \mathbb{T}^d$, $k \in \mathbb{Z}^d$.

Lemma B.1. *For each $n \in \mathbb{N}$, $j \in [d]$ and $k \in \mathbb{Z}_n^d$ the equalities*

$$\partial_{n,j} \varsigma_k = \mu_{k,j}^n \varsigma_k \quad \textit{and} \quad \Delta_n \varsigma_k = -\lambda_k^n \varsigma_k \tag{B.1}$$

hold with $\mu_{k,j}^n = \frac{(2n+1)}{2\pi} \left(e^{\mathrm{i} \frac{2\pi k_j}{2n+1}} - 1 \right)$ and $\lambda_k^n = \frac{(2n+1)^2}{2\pi^2} \sum_{j=1}^d \left[1 - \cos \frac{2\pi k_j}{2n+1} \right]$. Moreover,

$$\frac{|k_j|}{\sqrt{3}} \leq |\mu_{k,j}^n| \leq |k_j| \quad \textit{and} \quad \frac{|k|^2}{3} \leq \lambda_k^n \leq |k|^2. \tag{B.2}$$

Proof. The equalities (B.1) directly follows from simple computations. The inequalities (B.2) follows from

$$\frac{x^2}{3} \le \left|e^{\mathrm{i}x} - 1\right|^2 = (\cos x - 1)^2 + \sin^2 x \le x^2$$

and

$$\frac{x^2}{6} \le |1 - \cos x| \le \frac{x^2}{2}$$

for all $x \in [-\pi, \pi]$. □

We next recall that for each $f \in L_2(\mathbb{T}_n^d)$ and $g \in H_J$

$$\mathrm{ex}_n f = \sum_{k \in \mathbb{Z}_n^d} \langle f, \varsigma_k \rangle_n \varsigma_k \quad \text{and} \quad \mathrm{pr}_n g = \sum_{k \in \mathbb{Z}_n^d} \langle g, \varsigma_k \rangle \varsigma_k$$

that are smooth functions on $\mathbb{T}^d$. Moreover, the equality

$$\langle \mathrm{ex}_n f, g \rangle = \langle f, \mathrm{pr}_n g \rangle_n \tag{B.3}$$

holds. It directly follows from the fact that $\langle \varsigma_k, \varsigma_l \rangle = \delta_{k,l}$ for all $k, l \in \mathbb{Z}^d$ and $\langle \varsigma_k, \varsigma_l \rangle_n = \delta_{k,l}$ for all $k, l \in \mathbb{Z}_n^d$. We next collect the basic properties of the operator pr_n.

Lemma B.2. *The following statements holds.*

(i) For each $J \in \mathbb{R}$ and $g \in H_J$

$$\mathrm{pr}_n g \to g \quad \text{in } H_J \quad \text{and} \quad \|\mathrm{pr}_n g\|_{H_J} \le \|g\|_{H_J}.$$

(ii) Let $m \ge 0$ and $J > m + \frac{d}{2}$. Then every function $g \in H_J$ has m times continuously differentiable version, denoted also by g, such that

$$\|g\|_{\mathrm{C}^m} \le C_{m,J} \|g\|_{H_J}.$$

(iii) For each $J \ge 0$, $m := \lceil J \rceil$ and every $g \in \mathrm{C}^m(\mathbb{T}^d)$

$$\|g\|_{H_J} \le C_m \|g\|_{\mathrm{C}^m}.$$

(iv) For each $J, I \in \mathbb{R}$, $J < I$, $g \in H_I$ and $n \ge 1$

$$\|g - \mathrm{pr}_n g\|_{H_J} \le \frac{1}{n^{I-J}} \|g - \mathrm{pr}_n g\|_{H_I}.$$

In particular, for each $m \in \mathbb{N}_0$, $p \ge 0$ and $J > m + p + \frac{d}{2}$ one has

$$\|g - \mathrm{pr}_n g\|_{\mathrm{C}^m} \le \frac{C_{m,p,J}}{n^p} \|g\|_{H_J}.$$

Proof. The statement (i) directly follows from the definitions of $\mathrm{pr}_n g$ and the norm in H_J.

The statement (ii) is the well-known Sobolev embedding theorem.

Using integration-by-parts, we next estimate

$$\|g\|_{H_J}^2 = \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^J |\langle g, \varsigma_k \rangle|^2 \le \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^m |\langle g, \varsigma_k \rangle|^2$$

$$
\begin{aligned}
&\leq C_m \sum_{k\in\mathbb{Z}^d} \left(1+|k|^{2m}\right) |\langle g, \varsigma_k\rangle|^2 \\
&\leq C_m \sum_{k\in\mathbb{Z}^d} \left(1+\sum_{j=1}^d |k_j|^{2m}\right) |\langle g, \varsigma_k\rangle|^2 \\
&= C_m \sum_{k\in\mathbb{Z}^d} |\langle g, \varsigma_k\rangle|^2 + C_m \sum_{j=1}^d \sum_{k\in\mathbb{Z}^d} \left|\langle \partial_j^m g, \varsigma_k\rangle\right|^2 \\
&= C_m \|g\|^2 + C_m \sum_{j=1}^d \|\partial_j^m g\|^2 \leq C_m \|g\|_{\mathrm{C}^m}.
\end{aligned}
$$

This implies (iii).

According to the definition of $\mathrm{pr}_n g$, we have

$$
\begin{aligned}
\|g-\mathrm{pr}_n g\|_{H_J}^2 &= \sum_{k\notin\mathbb{Z}_n^d} \left(1+|k|^2\right)^J |\langle g, \varsigma_k\rangle|^2 \\
&= \sum_{k\notin\mathbb{Z}_n^d} \frac{\left(1+|k|^2\right)^I}{(1+|k|^2)^{I-J}} |\langle g, \varsigma_k\rangle|^2 \leq \frac{1}{n^{2(I-J)}} \|g-\mathrm{pr}_n g\|_{H_I}^2.
\end{aligned}
$$

The second part of (iv) directly follows from the first one and (ii). The proof of the lemma is complete. □

Lemma B.3. *The linear maps* $\mathrm{pr}_n : H_J \to L_2(\mathbb{T}_n^d)$ *and* $\mathrm{ex}_n : L_2(\mathbb{T}_n^d) \to H_J$ *are continuous for each* $J \in \mathbb{R}$*. Moreover,* $\mathrm{pr}_n \mathrm{ex}_n = \mathrm{id}$ *and* $\mathrm{ex}_n \mathrm{pr}_n = \mathrm{pr}_n$*, where* id *denotes the identity operator and* pr_n *in the right hand side of the second equality is considered as a map from* H_J *to* H_J*.*

Proof. We first show the continuity of ex_n, that will follow from its boundedness. Take $f \in L_2(\mathbb{T}_n^d)$ and estimate

$$
\begin{aligned}
\|\mathrm{ex}_n f\|_{H_J}^2 &= \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^J \left|\langle \mathrm{ex}_n f, \varsigma_k\rangle\right|^2 \\
&= \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^J \left|\langle f, \mathrm{pr}_n \varsigma_k\rangle_n\right|^2 \\
&= \sum_{k\in\mathbb{Z}_n^d} (1+|k|^2)^J \left|\langle f, \varsigma_k\rangle_n\right|^2 \\
&\leq \left[(1+|n|^2)^J \vee 1\right] \sum_{k\in\mathbb{Z}_n^d} \left|\langle f, \varsigma_k\rangle_n\right|^2 \\
&= \left[(1+|n|^2)^J \vee 1\right] \|f\|_n^2.
\end{aligned}
$$

Thus, ex_n is a bounded linear operator.

The boundedness of pr_n follows from the estimate

$$
\begin{aligned}
\|\mathrm{pr}_n g\|_n &= \sup_{f\in L_2(\mathbb{T}_n^d)} \frac{\langle \mathrm{pr}_n g, f\rangle_n}{\|f\|_n} = \sup_{f\in L_2(\mathbb{T}_n^d)} \frac{\langle g, \mathrm{ex}_n f\rangle}{\|f\|_n} \\
&\leq \sup_{f\in L_2(\mathbb{T}_n^d)} \|g\|_{H_J} \frac{\|\mathrm{ex}_n f\|_{H_{-J}}}{\|f\|_n} \leq \|g\|_{H_J} \left[(1+|n|^2)^{-J} \vee 1\right].
\end{aligned}
$$

for each $g \in H_J$, where we used the boundedness of ex_n in H_{-J}.

Now for $f \in L_2(\mathbb{T}_n^d)$ and $g \in H_J$ we get

$$\mathrm{pr}_n \mathrm{ex}_n f = \mathrm{pr}_n \sum_{k \in \mathbb{Z}_n^d} \langle f, \varsigma_k \rangle_n \varsigma_k = f$$

and

$$\begin{aligned} \mathrm{ex}_n \mathrm{pr}_n g = \mathrm{ex}_n \sum_{k \in \mathbb{Z}_n^d} \langle g, \varsigma_k \rangle \varsigma_k &= \sum_{k \in \mathbb{Z}_n^d} \langle g, \varsigma_k \rangle \mathrm{ex}_n \varsigma_k \\ &= \sum_{k,l \in \mathbb{Z}_n^d} \langle g, \varsigma_k \rangle \langle \varsigma_k, \varsigma_l \rangle_n \varsigma_l = \mathrm{pr}_n g. \end{aligned}$$

This completes the proof of the lemma. □

Corollary B.4. *Let $f_1, f_2 \in L_2(\mathbb{T}_n^d)$ and $g_1, g_2 \in H_J$. Then $\langle f_1, f_2 \rangle_n = \langle \mathrm{ex}_n f_1, \mathrm{ex}_n f_2 \rangle$ and $\langle \mathrm{pr}_n g_1, \mathrm{pr}_n g_2 \rangle_n = \langle \mathrm{pr}_n g_1, \mathrm{pr}_n g_2 \rangle$ for each $n \geq 1$.*

Proof. By Lemma B.3, $f_1 = \mathrm{pr}_n \mathrm{ex}_n f_1$. Thus, $\langle f_1, f_2 \rangle_n = \langle \mathrm{pr}_n \mathrm{ex}_n f_1, f_2 \rangle_n = \langle \mathrm{ex}_n f_1, \mathrm{ex}_n f_2 \rangle$ due to (B.3). The second equality follows from the first one by taking $f_i = \mathrm{pr}_n g_i$ and using the fact that $\mathrm{ex}_n \mathrm{pr}_n = \mathrm{pr}_n$. □

Remark B.5. The last two equalities in the proof of Lemma B.3 implies that for each $f \in L_2(\mathbb{T}_n^d)$ and $g \in H_J$

$$\mathrm{pr}_n \mathrm{ex}_n f(x) = \mathrm{ex}_n f(x) \quad \text{and} \quad \mathrm{ex}_n \mathrm{pr}_n g(x) = \mathrm{pr}_n g(x)$$

for all $x \in \mathbb{T}^d$.

We will next focus on the approximating properties of the operator ex_n. Recall that considering a function $f : \mathbb{T}^d \to \mathbb{R}$, we write $\mathrm{ex}_n f$ for the operator ex_n applied to the restriction of f to the set $\mathbb{T}_n^d$.

Lemma B.6. *Let $J \geq 0$ and $m \in \mathbb{N}$ such that $2m > J + 1 + \frac{d}{2}$. Then for each $f \in \mathrm{C}^{2m+1}(\mathbb{T}^d)$ one has*

$$\|\mathrm{ex}_n f - f\|_{H_J} \leq \frac{C_{J,m}}{n} \|f\|_{\mathrm{C}^{2m+1}}$$

for all $n \geq 1$.

Proof. Using the triangle inequality, we first get

$$\|\mathrm{ex}_n f - f\|_{H_J} \leq \|\mathrm{ex}_n f - \mathrm{pr}_n f\|_{H_J} + \|\mathrm{pr}_n f - f\|_{H_J},$$

where the second term in the right hand side of the estimate above can be bounded by

$$\|\mathrm{pr}_n f - f\|_{H_J} \leq \frac{1}{n} \|\mathrm{pr}_n f - f\|_{H_{J+1}} \leq \frac{2}{n} \|f\|_{H_{J+1}} \leq \frac{C_m}{n} \|f\|_{\mathrm{C}^{2m+1}},$$

according to Lemma B.2 and the fact that $\lceil J \rceil + 1 \leq 2m + 1$. The square of the first term can be rewritten as

$$\|\mathrm{ex}_n f - \mathrm{pr}_n f\|_{H_J}^2 = \sum_{k \in \mathbb{Z}_n^d} \left(1 + |k|^2\right)^J |\langle f, \varsigma_k \rangle_n - \langle f, \varsigma_k \rangle|^2 .$$

Thus, we will need to estimate the difference of discrete and continuous Fourier coefficients. Using the integration-by-parts formula and Lemma B.1, we get

$$\begin{aligned}
|\langle f, \varsigma_k\rangle_n - \langle f, \varsigma_k\rangle| &= \left| \frac{1}{(\lambda_k^n)^m} \langle f, \Delta_n^m \varsigma_k\rangle_n - \frac{1}{|k|^{2m}} \langle f, \Delta^m \varsigma_k\rangle \right| \\
&= \left| \frac{1}{(\lambda_k^n)^m} \langle \Delta_n^m f, \varsigma_k\rangle_n - \frac{1}{|k|^{2m}} \langle \Delta^m f, \varsigma_k\rangle \right| \\
&\le \left| \frac{1}{(\lambda_k^n)^m} - \frac{1}{|k|^{2m}} \right| |\langle \Delta_n^m f, \varsigma_k\rangle| \\
&\quad + \frac{1}{|k|^{2m}} |\langle \Delta_n^m f, \varsigma_k\rangle_n - \langle \Delta^m f, \varsigma_k\rangle|
\end{aligned}$$

for $k \in \mathbb{Z}_n^d \setminus \{0\}$.
Note that

$$\left| \frac{1}{(\lambda_k^n)^m} - \frac{1}{|k|^{2m}} \right| = \frac{1}{(\lambda_k^n)^m} \left| 1 - \left(\frac{(2n+1)^2}{2\pi^2|k|^2} \sum_{j=1}^{d} \left[1 - \cos\frac{2\pi k_j}{2n+1} \right] \right)^m \right|.$$

By Taylor's formula

$$\cos x = 1 - \frac{x^2}{2} + \frac{\cos\theta(x)}{4!} x^4,$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a function, we get for each $j \in [d]$

$$\begin{aligned}
\frac{(2n+1)^2}{2\pi^2|k|^2} \sum_{j=1}^{d} \left[1 - \cos\frac{2\pi k_j}{2n+1} \right] &= \frac{(2n+1)^2}{2\pi^2|k|^2} \sum_{j=1}^{d} \left[\frac{2\pi^2 k_j^2}{(2n+1)^2} + \frac{\cos\theta_j(n)}{4!} \frac{16\pi^4 k_j^4}{(2n+1)^4} \right] \\
&= 1 + \frac{\pi^2}{3|k|^2(2n+1)^2} \sum_{j=1}^{d} \cos\theta_j(n) k_j^4 =: 1 + z_k^n,
\end{aligned}$$

where $\theta_j(n) := \theta(2\pi k_j/(2n+1))$ and $|z_k^n| \le \frac{C|k|^2}{n^2}$ for all $k \in \mathbb{Z}_n^d \setminus \{0\}$, $n \ge 1$ and a constant $C > 0$ is independent of n and k. Consequently, using Taylor's formula again for the function $x \mapsto (1+x)^m$, we obtain

$$\left| \frac{1}{(\lambda_k^n)^m} - \frac{1}{|k|^{2m}} \right| = \frac{1}{(\lambda_k^n)^m} |1 - (1 + z_k^n)^m| \le \frac{C_m |k|^2}{(\lambda_k^n)^m n^2} \le \frac{C_m}{|k|^{2m-2} n^2} \tag{B.4}$$

for each $k \in \mathbb{Z}_n^d \setminus \{0\}$ and $n \in \mathbb{N}$, where we used Lemma B.1 in the last step.
By Taylor's formula, there exists a constant $C > 0$ such that

$$|\langle \Delta_n^m f, \varsigma_k\rangle_n - \langle \Delta^m f, \varsigma_k\rangle| \le \frac{C}{n} \|f\|_{\mathrm{C}^{2m+1}}.$$

Combining the obtained estimates together, we conclude

$$\begin{aligned}
|\langle f, \varsigma_k\rangle_n - \langle f, \varsigma_k\rangle| &\le \frac{C_m}{n^2|k|^{2m-2}} |\langle \Delta_n^m f, \varsigma_k\rangle| + \frac{C}{n|k|^{2m}} \|f\|_{\mathrm{C}^{2m+1}} \\
&\le \frac{C_m}{n|k|^{2m-1}} \|f\|_{\mathrm{C}^{2m+1}}
\end{aligned}$$

for all $k \in \mathbb{Z}_n^d \setminus \{0\}$ and $n \in \mathbb{N}$. Similarly, we can estimate

$$|\langle f, \varsigma_0\rangle_n - \langle f, \varsigma_0\rangle| \le \frac{C}{n} \|f\|_{\mathrm{C}^1}.$$

Consequently,

$$\begin{aligned}\|\mathrm{ex}_n f-\mathrm{pr}_n f\|_{H_J}^2 &\le \frac{C^2}{n^2}\|f\|_{\mathrm{C}^1}^2+\frac{C_m^2}{n^2}\|f\|_{\mathrm{C}^{2m+1}}^2\sum_{k\in\mathbb{Z}_n^d\setminus\{0\}}\frac{\left(1+|k|^2\right)^J}{|k|^{4m-2}}\\ &\le \frac{C_{J,m}}{n^2}\|f\|_{\mathrm{C}^{2m+1}}^2,\end{aligned}$$

since $2m-1-J>\frac{d}{2}$. This completes the proof of the statement. □

Lemma B.7. *For all $f\in L_2(\mathbb{T}_n^d)$ and $n\ge 1$ one has*

$$\left\|\mathrm{ex}_n f^2-(\mathrm{ex}_n f)^2\right\|\le \frac{C_m}{n}\,\|\mathrm{ex}_n f\|_{\mathrm{C}^{\lceil d/2\rceil+4}}^2.$$

Proof. We first note that $f(x)=\mathrm{ex}_n f(x)$ for all $x\in\mathbb{T}_n^d$, according to (A.8). Since $\mathrm{ex}_n f^2$ is only defined by values of f^2 on $\mathbb{T}_n^d$, $\mathrm{ex}_n f^2=\mathrm{ex}_n\left(\mathrm{ex}_n f\right)^2$. Hence, we can estimate

$$\begin{aligned}\left\|\mathrm{ex}_n f^2-(\mathrm{ex}_n f)^2\right\| &= \left\|\mathrm{ex}_n\left(\mathrm{ex}_n f\right)^2-(\mathrm{ex}_n f)^2\right\|\\ &\le \frac{C_m}{n}\left\|(\mathrm{ex}_n f)^2\right\|_{\mathrm{C}^{2m+1}}\le \frac{C_m}{n}\,\|\mathrm{ex}_n f\|_{\mathrm{C}^{2m+1}}^2\end{aligned}$$

due to Lemma B.6 with $J=0$ and $m\in\mathbb{N}$ satisfying $\frac{d}{2}+2<2m+1\le\lceil d/2\rceil+4$. This completes the proof of the statement. □

Lemma B.8. *For each $J\in\mathbb{R}$, $n\in\mathbb{N}$, $j\in[d]$ and $f\in L_2(\mathbb{T}_n^d)$, one has*

$$\|\mathrm{ex}_n\partial_{n,j}f\|_{H_J}\le\|\mathrm{ex}_n f\|_{H_{J+1}}.$$

Proof. Using Lemma B.1, we estimate

$$\begin{aligned}\|\mathrm{ex}_n\partial_{n,j}f\|_{H_J}^2 &= \sum_{k\in\mathbb{Z}^d}\left(1+|k|^2\right)^J|\langle\mathrm{ex}_n\partial_{n,j}f,\varsigma_k\rangle|^2=\sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^J|\langle\partial_{n,j}f,\varsigma_k\rangle_n|^2\\ &= \sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^J|\langle f,\partial_{n,j}\varsigma_k\rangle_n|^2=\sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^J\left|\mu_{k,j}^n\right|^2|\langle f,\varsigma_k\rangle_n|^2\\ &\le \sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^{J+1}|\langle f,\varsigma_k\rangle_n|^2=\|\mathrm{ex}_n f\|_{H_{J+1}}^2.\end{aligned}$$

□

Lemma B.9. *Let $J\in\mathbb{N}_0$. Then for each $f\in\mathrm{C}^J(\mathbb{T}^d)$ and $g\in H_J$ one has*

$$\|\mathrm{ex}_n(fg)\|_{H_J}\le C_J\|\mathrm{ex}_n f\|_{\mathrm{C}^J}\|\mathrm{ex}_n g\|_{H_J}.$$

Proof. Using Lemma B.1 and the integration-by-parts formula, we estimate

$$\begin{aligned}\|\mathrm{ex}_n(fg)\|_{H_J}^2 &\le \sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^J|\langle fg,\varsigma_k\rangle_n|^2\\ &\le 3^J\sum_{k\in\mathbb{Z}_n^d}\left(1+\sum_{j=1}^d|\mu_{k,j}^n|^2\right)^J|\langle fg,\varsigma_k\rangle_n|^2\\ &\le C_J\sum_{k\in\mathbb{Z}_n^d}\left(1+\sum_{j=1}^d|\mu_{k,j}^n|^{2J}\right)|\langle fg,\varsigma_k\rangle_n|^2\end{aligned}$$

$$= C_J \left[\|fg\|_n^2 + \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}_n^d} \left|\langle \partial_{n,j}^J (fg), \varsigma_k\rangle_n\right|^2 \right]$$

$$= C_J \left[\|fg\|_n^2 + \sum_{j=1}^{d} \left\|\partial_{n,j}^J (fg)\right\|_n^2 \right].$$

Iterating the equality $\partial_{n,j}(fg) = \partial_{n,j} f \tau_j^n g + f\partial_{n,j} g$, where $\tau_j^n f(x) = f(x+e_j^n)$, we get

$$\partial_{n,j}^J (fg) = \sum_{l=0}^{J} \binom{J}{l} \left[\partial_{n,j}^l f\right] \left[(\tau_j^n)^l \partial_{n,j}^{J-l} g\right].$$

Thus, using the fact that $\|\tau_{n,j} g\|_n = \|g\|_n$, we obtain

$$\|\mathrm{ex}_n (fg)\|_{H_J}^2 \le C_J \left[\|f\|_{n,\mathrm{C}}^2 \|g\|_n^2 + \sum_{j=1}^{d} \sum_{l=0}^{J} \left\|\partial_{n,j}^l f\right\|_{n,\mathrm{C}}^2 \left\|\partial_{n,j}^{J-l} g\right\|_n^2 \right].$$

We next note that $\mathrm{ex}_n f(x) = f(x)$ for all $x \in \mathbb{T}_n^d$. Thus, $\|f\|_{n,\mathrm{C}} \le \|\mathrm{ex}_n f\|_{\mathrm{C}}$. Moreover, applying Taylor's formula to $\mathrm{ex}_n f$, we get

$$\|\partial_{n,j}^l f\|_{\mathrm{C}} = \|\partial_{n,j}^l \mathrm{ex}_n f\|_{\mathrm{C}} \le C_l \|\partial_j^l \mathrm{ex}_n f\|_{\mathrm{C}}.$$

Consequently, we can continue the estimate as follows

$$\|\mathrm{ex}_n (fg)\|_{H_J}^2 \le C_J \left[\|\mathrm{ex}_n f\|_{\mathrm{C}}^2 \|\mathrm{ex}_n g\|^2 + \sum_{j=1}^{d} \sum_{l=0}^{J} \left\|\partial_j^l \mathrm{ex}_n f\right\|_{\mathrm{C}}^2 \left\|\mathrm{ex}_n \partial_{n,j}^{J-l} g\right\|^2 \right].$$

The statement now follows from Lemma B.8. □

Lemma B.10. *Let $J \in \mathbb{R}$, $n \in \mathbb{N}$, $j \in [d]$ and $f \in L_2(\mathbb{T}_n^d)$. Then*

$$\|\mathrm{ex}_n \tau_j^n f\|_{H_J} = \|\mathrm{ex}_n f\|_{H_J}.$$

Proof. The statement directly follows from the following computation

$$\begin{aligned} \|\mathrm{ex}_n \tau_j^n f\|_{H_J}^2 &= \sum_{k\in\mathbb{Z}_n^d} \left(1+|k|^2\right)^J \left|\langle \tau_j^n f, \varsigma_k\rangle_n\right|^2 \\ &= \sum_{k\in\mathbb{Z}_n^d} \left(1+|k|^2\right)^J \left|\left\langle f, (\tau_k^n)^{-1} \varsigma_k \right\rangle_n\right|^2 \\ &= \sum_{k\in\mathbb{Z}_n^d} \left(1+|k|^2\right)^J \left|\langle f, \varsigma_k\rangle_n\right|^2 = \|\mathrm{ex}_n f\|_{H_J}. \end{aligned}$$

□

Lemma B.11. *There exists a constant $C > 0$ such that for each $J \in \mathbb{R}$ and $g \in H_{J+2}$ the inequality*

$$\|\mathrm{ex}_n \Delta_n \mathrm{pr}_n g - \mathrm{pr}_n \Delta g\|_{H_J} \le \frac{C}{n} \|g\|_{H_{J+2}}$$

holds.

Proof. Using integration-by-parts formula Lemma B.1 and (A.9), we compute

$$\begin{aligned}\|\mathrm{ex}_n\Delta_n\mathrm{pr}_n g-\mathrm{pr}_n\Delta g\|_{H_J}^2&=\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^J\,|\langle \mathrm{ex}_n\Delta_n\mathrm{pr}_n g,\varsigma_k\rangle-\langle \mathrm{pr}_n\Delta g,\varsigma_k\rangle|^2\\&=\sum_{k\in\mathbb{Z}_n^d}(1+|k|^2)^J\,|\langle \Delta_n\mathrm{pr}_n g,\varsigma_k\rangle_n-\langle \Delta g,\varsigma_k\rangle|^2\\&=\sum_{k\in\mathbb{Z}_n^d}(1+|k|^2)^J\,\left|\lambda_k^n\langle \mathrm{pr}_n g,\varsigma_k\rangle_n-|k|^2\langle g,\varsigma_k\rangle\right|^2\\&=\sum_{k\in\mathbb{Z}_n^d}(1+|k|^2)^J\,\left|\lambda_k^n-|k|^2\right|^2|\langle g,\varsigma_k\rangle|^2,\end{aligned}$$

where we used the equality $\langle \mathrm{pr}_n g,\varsigma_k\rangle_n=\langle g,\mathrm{ex}_n\varsigma_k\rangle=\langle g,\varsigma_k\rangle$ in the last step. Using Taylor's expansion, we get

$$\lambda_k^n=\frac{(2n+1)^2}{2\pi^2}\sum_{j=1}^d\left[1-\cos\frac{2\pi k_j}{2n+1}\right]=|k|^2+\frac{1}{(2n+1)^2}\sum_{j=1}^d k_j^4a_k^n,$$

where the family $|a_k^n|$, $k\in\mathbb{Z}_n^d$, $n\geq 1$, is bounded by an universal constant. Thus,

$$\begin{aligned}\|\mathrm{ex}_n\Delta_n\mathrm{pr}_n g-\mathrm{pr}_n\Delta g\|_{H_J}^2&\leq\frac{C}{(2n+1)^2}\sum_{k\in\mathbb{Z}_n^d}(1+|k|^2)^J\,|k|^4\,|\langle g,\varsigma_k\rangle|^2\\&\leq\frac{C}{n^2}\|g\|_{H_{J+2}}^2.\end{aligned}$$

This completes the proof of the lemma. □

Corollary B.12. *Let $(\rho_t^\infty)_{t\geq0}$ and $(\rho_t^n)_{t\geq0}$, $n\geq1$, be solutions to (3.1) and (2.2), respectively. Let also $\rho_0^\infty\in H_2$. Then for each $T>0$ there exists a constant $C_T>0$ such that*

$$\|\mathrm{ex}_n\rho_t^n-\mathrm{pr}_n\rho_t^\infty\|\leq C_T\|\mathrm{ex}_n\rho_0^n-\rho_0^\infty\|+\frac{C_T}{n}\|\rho_0^\infty\|_{H_2}$$

for all $t\in[0,T]$ and $n\geq1$.

Proof. Using Corollary B.4, we easily get

$$\begin{aligned}\|\mathrm{ex}_n\rho_t^n-\mathrm{pr}_n\rho_t^\infty\|^2&=\|\rho_t^n-\mathrm{pr}_n\rho_t^\infty\|_n^2=\\&=\|\rho_0^n-\mathrm{pr}_n\rho_0^\infty\|_n^2+4\pi^2\int_0^t\langle\rho_s^n-\mathrm{pr}_n\rho_s^\infty,\Delta_n\rho_s^n-\mathrm{pr}_n\Delta\rho_s^\infty\rangle_n ds.\end{aligned}$$

Integrating-by-parts and using the Cauchy-Schwarz inequality, we can estimate

$$\begin{aligned}\langle\rho_s^n-\mathrm{pr}_n\rho_s^\infty,\Delta_n\rho_s^n-\mathrm{pr}_n\Delta\rho_s^\infty\rangle_n&=\langle\rho_s^n-\mathrm{pr}_n\rho_s^\infty,\Delta_n\rho_s^n-\Delta_n\mathrm{pr}_n\rho_s^\infty\rangle_n\\&\quad+\langle\rho_s^n-\mathrm{pr}_n\rho_s^\infty,\Delta_n\mathrm{pr}_n\rho_s^\infty-\mathrm{pr}_n\Delta\rho_s^\infty\rangle_n\\&\leq-\sum_{j=1}^d\|\partial_{n,j}\left(\rho_s^n-\mathrm{pr}_n\rho_s^\infty\right)\|_n^2\\&\quad+\|\rho_s^n-\mathrm{pr}_n\rho_s^\infty\|_n\,\|\Delta_n\mathrm{pr}_n\rho_s^\infty-\mathrm{pr}_n\Delta\rho_s^\infty\|_n\\&\leq\frac12\|\rho_s^n-\mathrm{pr}_n\rho_s^\infty\|_n^2+\frac12\|\Delta_n\mathrm{pr}_n\rho_s^\infty-\mathrm{pr}_n\Delta\rho_s^\infty\|_n^2.\end{aligned}$$

According to Corollary B.4 and Lemma B.11, the bounds

$$\begin{aligned}\|\Delta_n \mathrm{pr}_n \rho_s^\infty - \mathrm{pr}_n \Delta \rho_s^\infty\|_n^2 &= \|\mathrm{ex}_n \Delta_n \mathrm{pr}_n \rho_s^\infty - \mathrm{pr}_n \Delta \rho_s^\infty\|^2 \\ &\leq \frac{C}{n^2}\|\rho_s^\infty\|_{H_2}^2 \leq \frac{C}{n^2}\|\rho_0^\infty\|_{H_2}^2\end{aligned}$$

hold. Consequently, we obtain

$$\|\mathrm{ex}_n \rho_t^n - \mathrm{pr}_n \rho_t^\infty\|^2 \leq \|\rho_0^n - \mathrm{pr}_n \rho_0^\infty\|_n^2 + 2\pi^2 \int_0^t \|\rho_s^n - \mathrm{pr}_n \rho_s^\infty\|_n^2 \, ds + \frac{Ct}{n^2}\|\rho_0^\infty\|_{H_2}$$

for all $t \geq 0$. Using Grönwall's inequality, we conclude

$$\|\mathrm{ex}_n \rho_t^n - \mathrm{pr}_n \rho_t^\infty\|^2 \leq C_T \|\mathrm{ex}_n \rho_0^n - \mathrm{pr}_n \rho_0^\infty\|^2 + \frac{C_T}{n^2}\|\rho_0^\infty\|_{H_2}^2$$

that completes the proof of the corollary. □

B.2 Multilinear operators on Sobolev spaces

Recall that $\mathcal{L}_m(H_J)$ denotes the space of all continuous multilinear operators from $(H_J)^m$ to $\mathbb{R}$ equipped with the norm

$$\|A\|_{\mathcal{L}_m} = \sup_{\|f_j\|_{H_J} \leq 1} |A[f_1, \ldots, f_m]|,$$

and the subset of $\mathcal{L}_m(H_J)$ consisting of multilinear operators with finite Hilbert-Schmidt norm (A.1) is denoted by $\mathcal{L}_m^{HS}(H_J)$.

Since for each $J < I$ one has $H_I \subset H_J$ and $\|\cdot\|_{H_J} \leq \|\cdot\|_{H_I}$, the space $\mathcal{L}_m(H_J)$ is continuously embedded into $\mathcal{L}_m(H_I)$ and $\|\cdot\|_{\mathcal{L}_m(H_I)} \leq \|\cdot\|_{\mathcal{L}_m(H_J)}$. We next show the continuous embedding of $\mathcal{L}_m^{HS}(H_J)$ into $\mathcal{L}_m(H_I)$.

Lemma B.13. *For each $I, J \in \mathbb{R}$ with $I > J + \frac{d}{2}$ one has*

$$\|A\|_{\mathcal{L}_m^{HS}(H_I)} \leq C_{I-J,m}\|A\|_{\mathcal{L}_m(H_J)}$$

for all $A \in \mathcal{L}_m^{HS}(H_I)$. In particular, the space $\mathcal{L}_m^{HS}(H_I)$ is continuously embedded into $\mathcal{L}_m(H_J)$.

Proof. The statement follows from the straightforward estimate

$$\begin{aligned}\|A\|_{\mathcal{L}_m^{HS}(H_I)}^2 &= \sum_{k_1,\ldots,k_m \in \mathbb{Z}^d} \prod_{j=1}^m \left(1+|k_j|^2\right)^{-I} |A[\tilde{\varsigma}_{k_1}, \ldots, \tilde{\varsigma}_{k_m}]|^2 \\ &\leq \|A\|_{\mathcal{L}_m(H_J)} \sum_{k_1,\ldots,k_m \in \mathbb{Z}^d} \prod_{j=1}^m \left(1+|k_j|^2\right)^{-I} \|\tilde{\varsigma}_j\|_{H_J}^2 \\ &\leq \|A\|_{\mathcal{L}_m(H_J)} \sum_{k_1,\ldots,k_m \in \mathbb{Z}^d} \prod_{j=1}^m \left(1+|k_j|^2\right)^{-I+J}.\end{aligned}$$

This completes the proof of the lemma. □

We will further focus on the case $m = 2$. Take $a \in \mathrm{C}^{J,J}(\mathbb{T}^d, \mathbb{T}^d)$ for some even $J \in \mathbb{N}_0$ and define the multilinear operator K_a with kernel a by

$$
\begin{aligned}
K_a(f,g) := \langle f \otimes g, a \rangle &= \sum_{k,l \in \mathbb{Z}^d} \langle \varsigma_{(k,l)}, a \rangle \langle f, \varsigma_k \rangle \langle g, \varsigma_l \rangle \\
&= \sum_{k,l \in \mathbb{Z}^d} \langle a, \varsigma_{(-k,-l)} \rangle \langle f, \varsigma_k \rangle \langle g, \varsigma_l \rangle, \quad f, g \in \mathrm{C}(\mathbb{T}^d),
\end{aligned}
$$

where $\varsigma_{(k,l)}(x,y) = \varsigma_k \otimes \varsigma_l(x,y) = \varsigma_k(x)\varsigma_l(y)$, $x, y \in \mathbb{T}^d$. Then the operator K_a can be uniquely extended to a multilinear operator on H_{-J}, denoted also by K_a. Moreover, it is a Hilbert-Schmidt operator satisfying

$$
\|K_a\|_{\mathcal{L}_2^{HS}(H_{-J})} \leq C \|a\|_{\mathrm{C}^{J,J}}. \tag{B.5}
$$

Indeed, this directly follows from the following computation

$$
\begin{aligned}
\|K_a\|^2_{\mathcal{L}_2^{HS}} &= \sum_{k,l \in \mathbb{Z}^d} (1 + |k|^2)^J (1 + |l|^2)^J \left| \langle a, \varsigma_{(k,l)} \rangle \right|^2 \\
&= \sum_{k,l \in \mathbb{Z}^d} \left| \left\langle a, (1+\Delta)^{J/2} \otimes (1+\Delta)^{J/2} \varsigma_k \otimes \varsigma_l \right\rangle \right|^2 \\
&= \sum_{k,l \in \mathbb{Z}^d} \left| \left\langle (1+\Delta)^{J/2} \otimes (1+\Delta)^{J/2} a, \varsigma_k \otimes \varsigma_l \right\rangle \right|^2 \\
&= \left\| (1+\Delta)^{J/2} \otimes (1+\Delta)^{J/2} a \right\|^2 \leq C \|a\|^2_{\mathrm{C}^{J,J}},
\end{aligned}
$$

where we have used the integration-by-parts passing from the second to the third line.
Since we usually work with the Fourier basis $\{\varsigma_k, k \in \mathbb{Z}^d\}$ instead of $\{\tilde{\varsigma}_k, k \in \mathbb{Z}^d\}$, we will extend $A \in \mathcal{L}_m(H_J)$ linearly with respect to each component to the set of complex valued square integrable function, following e.g. the definition of the kernel operator K_a. In this case, a simple computation shows that

$$
\mathrm{pr}_n^{\otimes m} A := \sum_{k \in (\mathbb{Z}_n^d)^m} A[\varsigma_{-(\times k)}] \varsigma_k = \sum_{k \in (\mathbb{Z}_n^d)^m} A[\tilde{\varsigma}_{\times k}] \tilde{\varsigma}_k
$$

for all $n \in \mathbb{N}$, where $\varsigma_{\times k} = (\varsigma_{k_j})_{j \in [m]}$, $\tilde{\varsigma}_{\times k} = \left(\tilde{\varsigma}_{k_j}\right)_{j \in [m]}$, $\varsigma_k(x) = \prod_{j=1}^m \varsigma_{k_j}(x_j)$ and $\tilde{\varsigma}_k(x) = \prod_{j=1}^m \tilde{\varsigma}_{k_j}(x_j)$ for $k = (k_j)_{j \in [m]}$ and $x = (x_j)_{j \in [m]}$. Thus, for each $f = (f_j)_{j \in [m]} \in (H_J)^m$ we have

$$
\langle f^{\otimes}, \mathrm{pr}_n^{\otimes m} A \rangle = \sum_{k \in (\mathbb{Z}_n^d)^m} A[\varsigma_{\times k}] \langle f, \varsigma_k \rangle = A[\mathrm{pr}_n f] \to A[f], \quad n \to \infty, \tag{B.6}
$$

where $\mathrm{pr}_n f := (\mathrm{pr}_n f_j)_{j \in [m]}$ and $f^{\otimes}(x) = \prod_{j=1}^m f_j(x_j)$, $x = (x_j)_{j \in [m]}$, according to the continuity of A and the convergence $\mathrm{pr}_n f_j \to f_j$ in H_J. Thus, we can consider $\mathrm{pr}_n^{\otimes m}$ as an analog of the operator pr_n.
The following statement is an analog of Lemma B.2 (iv).

Lemma B.14. *For each $J, I \in \mathbb{R}$, $J < I$, a multi-linear operator $A \in \mathcal{L}_2^{HS}(H_J)$ and $n \geq 1$ the kernel operator $K_{\mathrm{pr}_n^{\otimes 2} A}$ belongs to $\mathcal{L}_2^{HS}(H_J)$ and*

$$
\left\| A - K_{\mathrm{pr}_n^{\otimes 2} A} \right\|^2_{\mathcal{L}_2^{HS}(H_I)} \leq \frac{1}{n^{I-J}} \left\| A - K_{\mathrm{pr}_n^{\otimes 2} A} \right\|^2_{\mathcal{L}_2^{HS}(H_J)}.
$$

Proof. The fact that $K_{\mathrm{pr}_n^{\otimes 2}A} \in \mathcal{L}_2^{HS}(H_J)$ follows from the definitions of the kernel operator and (B.5). We next estimate

$$\begin{aligned}\left\|A - K_{\mathrm{pr}_n^{\otimes 2}A}\right\|^2_{\mathcal{L}_2^{HS}(H_I)} &= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{-I}(1+|l|^2)^{-I}\left|A[\varsigma_k,\varsigma_l] - \langle \varsigma_{(k,l)}, \mathrm{pr}_n^{\otimes 2}A\rangle\right|^2 \\ &= \sum_{(k,l)\notin(\mathbb{Z}_n^d)^2} (1+|k|^2)^{-I}(1+|l|^2)^{-I}|A[\varsigma_k,\varsigma_l]|^2 \\ &\leq \frac{1}{n^{2(I-J)}} \sum_{(k,l)\notin(\mathbb{Z}_n^d)^2} (1+|k|^2)^{-J}(1+|l|^2)^{-J}|A[\varsigma_k,\varsigma_l]|^2 \\ &= \frac{1}{n^{2(I-J)}}\left\|A - K_{\mathrm{pr}_n^{\otimes 2}A}\right\|^2_{\mathcal{L}_2^{HS}(H_J)}.\end{aligned}$$

This completes the proof of the lemma. □

We next define a bounded linear operator $\mathrm{Tr} : \mathcal{L}_2^{HS}(H_{-J}) \to H_I$ for some J and I such that $\mathrm{Tr}K_a = a(x,x)$ for a kernel a. Note that the δ_x-function belongs to H_{-J} for $J > \frac{d}{2}$ due to the inequality

$$\begin{aligned}\|\delta_x\|^2_{H_{-J}} &= \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-J}\left|\langle\delta_x,\varsigma_k\rangle\right|^2 = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-J}|\varsigma_k(x)|^2 \\ &= \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{-J} < \infty.\end{aligned}$$

Lemma B.15. *Let $J > \frac{d}{2}$ and $I < J - \frac{d}{2}$. Then for each $A \in \mathcal{L}_2^{HS}(H_{-J})$ the function* $\mathrm{Tr}A$*, defined by*

$$\mathrm{Tr}A(x) := A[\delta_x, \delta_x]$$

is continuous, belongs to H_I and

$$\mathrm{Tr}A(x) = \sum_{l\in\mathbb{Z}^d}\left(\sum_{k\in\mathbb{Z}^d} A[\varsigma_{k-l},\varsigma_{-k}]\right)\varsigma_l(x) \tag{B.7}$$

for all $x \in \mathbb{T}^d$. Moreover, $\mathrm{Tr} : \mathcal{L}_2^{HS}(H_{-J}) \to H_I$ *is a bounded linear operator satisfying* $\mathrm{Tr}K_a(x) = a(x,x)$*, $x \in \mathbb{T}^d$, for each $a \in \mathrm{C}^{m,m}(\mathbb{T}^d,\mathbb{T}^d)$, where $m \geq J$ is an even number.*

Proof. The continuity of $\mathrm{Tr}A$ as a map from $\mathbb{T}^d$ to $\mathbb{R}$ follows from the continuity of $A : (H_{-J})^2 \to \mathbb{R}$ and $\delta_\cdot : \mathbb{T}^d \to H_{-J}$. By the definition of Tr and (B.6), we have

$$\begin{aligned}\mathrm{Tr}A(x) &= \lim_{n\to\infty}\langle\delta_x\otimes\delta_x, \mathrm{pr}_n^{\otimes 2}A\rangle = \sum_{k,l\in\mathbb{Z}^d} A[\varsigma_k,\varsigma_l]\,\langle\delta_x,\varsigma_k\rangle\,\langle\delta_x,\varsigma_l\rangle \\ &= \sum_{k,l\in\mathbb{Z}^d} A[\varsigma_k,\varsigma_l]\varsigma_{-k}(x)\varsigma_{-l}(x).\end{aligned}$$

The series above converges absolutely because

$$\begin{aligned}\sum_{k,l\in\mathbb{Z}^d}|A(\varsigma_k,\varsigma_l)\varsigma_{-k}(x)\varsigma_{-l}(x)| = \sum_{k,l\in\mathbb{Z}^d}&\frac{1}{(1+|k|^2)^{J/2}(1+|l|^2)^{J/2}} \\ &\cdot(1+|k|^2)^{J/2}(1+|l|^2)^{J/2}\,|A[\varsigma_k,\varsigma_l]|\end{aligned}$$

$$\leq \left(\sum_{k,l\in\mathbb{Z}^d} \frac{1}{(1+|k|^2)^J(1+|l|^2)^J} \right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^J(1+|l|^2)^J \, |A[\varsigma_k,\varsigma_l]|^2 \right)^{\frac{1}{2}}$$

$$\leq C_J \|A\|_{\mathcal{L}_2^{HS}(H_{-J})},$$

where we have used the Cauchy-Schwarz inequality in the second step. Thus, we may interchange the summands in the series, to get the expression (B.7).
We next show that $\mathrm{Tr}A \in H_I$ for $I < J - \frac{d}{2}$. Similarly to the estimate above, we conclude

$$\left| \sum_{k\in\mathbb{Z}^d} A[\varsigma_{k-l},\varsigma_{-k}] \right|^2 \leq \|A\|^2_{\mathcal{L}_2^{HS}} \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k-l|^2)^J(1+|k|^2)^J}$$

since $J > \frac{d}{2}$. Thus,

$$\begin{aligned}
\|\mathrm{Tr}A\|^2_{H_I} &= \sum_{l\in\mathbb{Z}^d} (1+|l|^2)^I \left| \sum_{k\in\mathbb{Z}^d} A[\varsigma_{k-l},\varsigma_{-k}] \right|^2 \\
&\leq \|A\|^2_{\mathcal{L}_2^{HS}} \sum_{l\in\mathbb{Z}^d} (1+|l|^2)^I \sum_{k\in\mathbb{Z}^d} \frac{1}{(1+|k-l|^2)^J(1+|k|^2)^J} \\
&\leq \|A\|^2_{\mathcal{L}_2^{HS}} \sum_{k,l\in\mathbb{Z}^d} \frac{(1+|l+k|^2)^I}{(1+|l|^2)^J(1+|k|^2)^J} < \infty
\end{aligned}$$

due to $J - I > \frac{d}{2}$.
We note that $K_a \in \mathcal{L}_2^{HS}(H_{-J})$, by (B.5), and trivially $\mathrm{Tr}K_a(x) = K_a[\delta_x,\delta_x] = a(x,x)$ for all $x \in \mathbb{T}^d$. This completes the proof of the lemma. □

Define the mixed derivative of a multilinear operator from $\mathcal{L}_2(H_J)$ by

$$\partial_j^{\otimes 2} A[f,g] = -A[\partial_j f, \partial_j g], \quad f,g \in H_{J+1}.$$

The following statement easily follows from the definition of ∂_j on H_J.

Lemma B.16. *For each $J \in \mathbb{R}$ and $A \in \mathcal{L}_2^{HS}(H_J)$ the multilinear operator $\partial_j^{\otimes 2} A$ is well-defined and belongs to $\mathcal{L}_2^{HS}(H_{J+1})$. Moreover,*

$$\left\|\partial_j^{\otimes 2} A\right\|_{\mathcal{L}_2^{HS}(H_{J+1})} \leq \|A\|_{\mathcal{L}_2^{HS}(H_J)}.$$

Remark B.17. (i) The statement of Lemma B.16 remains true, if we replace $\mathcal{L}_2^{HS}$ by $\mathcal{L}_2$ and $\|\cdot\|_{\mathcal{L}_2^{HS}}$ by $\|\cdot\|_{\mathcal{L}_2}$.
(ii) According integration-by-parts formula, we have the equality $\partial_j^{\otimes 2} K_a = K_{\partial_j^{\otimes 2} a}$ for each $j \in [d]$.

With some abuse of notation we will also set

$$\mathrm{Tr}f(x) = f(x,x), \quad x \in \mathbb{T}_n^d,$$

if $f \in (L_2(\mathbb{T}_n^d))^2$.

Proposition B.18. *Let $J-1-\frac{d}{2}>I\geq 0$ and $j\in[d]$. Then for every $A\in\mathcal{L}_2^{HS}(H_{-J})$ and $n\in\mathbb{N}$ the estimate*

$$\left\|\mathrm{ex}_n\left[\mathrm{Tr}\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A\right]\right\|_{H_I}\leq C_{J,I}\|A\|_{\mathcal{L}_2^{HS}(H_{-J})} \tag{B.8}$$

holds. Moreover, if $J>d+2$ then for each $A\in\mathcal{L}_2^{HS}(H_{-J})$ and $n\in\mathbb{N}$

$$\max_{x\in\mathbb{T}_n^d}\left|\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x)-\mathrm{pr}_n\mathrm{Tr}\left(\partial_j^{\otimes 2}A\right)(x)\right|\leq\frac{C_J}{n}\|A\|_{\mathcal{L}_2^{HS}(H_{-J})}.$$

Proof. We first prove the estimate (B.8). Setting

$$R_n(x):=\mathrm{Tr}\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x)=\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x),\quad x\in\mathbb{T}_n^d,$$

and using (A.10), we get

$$\begin{aligned}\|\mathrm{ex}_nR_n\|_{H_I}^2&=\sum_{k\in\mathbb{Z}^d}\left(1+|k|^2\right)^I|\langle\mathrm{ex}_nR_n,\varsigma_k\rangle|^2=\sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^I|\langle R_n,\varsigma_k\rangle_n|^2\\&=\sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^I\left|\left\langle\mathrm{Tr}\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A,\varsigma_k\right\rangle_n\right|^2.\end{aligned}$$

By (B.1), we can write for $x\in\mathbb{T}_n^d$

$$\begin{aligned}\mathrm{Tr}\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x)&=\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}A[\varsigma_{-l},\varsigma_{-\tilde{l}}]\left(\partial_{n,j}\varsigma_l(x)\partial_{n,j}\varsigma_{\tilde{l}}(x)\right)\\&=\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\mu_{l,j}\mu_{\tilde{l},j}A[\varsigma_{-l},\varsigma_{-\tilde{l}}]\varsigma_{l+\tilde{l}}(x)\end{aligned}$$

and thus, using the periodicity of $\varsigma_{l+\tilde{l}}$ on $\mathbb{T}_n^d$, we estimate

$$\begin{aligned}\left|\left\langle\mathrm{Tr}\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A,\varsigma_k\right\rangle_n\right|^2&=\left|\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\mu_{l,j}\mu_{\tilde{l},j}A[\varsigma_l,\varsigma_{\tilde{l}}]\langle\varsigma_{l+\tilde{l}},\varsigma_k\rangle_n\right|^2\\&\leq\left|\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\mu_{l,j}\mu_{\tilde{l},j}A[\varsigma_l,\varsigma_{\tilde{l}}]\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}\right|^2\\&\leq\|A\|_{\mathcal{L}_2^{HS}(H_{-J})}^2\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\frac{|l|^2|\tilde{l}|^2}{(1+|l|^2)^J(1+|\tilde{l}|^2)^J}\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}\\&\leq\|A\|_{\mathcal{L}_2^{HS}(H_{-J})}^2\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\frac{1}{(1+|l|^2)^{J-1}(1+|\tilde{l}|^2)^{J-1}}\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}.\end{aligned} \tag{B.9}$$

Combining the estimates together and using the fact that $|k|\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}\leq|l+\tilde{l}|\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}$, we obtain

$$\begin{aligned}\|\mathrm{ex}_nR_n\|_{H_I}^2\leq\|A\|_{\mathcal{L}_2^{HS}(H_{-J})}^2&\sum_{k\in\mathbb{Z}_n^d}\left(1+|k|^2\right)^I\\&\cdot\sum_{l,\tilde{l}\in\mathbb{Z}_n^d}\frac{1}{(1+|l|^2)^{J-1}(1+|\tilde{l}|^2)^{J-1}}\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}\end{aligned}$$

$$
\begin{aligned}
&\leq \|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})} \sum_{k\in\mathbb{Z}_n^d}\sum_{l,\tilde{l}\in\mathbb{Z}_n^d} \frac{\left(1+|l+\tilde{l}|^2\right)^I}{(1+|l|^2)^{J-1}(1+|\tilde{l}|^2)^{J-1}}\mathbb{I}_{\{l+\tilde{l}=k \mod (2n+1)\}}\\
&= 3^d\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})} \sum_{l,\tilde{l}\in\mathbb{Z}_n^d} \frac{\left(1+|l+\tilde{l}|^2\right)^I}{(1+|l|^2)^{J-1}(1+|\tilde{l}|^2)^{J-1}} < C_{J,I}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})},
\end{aligned}
$$

since $J-1-I>\frac{d}{2}$.

To get the second part of the statement, we first use (B.1) and the Cauchy-Schwarz inequality to estimate for each $x,y\in\mathbb{T}_n^d$

$$
\begin{aligned}
&\left|\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,y)-\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,y)\right|^2\\
&\quad=\left|\sum_{k,l\in\mathbb{Z}_n^d}A[\varsigma_{-k},\varsigma_{-l}]\partial_{n,j}\varsigma_k(x)\partial_{n,j}\varsigma_l(y)-\sum_{k,l\in\mathbb{Z}_n^d}A[\varsigma_{-k},\varsigma_{-l}]\partial_j\varsigma_k(x)\partial_j\varsigma_l(y)\right|^2\\
&\quad\leq\left[\sum_{k,l\in\mathbb{Z}_n^d}|A[\varsigma_{-k},\varsigma_{-l}]|\left|\mu_{k,j}^n\mu_{l,j}^n+k_jl_j\right|\right]^2\\
&\quad\leq\sum_{k,l\in\mathbb{Z}_n^d}|A[\varsigma_k,\varsigma_l]|^2(1+|k|^2)^J(1+|l|^2)^J\\
&\qquad\qquad\cdot\sum_{k,l\in\mathbb{Z}_n^d}(1+|k|^2)^{-J}(1+|l|^2)^{-J}\left|\mu_{k,j}^n\mu_{l,j}^n+k_jl_j\right|^2\\
&\quad\leq 2\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}\sum_{k,l\in\mathbb{Z}_n^d}(1+|k|^2)^{-J}(1+|l|^2)^{-J}\\
&\qquad\qquad\cdot\left[|\mu_{l,j}^n|^2\left|\mu_{k,j}^n-\mathrm{i}k_j\right|^2+|k_j|^2|\mu_{l,j}^n-\mathrm{i}l_j|^2\right].
\end{aligned}
$$

By Lemma B.1 and Taylor's expansion

$$
\mu_{k,j}^n=\mathrm{i}k_j-\frac{k_j^2}{n}\theta_{n,j}
$$

for some $\theta_{j,n}\in\mathbb{C}$ such that $|\theta_{j,n}|\leq 1$, we can continue the estimate as follow

$$
\begin{aligned}
&\frac{2}{n^2}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}\sum_{k,l\in\mathbb{Z}_n^d}(1+|k|^2)^{-J}(1+|l|^2)^{-J}\left[l_j^2k_j^4+k_j^2l_j^4\right]\\
&\qquad\leq\frac{4}{n^2}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}\sum_{k,l\in\mathbb{Z}_n^d}(1+|k|^2)^{-J+2}(1+|l|^2)^{-J+2}\\
&\qquad\leq\frac{C_J}{n^2}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}
\end{aligned}
$$

since $J>\frac{d}{2}+2$. Thus,

$$
\max_{x\in\mathbb{T}_n^d}\left|\partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x)-\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x)^2\right|\leq\frac{C_J}{n}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}. \tag{B.10}
$$

We next compute the norm

$$
\left\|K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A}-\partial_j^{\otimes 2}A\right\|^2_{\mathcal{L}_2^{HS}(H_{-J+2})}
$$

$$= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{J-2}(1+|l|^2)^{J-2} \left| K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A}[\varsigma_k,\varsigma_l] - \partial_j^{\otimes 2}A[\varsigma_k,\varsigma_l] \right|^2 \tag{B.11}$$

$$= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{J-2}(1+|l|^2)^{J-2} \left| \left\langle \varsigma_k\otimes\varsigma_l, \partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A \right\rangle - A[\partial_j\varsigma_k,\partial_j\varsigma_l] \right|^2.$$

Using the definition of $\mathrm{pr}_n^{\otimes 2}$ and (B.1), we continue the equality as follows

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{J-2}(1+|l|^2)^{J-2} \left| \left\langle \partial_j\varsigma_k\otimes\partial_j\varsigma_l, \mathrm{pr}_n^{\otimes 2}A \right\rangle - A[\partial_j\varsigma_k,\partial_j\varsigma_l] \right|^2 \\
&\quad = \sum_{k,l\notin\mathbb{Z}_n^d} (1+|k|^2)^{J-2}(1+|l|^2)^{J-2} k_j^2 l_j^2 \left| A[\varsigma_k,\varsigma_l] \right|^2 \\
&\quad \le \sum_{k,l\notin\mathbb{Z}_n^d} (1+|k|^2)^{J-1}(1+|l|^2)^{J-1} \left| A[\varsigma_k,\varsigma_l] \right|^2 \le \frac{1}{n^2}\|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})}.
\end{aligned}$$

Now, taking $\tilde{I} \in \left(\frac{d}{2}, J-2-d/2\right)$ and applying Lemmas B.2, B.15, B.16 and (B.11), we obtain

$$\begin{aligned}
\left\| \mathrm{Tr}K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} - \mathrm{pr}_n\mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{\mathrm{C}} &\le \left\| \mathrm{Tr}K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} - \mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{\mathrm{C}} \\
&\quad + \left\| \mathrm{Tr}\partial_j^{\otimes 2}A - \mathrm{pr}_n\mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{\mathrm{C}} \\
&\le C_J \left\| \mathrm{Tr}K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} - \mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{H_{\tilde{I}}} + \left\| \mathrm{Tr}\partial_j^{\otimes 2}A - \mathrm{pr}_n\mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{H_{\tilde{I}}} \\
&\le C_{J,\tilde{I}} \left\| K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} - \partial_j^{\otimes 2}A \right\|_{\mathcal{L}_2^{HS}(H_{-J+2})} + \frac{C_{\tilde{I}}}{n} \left\| \mathrm{Tr}\partial_j^{\otimes 2}A \right\|_{H_{\tilde{I}+1}} \\
&\le \frac{C_{J,\tilde{I}}}{n}\|A\|_{\mathcal{L}_2^{HS}(H_{-J})} + \frac{C_{\tilde{I}}}{n}\|\partial_j^{\otimes 2}A\|_{\mathcal{L}_2^{HS}(H_{-J+1})} \\
&\le \frac{C_{J,\tilde{I}}}{n}\|A\|_{\mathcal{L}_2^{HS}(H_{-J})}.
\end{aligned} \tag{B.12}$$

Thus,

$$\begin{aligned}
\max_{x\in\mathbb{T}_n^d} \left| \partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x) - \mathrm{pr}_n\mathrm{Tr}\partial_j^{\otimes 2}A(x) \right| &\le \max_{x\in\mathbb{T}_n^d} \left| \partial_{n,j}^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x) - \partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x) \right| \\
&\quad + \|\mathrm{Tr}K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} - \mathrm{pr}_n\mathrm{Tr}\partial_j^{\otimes 2}A\|_{\mathrm{C}} \le \frac{C}{n}\|A\|_{\mathcal{L}_2^{HS}(H_{-J})},
\end{aligned}$$

due to the triangle inequality the estimates (B.10), (B.12) and the fact that $\mathrm{Tr}K_{\partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A} = \partial_j^{\otimes 2}\mathrm{pr}_n^{\otimes 2}A(x,x)$, $x\in\mathbb{T}^d$ (see Lemma B.15). This completes the proof of the proposition. □

For $A \in \mathcal{L}_2^{HS}(H_{-J})$ and $B \in \mathcal{L}_2^{HS}(H_J)$ we define

$$A : B := \sum_{k,l\in\mathbb{Z}^d} A[\tilde{\varsigma}_k,\tilde{\varsigma}_l]B[\tilde{\varsigma}_k,\tilde{\varsigma}_l],$$

The series above absolutely converges and

$$\begin{aligned}
|A:B|^2 &\le \left(\sum_{k,l\in\mathbb{Z}^d} |A[\tilde{\varsigma}_k,\tilde{\varsigma}_l]B[\tilde{\varsigma}_k,\tilde{\varsigma}_l]| \right)^2 \\
&= \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^{-J}(1+|l|^2)^{-J} \left| A[\tilde{\varsigma}_k,\tilde{\varsigma}_l] \right|^2
\end{aligned} \tag{B.13}$$

$$\cdot \sum_{k,l\in\mathbb{Z}^d} (1+|k|^2)^J (1+|l|^2)^J \, |B[\tilde{\varsigma}_k, \tilde{\varsigma}_l]|^2$$
$$= \|A\|^2_{\mathcal{L}_2^{HS}(H_{-J})} \|B\|^2_{\mathcal{L}_2^{HS}(H_J)},$$

where we Lemma (3.2) and Hölder's inequality.

B.3 Differentiable functions on H_J

In this section, we will investigate some differential properties of functions defined on Sobolev spaces.

Recall that for $J < I$ we have $H_I \subset H_J$. Abusing notation, the restrictions of a function $F : H_J^k \to \mathbb{R}$ to H_I^k will be denoted also by F for $k \in \mathbb{N}$. Let $(E, \|\cdot\|_E)$ denote a Banach space. The following statement directly follows from the inequality $\|\cdot\|_{H_J} \leq \|\cdot\|_{H_I}$.

Lemma B.19. *For each $J, I \in \mathbb{R}$, $J < I$, and $k \in \mathbb{N}$ the space $\mathcal{L}_k(H_J; E)$ is continuously embedded into $\mathcal{L}_k(H_I; E)$ and*

$$\|A\|_{\mathcal{L}_k(H_I;E)} \leq \|A\|_{\mathcal{L}_k(H_J;E)}$$

for all $A \in \mathcal{L}_k(H_J; E)$.

Recall that a function $F : H_J \to E$ is differentiable[8] at $f \in H_J$ if it is continuous at f and there exists a bounded linear map $DF(f)$ from H_J to E such that

$$F(f+h) = F(f) + DF(f)[h] + o(\|h\|_{H_J})$$

as $h \to 0$. Following [9, Section 5], we defined the k-th derivative $D^kF(f)$ of a map $F : H_J \to \mathbb{R}$ at $f \in H_J$ as an element of $\mathcal{L}_k(H_J)$ that is identified with the derivative of $D^{k-1}F : H_J \to \mathcal{L}_{k-1}(H_J)$ at f. Considering a differentiable function defined on a Sobolev space H_J, we often consider its restriction to a smaller Sobolev space H_I with $J < I$. The following statement guarantees the preservation of the differentiability. To point out that D^kF is the derivative of F with respect to the topology of the space H_J in the next statements, we will write D_J^kF instead.

Lemma B.20. *Let $J, I \in \mathbb{R}$, $J < I$, and $F : H_J \to E$ be a differentiable function at $f \in H_I$ in the space H_J. Then F is differentiable at f in the topology of the space H_I, $D_IF(f)$ coincide with the restriction of $D_JF(f)$ to H_I and*

$$\|D_IF(f)\|_{\mathcal{L}_1(H_I;E)} \leq \|D_JF(f)\|_{\mathcal{L}_1(H_J;E)}.$$

Proof. The continuity of F at f in the space H_I trivially follows from the continuous embedding of H_I into H_J. According to Lemma B.19, $D_JF(f) \in \mathcal{L}_k(H_I; E)$ and

$$\|D_JF(f)\|_{\mathcal{L}_k(H_I;E)} \leq \|D_JF(f)\|_{\mathcal{L}_k(H_J;E)}.$$

We have only to show that $D_IF(f)[h] = D_JF(f)[h]$, $h \in H_I$. Using the differentiability of F in H_J at f and the fact that $\|\cdot\|_{H_J} \leq \|\cdot\|_{H_I}$, we get

$$\begin{aligned} F(f+h) &= F(f) + D_J(f)[h] + o(\|h\|_{H_J}) \\ &= F(f) + D_J(f)[h] + o(\|h\|_{H_I}). \end{aligned}$$

This completes the proof of the lemma. ☐

[8]see [9, Definition on p. 25]

The following corollary is the direct consequence of Lemma B.20.

Corollary B.21. *For each $J, I \in \mathbb{R}$, $J < I$, and $m \in \mathbb{N}_0$ the space $\mathrm{C}^m(H_J)$ is a subset of $\mathrm{C}^m(H_I)$. Moreover, for each $F \in \mathrm{C}^m(H_J)$, $k \in [m]$, $f \in H_I$ the derivatives $D_J^k F(f)$ and $D_I^k F(f)$ coincide on H_I^k and*

$$\|D_I^k F(f)\|_{\mathcal{L}_k(H_I)} \leq \|D_J^k F(f)\|_{\mathcal{L}_k(H_J)}.$$

Our further goal will be to investigate the differentiability of $F \circ \mathrm{ex}_n : L_2(\mathbb{T}_n^d) \to \mathbb{R}$ for $F \in \mathrm{C}^m(H_J)$. Using the fact that $\mathrm{ex}_n : L_2(\mathbb{T}_n^d) \to H_J$ is a continuous linear operator, it is continuously differentiable with

$$D\mathrm{ex}_n(f)[h] = \mathrm{ex}_n h$$

for each $f, h \in L_2(\mathbb{T}_n^d)$.

Lemma B.22. *Let $F \in \mathrm{C}^1(H_J)$ for some $m \in \mathbb{N}$ and $J \in \mathbb{R}$. Then the function $F \circ \mathrm{ex}_n$ belongs to $\mathrm{C}^1(L_2(\mathbb{T}_n^d))$ and*

$$D(F \circ \mathrm{ex}_n) = \mathrm{pr}_n DF \circ \mathrm{ex}_n.$$

Proof. The differentiability of $F \circ \mathrm{ex}_n$ follows from [9, Theorem 2.2.1] and the differentiability of $F : H_J \to \mathbb{R}$ and $\mathrm{ex}_n : L_2(\mathbb{T}_n^d) \to H_J$. We will only compute the derivative of $D(F \circ \mathrm{ex}_n)$. Taking $f, h \in L_2(\mathbb{T}_n^d)$ and using the chain rule and (A.9), we compute

$$\begin{aligned} D(F \circ \mathrm{ex}_n)(f)[h] &= DF(\mathrm{ex}_n f)\left[D\mathrm{ex}_n(f)[h]\right] = DF(\mathrm{ex}_n f)[\mathrm{ex}_n h] \\ &= \langle DF(\mathrm{ex}_n f), \mathrm{ex}_n h\rangle = \langle \mathrm{pr}_n (DF)(\mathrm{ex}_n f), h\rangle_n. \end{aligned}$$

This completes the proof of the statement. □

C Some additional facts and proofs

We recall that Sym_n denotes the Hilbert space of symmetric matrices $A = (A_{k,l})_{k,l \in \mathbb{Z}_n^d}$ with real-valued entries equipped with the inner product

$$A : B = \sum_{k,l \in \mathbb{Z}_n^d} A_{k,l} B_{k,l}.$$

An open subset of positively defined matrices from Sym_n is denoted by Sym_n^+.

Lemma C.1. *Let $A = (A_{k,l})_{k,l \in \mathbb{Z}_n^d} \in \mathrm{Sym}_n^+$ and $B \in \mathrm{Sym}_n$. Then for a standard Gaussian vector ζ in $\mathbb{R}^{\mathbb{Z}_n^d}$ and $f \in \mathrm{C}_l^2(\mathbb{R}^{\mathbb{Z}_n^d})$ the integration-by-parts formula*

$$\mathbb{E}\left[Df(A\zeta) \cdot (B\zeta)\right] = \mathbb{E}\left[D^2 f(A\zeta) : (BA)\right]$$

holds.

Proof. Setting $R := BA^{-1}$, $\eta := A\zeta$, and using the integration-by-parts formula, we get

$$\begin{aligned} \mathbb{E}\left[Df\left(A\zeta\right) \cdot (B\zeta)\right] &= \mathbb{E}\left[Df\left(A\zeta\right) \cdot \left(BA^{-1}A\zeta\right)\right] \\ &= \sum_{k,l \in \mathbb{Z}_n^d} \mathbb{E}\left[\frac{\partial f}{\partial x_k}(\eta) R_{k,l} \eta_l\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l,\tilde{k}\in\mathbb{Z}_n^d} \mathbb{E}\left[\frac{\partial^2 f}{\partial x_k \partial x_{\tilde{k}}}(\eta) R_{k,l} \mathrm{Cov}(\eta_l, \eta_{\tilde{k}})\right] \\
&= \sum_{k,l,\tilde{k}\in Z_n^d} \mathbb{E}\left[\frac{\partial^2 f}{\partial x_k \partial x_{\tilde{k}}}(\eta) R_{k,l} (A^2)_{l,\tilde{k}}\right] \\
&= \sum_{k,\tilde{k}\in Z_n^d} \mathbb{E}\left[\frac{\partial^2 f}{\partial x_k \partial x_{\tilde{k}}}(\eta) (RA^2)_{k,\tilde{k}}\right] \\
&= \mathbb{E}\left[D^2 f(A\zeta) : (BA)\right].
\end{aligned}
$$

This completes the proof of the lemma. □

For completeness of the presentation, we next provide the estimate of the term

$$
I := \frac{1}{(2n+1)^d} \sum_{x\neq y\in\mathbb{T}_n^d} \mathbb{E}\left[\left(\eta_t^n(x) - \rho_t^n(x)\right)\left(\eta_t^n(y) - \rho_t^n(y)\right)\right] \varphi_n(x)\varphi_n(y)
$$

in the proof of Lemma [56, p. 32], following the proof of the main theorem in [56, p. 32].

Proof. (Estimate of off-diagonal sum in the proof of Lemma 2.4) . Set for $x, y \in \mathbb{T}_n^d$

$$
\begin{aligned}
V(t,x,y) &:= \frac{1}{(2n+1)^d} \mathbb{E}(\eta_t^n(x) - \rho_t^n(x))(\eta_t^n(y) - \rho_t^n(y)) \\
&= \mathbb{E}\eta_t^n(x)\eta_t^n(y) - \rho_t^n(x)\rho_t^n(y),
\end{aligned}
$$

where the letter equality follows from the definition of ρ_t^n. Applying the generator $2\pi^2\Delta_n \otimes \mathcal{G}_n^{EP}$ of the Markov process (ρ_t^n, η_t^n), $t \geq 0$, to the function $G(x,y;\eta,\rho) = \eta(x)\eta(y) - \rho(x)\rho(y)$ for fixed $x, y \in \mathbb{T}_n^d$, $x \neq y$, we get

$$
\left(2\pi^2\Delta_n \otimes \mathcal{G}_n^{EP}\right) G(x,y;\cdot,\cdot)(\eta,\rho) = \mathcal{G}_n^{EP}\left[\eta(x)\eta(y)\right] - 2\pi^2\Delta_n\left[\rho(x)\rho(y)\right].
$$

Now we separately rewrite

$$
\begin{aligned}
\mathcal{G}_n^{EP}\left[\eta(x)\eta(y)\right] &= \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \sum_{z\in\mathbb{T}_n^d} \left(\eta^{z\leftrightarrow z+e_j}(x)\eta^{z\leftrightarrow z+e_j}(y) - \eta(x)\eta(y)\right) \\
&= \frac{(2n+1)^2}{2} \sum_{e\in E_1} \left(\eta(x+e)\eta(y) - \eta(x)\eta(y)\right)\left(1 - \mathbb{I}_{\{x+e=y\}}\right) \\
&+ \frac{(2n+1)^2}{2} \sum_{e\in E_1} \left(\eta(x)\eta(y-e) - \eta(x)\eta(y)\right)\left(1 - \mathbb{I}_{\{x+e=y\}}\right),
\end{aligned}
$$

where the summation is taken over $E_1 := \{\pm e_j, j \in [d]\}$. We also note that

$$
\begin{aligned}
2\pi^2\Delta_n\left[\rho(x)\rho(y)\right] &= \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \left(\rho(x+e_j)\rho(y) + \rho(x-e_j)\rho(y) - 2\rho(x)\rho(y)\right) \\
&+ \frac{(2n+1)^2}{2} \sum_{j=1}^{d} \left(\rho(x)\rho(y+e_j) + \rho(x)\rho(y-e_j) - 2\rho(x)\rho(y)\right) \\
&= \frac{(2n+1)^2}{2} \sum_{e\in E_1} \left(\rho(x+e)\rho(y) - \rho(x)\rho(y)\right)
\end{aligned}
$$

$$+\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\rho(x)\rho(y-e)-\rho(x)\rho(y)\right).$$

Hence

$$\begin{aligned}
&\left(2\pi^2\Delta_n\otimes\mathcal{G}_n^{EP}\right)G(x,y;\eta,\rho)\\
&=\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\eta(x+e)\eta(y)-\eta(x)\eta(y)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right)\\
&+\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\eta(x)\eta(y-e)-\eta(x)\eta(y)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right)\\
&-\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\rho(x+e)\rho(y)-\rho(x)\rho(y)\right)\\
&-\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\rho(x)\rho(y-e)-\rho(x)\rho(y)\right)\\
&=\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(G(x+e,y;\eta,\rho)-G(x,y;\eta,\rho)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right)\\
&+\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(G(x,y-e;\eta,\rho)-G(x,y;\eta,\rho)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right)\\
&-\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\rho(x)-\rho(y)\right)^2\mathbb{I}_{\{x+e=y\}}.
\end{aligned}$$

This implies that the function

$$V(t,x,y)=\mathbb{E}\left[\eta_t^n(x)\eta_t^n(y)\right]-\rho_t^n(x)\rho_t^n(y)$$

is a solution to the following differential equation

$$\frac{d}{dt}V(t,x,y)=\mathcal{L}V(t,x,y)-\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(\rho_t^n(x)-\rho_t^n(y)\right)^2\mathbb{I}_{\{x+e=y\}},$$

where

$$\begin{aligned}
\mathcal{L}V(x,y)=&\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(V(x+e,y)-V(x,y)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right)\\
&+\frac{(2n+1)^2}{2}\sum_{e\in E_1}\left(V(x,y-e)-V(x,y)\right)\left(1-\mathbb{I}_{\{x+e=y\}}\right).
\end{aligned}$$

Note that $\mathcal{L}$ is the generator of the process $\{X_t,Y_t\}$, $t\geq 0$, on $\mathbb{T}_n^d\times\mathbb{T}_n^d$ that evolves as an exclusion process with two particles. Let $P_t(x,y;u,v)$, $u,v\in\mathbb{T}_n^d$, be its semigroup. Then

$$\begin{aligned}
V(t,x,y)&=P_tV(0,x,y)\\
&-\frac{(2n+1)^2}{2}\int_0^t\sum_{u,v\in\mathbb{T}_n^d}\sum_{e\in E_1}P_{t-s}(x,y;u,v)\left(\rho_s^n(u)-\rho_s^n(v)\right)^2\mathbb{I}_{\{u+e=v\}}ds\\
&=P_tV(0,x,y)
\end{aligned}$$

$$- \frac{(2n+1)^2}{2} \int_0^t \sum_{e \in E_1} \sum_{u \in \mathbb{T}_n^d} P_{t-s}(x,y;u,u+e) \left(\rho_s^n(u) - \rho_s^n(u+e)\right)^2 ds.$$

Due to the independents of $\eta_0^n(x)$ and $\eta_0^n(y)$ we conclude $V(0,x,y) = 0$. Therefore, $P_t V(0,x,y) = 0$. Thus,

$$V(t,x,y) = - \frac{(2n+1)^2}{2} \int_0^t \sum_{e \in E_1} \sum_{u \in \mathbb{T}_n^d} P_{t-s}(x,y;u,u+e) \left(\rho_s^n(u) - \rho_s^n(u+e)\right)^2 ds.$$

Consequently, we can estimate

$$\begin{aligned}
|I| &\leq \frac{(2n+1)^2}{2} \frac{1}{(2n+1)^d} \sum_{e \in E_1} \sum_{u \in \mathbb{T}_n^d} \sum_{x \neq y} |\varphi_n(x)||\varphi_n(y)| \\
&\qquad \cdot \int_0^t P_{t-s}(x,y;u,u+e) \left(\rho_s^n(u) - \rho_s^n(u+e)\right)^2 ds \\
&\leq \frac{2\pi^2}{(2n+1)^d} \sup_{s \in [0,t]} \max_{u \in \mathbb{T}_n^d} |\nabla_n \rho_s^n(u)|^2 \\
&\qquad \cdot \sum_{e \in E_1} \sum_{u \in \mathbb{T}_n^d} \sum_{x \neq y} |\varphi_n(x)||\varphi_n(y)| \int_0^t P_{t-s}(x,y;u,u+e) ds.
\end{aligned}$$

Using the duality of the SSEP, we get for each $e \in E_1$

$$\begin{aligned}
&\frac{1}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \neq y} |\varphi_n(x)||\varphi_n(y)| \int_0^t P_{t-s}(x,y;u,u+e) ds \\
&\quad = \frac{1}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \neq y} |\varphi_n(x)||\varphi_n(y)| \int_0^t P_{t-s}(u,u+e;x,y) ds \\
&\quad \leq \frac{\|\varphi_n\|_{n,\mathrm{C}}}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \in \mathbb{T}_n^d} |\varphi_n(x)| \int_0^t P_{t-s}(u,u+e;x,\mathbb{T}_n^d) ds.
\end{aligned}$$

Since

$$P_{t-s}(u,u+e;x,\mathbb{T}_n^d) = P_{t-s}^0(u;x) + P_{t-s}^0(u+e;x),$$

where P_t^0 is the transition kernel for a single particle executing a random walk in $\mathbb{T}_n^d$, we get for $e \in E_1$

$$\begin{aligned}
&\frac{1}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \in \mathbb{T}_n^d} |\varphi_n(x)| \int_0^t P_{t-s}(u,u+e;x,\mathbb{T}_n^d) ds \\
&\quad = \frac{1}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \in \mathbb{T}_n^d} |\varphi_n(x)| \int_0^t \left[P_{t-s}^0(u;x) + P_{t-s}^0(u+e;x)\right] ds \\
&\quad = \frac{1}{(2n+1)^d} \sum_{u \in \mathbb{T}_n^d} \sum_{x \in \mathbb{T}_n^d} |\varphi_n(x)| \int_0^t \left[P_{t-s}^0(x;u) + P_{t-s}^0(x;u+e)\right] ds \\
&\quad = \frac{t}{(2n+1)^d} \sum_{x \in \mathbb{T}_n^d} |\varphi_n(x)| \leq t \|\varphi_n\|_{n,\mathrm{C}}.
\end{aligned}$$

Combining the estimates above, we conclude

$$|I| \leq 2\pi^2 \sup_{s \in [0,t]} \max_{u \in \mathbb{T}_n^d} |\nabla_n \rho_s^n(u)|^2 \, \|\mathrm{pr}_n \varphi\|_{n,\mathrm{C}}^2 t.$$

This completes the estimate. □

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